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# Kato's inequality when $\Delta_p u$ is a measure and related topics

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# 1 Introduction

$\Omega$ : a bounded domain of $\mathbf{R}^N$ $(N \geq 1)$.

$$\Delta u = \operatorname{div}(\nabla u), \quad \nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \ldots, \partial u/\partial x_N).$$

## 1.1 Convex type inequality

**Lemma 1** *(The classical convex type Kato's inequality ) Let* $u \in L^1_{\text{loc}}(\Omega)$ *s.t.* $\Delta u \in L^1_{\text{loc}}(\Omega)$, *then* $\Delta|u|$ *and* $\Delta u^+$ *are Radon measures and we have*

$$\Delta|u| \geq \operatorname{sgn}(u)\Delta u \qquad in\ D'(\Omega), \tag{1}$$

$$\Delta u^+ \geq \chi_{[u\geq 0]}\Delta u \qquad in\ D'(\Omega), \tag{2}$$

*where* $\operatorname{sgn}(s) = 1$ *if* $s > 0$, $-1$ *if* $s < 0$ *and zero at* $s = 0$ $u^+ = \max[u, 0]$.

**Remark 1.1** *1. If we assume in addition that u is continuous in* $\Omega$, *then we have*

$$\Delta|u| = \operatorname{sgn}(u)\Delta u \qquad in\ D'([u \neq 0]). \tag{3}$$

*The inequality (1) ;* $\Delta|u| \geq \operatorname{sgn}(u)\Delta u \quad in\ D'(\Omega)$

*is a consequence of the fact that* $|u|$ *takes its minimum on the set* $[u = 0]$.

*2. Similar inequalities hold*

*when* $\Delta u$ *is replaced by elliptic operator* $M(x, \partial_x)$*:*

$$M(x, \partial_x)u = \sum_{j.k=1}^{N} \partial_{x_j}\left(a_{j,k}(x)\partial_{x_k} u\right),$$

*where* $a_{j,k}(x) \in C^1$, *and for some* $C > 0$

$$\sum_{j.k=1} a_{j,k}(x)\xi_j\xi_k \geq C|\xi|^2, \quad for\ any\ \xi \in \mathbf{R}^N$$

## 1.2 Concave type inequality

**Definition 1** (Truncation) *:* $T_k(s)$ : *Given* $k > 0$, *we denote by* $T_k$:$\mathbf{R} \to \mathbf{R}$ *a truncation function*

$$T_k(s) := \begin{cases} k & if\ s \geq k, \\ s & if\ -k < s < k, \\ -k & if\ s \leq -k. \end{cases} \tag{4}$$

Since $T_k|_{\mathbf{R}_+}$ is concave, we have the following lemma:

**Lemma 2** *Assume that* $u \in L^1_{\rm loc}(\Omega)$, $\Delta u \in L^1_{\rm loc}(\Omega)$ *and* $u \geq 0$ *a.e. in* $\Omega$. *Then, for any* $k \geq 0$ *we have*

$$\Delta(T_k(u)) \leq \chi_{[0 \leq u \leq k]} \Delta u \quad in\ D'(\Omega), \tag{5}$$

*where* $\chi_S(x)$ *is a characteristic function of* $S \subset \mathbf{R}$.

Moreover, when$\Delta u$ can be replaced by $\Delta_p u$ under additional assumptions on distributional derivatives of $u \in L^1_{\rm loc}(\Omega)$.
Here, $p$-Laplace operator is defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u),$$

**Example 1** *(Classical)*
*Let* $1 < p < \infty$. *For* $u \in K_p(\Omega)$ *we have*

1. *(Convex type):*

$$\Delta_p |u| \geq \operatorname{sgn}(u) \Delta_p u \qquad in\ D'(\Omega), \tag{6}$$

$$\Delta_p u^+ \geq \chi_{[u \geq 0]} \Delta_p u \qquad in\ D'(\Omega). \tag{7}$$

2. *(Concave type): If* $u \geq 0$, *the we have*

$$\Delta_p T_k(u) \leq \chi_{[0 \leq u \leq k]} \Delta_p u \qquad in\ D'(\Omega). \tag{8}$$

*Here* $K_p(\Omega)$ *is given by*

$$K_p(\Omega) = \{u \in L^1_{loc}(\Omega) : \partial_j u, \partial^2_{j,k} u \in L^{p^*}_{loc}(\Omega),$$
$$|\nabla u|^{p-2} |\partial^2_{j,k} u| \in L^1_{loc}(\Omega) \ for\ j,k = 1,2,...,N\},$$

*where* $p^* = \max[(p-1), 1]$.

## 2 Main Aim

Consider a class of second order elliptic operators $\mathcal{A}$ including $\Delta_p$ and establish improved Kato's inequalities when $\mathcal{A}u$ is a Radon measure.

$$\mathcal{A}u = \operatorname{div} A(x, \nabla u), \tag{9}$$

where $A : \Omega \times R^N \mapsto R^N$ satisfies the following assumptions for some positive numbers $c_1, c_2$ and $c_3$:

1. the function $x \mapsto A(x,\xi)$ is bounded measurable for ${}^\forall \xi \in R^N$,

2. the function $\xi \mapsto A(x,\xi)$ is continuous for a.e. $x \in \Omega$,

3. $$|A(x,\xi) - A(x,\eta)| \leq c_2(|\xi| + |\eta|)^{p-2}|\xi - \eta|, \quad {}^\forall \xi, \eta \in R^N, \text{ a.e. } x \in \Omega,$$

4. $$(A(x,\xi) - A(x,\eta)) \cdot (\xi - \eta) \geq c_3(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2, \ {}^\forall \xi, \eta \in R^N, \text{ a.e. } x \in \Omega,$$

5. $$A(x, \lambda\xi) = \lambda|\lambda|^{p-2}A(x,\xi), \quad \text{for all } \lambda \in R, \lambda \neq 0.$$

**Remark 2.1** *1. It follows from the assumption 4 that we have*

$$A(x,\xi) \cdot \xi \geq c_1|\xi|^p \quad \textit{for all } \xi \in R^N \textit{ and a.e. } x \in \Omega.$$

*2. For some* $C > 0$

$$\sum_{j,k=1}^{N} \left| \frac{\partial A_j}{\partial \xi_k}(x,\xi) \right| \leq C|\xi|^{p-2}, \quad {}^\forall \xi \in R^N \setminus \{0\}, \ \textit{a.e. } x \in \Omega, \tag{10}$$

*Then* $A$ *satisfies the assumptions 3 and 4.*

**Example 2** *1. In the case of* $\Delta_p$, $A = A(\xi) = |\xi|^{p-2}\xi$, *and* $A$ *satisfies the estimate (10).*

*2. Assume that* $a_{j,k} \in L^\infty(\Omega)$, $a_{j,k} = a_{k,j}$ *for* $j,k = 1,2,\ldots,N$ *and* $\{a_{j,k}\}$ *satisfies the uniformly elliptic estimate:*

$$\sum_{j,k=1}^{N} a_{j,k}\xi_j\xi_k \geq C|\xi|^2 \quad \textit{for any} \quad \xi \in R^N.$$

$$\mathcal{B}u = \sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\left(a_{j,k}(x)|\nabla u|^{p-2}\frac{\partial u}{\partial x_k}\right). \tag{11}$$

*If* $p$ *is sufficiently close to 2, then the operator* $\mathcal{B}$ *satisfies the assumptions* $1 \sim 5$ *with* $A_j(x,\xi) = \sum_{k=1}^{N}\left(a_{j,k}(x)|\xi|^{p-2}\xi_k\right)$.

**Definition 2** *(* $M(\Omega)$*: the space of Radon measure):*

$\mu \in M(\Omega) \iff$ *For every open set* $\omega \subset\subset \Omega$, ${}^\exists C_\omega > 0$ *s.t.* $|\int_\Omega \varphi \, d\mu| \leq C_\omega ||\varphi||_{L^\infty}$, *for* ${}^\forall \varphi \in C_0^\infty(\omega)$.

*We do not assume the finiteness of the total measure* $|\mu|(\Omega) < \infty$ *but assume* $|\mu|(\omega) < \infty$ *for each* $\omega \subset\subset \Omega$.

## 3 Decomposition of Radon measures

For any $\mu \in M(\Omega)$, $\mu$ can be uniquely decomposed as a sum of two Radon measures on $\Omega$ (see e.g. [7, 10]) : $\mu = \mu_d + \mu_c$, where

$$\begin{cases} \mu_d(A) = 0 & \text{for any Borel set } A \subset \Omega \text{ s.t } C_p(A, \Omega) = 0, \\ |\mu_c|(\Omega \setminus F) = 0 & \text{for some Borel set } F \subset \Omega \text{ s.t } C_p(F, \Omega) = 0. \end{cases}$$

Total measure: $|\mu| = \mu^+ + \mu^-$.

**Definition 3** *(A p-capacity relative to $\Omega$)*
*For each compact set $K \subset \Omega$,*

$$C_p(K, \Omega) = \inf\{\int_\Omega |\nabla\varphi|^p : \varphi \in C_0^\infty(\Omega), \varphi \geq 1 \textit{ in some nbd of } K\}.$$

Note that $(\mu_d)^+ = (\mu^+)_d$ and $(\mu_c)^+ = (\mu^+)_c$ by the definition.

## 4 Definition of admissible class

**Definition 4 (Admissible class in $W^{1,p^*}_{\text{loc}}(\Omega)$)**
*Let $p^* = \max(1, p-1)$.*

*A function $u \in W^{1,p^*}_{\text{loc}}(\Omega)$ is said to be admissible iff $\mathcal{A}u \in M(\Omega)$ and there exists a sequence $\{u_n\}_{n=1}^\infty \subset W^{1,p}_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ s.t:*

1. *$u_n \to u$ a.e. in $\Omega$, $u_n \to u$ in $W^{1,p^*}_{\text{loc}}(\Omega)$ as $n \to \infty$.*
2. *$\mathcal{A}u_n \in L^1_{\text{loc}}(\Omega)$ $(n = 1, 2, \cdots)$ and*

$$\sup_n |\mathcal{A}u_n|(\omega) = \sup_n \int_\omega |\mathcal{A}u_n| < \infty \qquad \textit{for every } \omega \subset\subset \Omega. \tag{12}$$

## 5 Some results on the admissibility

1. If $u \in W^{1,p^*}_{\text{loc}}(\Omega)$ is admissible $\Longrightarrow$ $u^+ = \max[u, 0]$, $u^- = \max[-u, 0]$, $T_k(u)$ are admissible.
2. $T_k(u) \in W^{1,p}_{loc}(\Omega)$ for $^\forall k > 0$. Moreover, given $\omega \subset\subset \omega' \subset\subset \Omega$, $^\exists C > 0$ independent on $u$ s.t
$$\begin{cases} \int_\omega |\nabla T_k(u)|^2 \leq k\left(\int_{\omega'} |\Delta u| + C\int_{\omega'} |u|\right), & \text{if } p = 2, \\ \int_\omega |\nabla T_k(u)|^p \leq Ck\left(\int_{\omega'} |\Delta_p u| + \int_{\omega'} |\nabla u|^{p-1}\right) & \text{if } p \neq 2, \end{cases}$$
3. When $p = 2$ and $\mathcal{A} = \Delta$,
$u \in W^{1,1}_{\text{loc}}(\Omega)$, $\Delta u \in M(\Omega) \Longrightarrow$ u is admissible.
4. $u \in W^{1,p}_0(\Omega)$, $\mathcal{A}u \in M(\Omega) \Longrightarrow$ u is admissible.

# 6 Counter-example due to J.Serrin

Let $\Omega$ be a unit ball $B_1 = \{x \in R^N : |x| < 1\}$, and set

$$a_{i,j} = \delta_{i,j} + (a-1)\frac{x_i x_j}{r^2}, \qquad (r = |x|), \tag{13}$$

$$\mathcal{B}u = \sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\left(a_{j,k}(x)\frac{\partial U}{\partial x_k}\right) = 0. \tag{14}$$

Then we have a pathological weak solution of the form

$$U(x) = x_1 r^{-\alpha}, \qquad \text{where } \alpha = \frac{N}{2} + \sqrt{\left(\frac{N}{2}-1\right)^2 + \frac{N-1}{a}}. \tag{15}$$

If $a > 1 \Longrightarrow N-1 < \alpha < N$.

**Proposition 1** *Assume that $a > 1$. Then $U \in W^{1,1}_{\rm loc}(B_1)$ and $\mathcal{B}U = 0$ in $D'(B_1)$. But $U$ is not admissible, and $\mathcal{B}(U^+)$ is not a Radon measure.*

# 7 Main results and Applications

In the rest of this note, we assume for the sake of simplicity

$$\mathcal{A} = \Delta_p.$$

## 7.1 Improved Concave type inequality

**Theorem 1** *[15, 16]*
*Assume that $u \in W^{1,p^*}_{\rm loc}(\Omega)$ and $u$ is addmissible.*

$$\Downarrow\Downarrow\Downarrow$$

*If $u \geq 0$ a.e. in $\Omega$, then $\Delta_p(T_k(u))$ is a Radon measure for every $k > 0$. Moreover, we have*

$$\Delta_p(T_k(u)) \leq (\Delta_p u)^+. \tag{16}$$

## 7.2 Application to Strong Maximum Principle

**Theorem 2** *[15] Let $\Omega$ be a bounded domain of $\mathbf{R}^N$. Assume that $u \in W^{1,p^*}_{\rm loc}(\Omega)$, $u \geq 0$ a.e. and $u$ is admissible. Then*

1. *There exists a quasicontinuous function ( w.r.t. $C_p$ ) $\tilde{u} : \Omega \mapsto \mathbf{R}$ such that $u = \tilde{u}$ a.e. in $\Omega$.*

*2. Assume that*

$$-\Delta_p u \geq 0 \text{ in } \Omega \quad \text{in the sense of measures.} \tag{17}$$

*If $\tilde{u} = 0$ on some $K \subset \Omega$ with $C_p(K, \Omega) > 0$, then $u = 0$ a.e. in $\Omega$.*

**Remark 7.1** *$-\Delta_p u$ can be replaced by $-\Delta_p u + a u^q$, where $0 \leq a \in L^1_{loc}(\Omega)$ and $q \geq p-1$.*

**Example 3** *$U = x_1/|x|^\alpha$ is not admissible in $\Omega = B_1$. Moreover $U = 0$ on $\{x_1 = 0\} \cap B_{1/2}$ which has positive $p$-capacity.*

### 7.3 A quick sketch of the proof of Theorem 2

$$\text{Since} \quad \Delta_p u \leq 0 \text{ in } \Omega \quad \text{in the measure sense,}$$

$$\Downarrow$$

$$(\Delta_p u)^+_d = 0$$

$$\Downarrow$$

Since $T_k(u) \in W^{1,p}_{\text{loc}}(\Omega)$, $\Delta_p(T_k(u)) \in M(\Omega)$ for any $k > 0$,

$$\Delta_p(T_k(u)) \leq (\Delta_p u)^+_d = 0 \quad \text{in } \mathcal{D}'(\Omega), \quad \forall k > 0.$$

$$\Downarrow$$

$$\text{Now we can assume that} \quad u \in L^\infty(\Omega)$$

$$\Downarrow$$

As a test function, using $\varphi_0^p/(u+\delta)^{p-1}$ with $\varphi_0 = 1$ on $\omega$,

$$\int_\omega \left|\nabla \log\left(\frac{u}{\delta} + 1\right)\right|^p \leq C \int_\Omega (\varphi_0^p + |\nabla \varphi_0|^p).$$

$$\Downarrow$$

Let $E \subset \Omega$ with $C_p(E, \Omega) > 0$ s.t. $\tilde{u} = 0$ on $E \subset \omega \subset\subset \Omega$.

By **the Poincare's inequality**

$$\int_\omega \left|\log\left(\frac{u}{\delta} + 1\right)\right|^p \leq C \int \varphi_0^p + |\nabla \varphi_0|^p \qquad \forall \delta > 0.$$

$$\Downarrow$$

We conclude that $u = 0$ a.e. in $\Omega$. □

## 7.4 Convex type Kato's inequality

**Theorem 3** *[15, 16] Let $\Phi$ be a $C^1$ convex function s.t $0 \leq \Phi' < \infty$. Assume $u \in W^{1,p^*}_{\text{loc}}(\Omega)$ and $u$ is addmissible. Then we have*

$$\Delta_p\Phi(u) \geq \Phi'(u)^{p-1}(\Delta_p u)_d - ||\Phi'||_{L^\infty(\mathbf{R})}(\Delta_p u)_c^- \quad in \quad D'(\Omega). \tag{18}$$

**Corollary 1** *Assume the same assumptions in Theorem 3.*
*Then it holds that*

$$\Delta_p(u^+) \geq \chi_{[u\geq 0]}(\Delta_p u)_d - (\Delta_p u)_c^- \qquad in\ D'(\Omega), \tag{19}$$

$$\Delta_p|u| \geq \operatorname{sgn}(u)(\Delta_p u)_d - |\Delta_p u|_c \qquad in\ D'(\Omega), \tag{20}$$

*where* $\operatorname{sgn}(t) = 1$ *for* $t > 0$, $\operatorname{sgn}(t) = -1$ *for* $t < 0$, *and* $\operatorname{sgn}(0) = 0$.

**Example 4** *Let $u = |x|^\alpha$ for $\alpha = (p - N)/(p - 1)$ and $0 \in \Omega$.*

1. *$u$ satisfies $\Delta_p u = \alpha|\alpha|^{p-2}c_N\delta$, $\delta$ : a Dirac mass, $c_N$ : the surface area of the unit ball $B_1$. If $p > 2 - 1/N$, then $|\nabla u| \in L^1_{loc}(\Omega)$ and $u$ is addimible.*

   *Recall* $\quad \Delta_p(u^+) \geq \chi_{[u\geq 0]}(\Delta_p u)_d - (\Delta_p u)_c^- \quad inD'(\Omega). \quad (19)$

2. *If $2 - 1/N \leq p \leq N$, then $\alpha \leq 0$, $C_p(\{0\}, \Omega) = 0$ ($\Delta_p(u^+)$ is concentrated) $(\Delta_p(u^+))_c = (\Delta_p u)_c = -(\Delta_p u)_c^- = \alpha|\alpha|^{p-2}c_N\delta \leq 0$.*

   *If $p > N$, then $\alpha > 0$, $C_p(\{0\}, \Omega) > 0$ ($\Delta_p(u^+)$ is diffuse) $(\Delta_p(u^+))_c = (\Delta_p u)_c = (\Delta_p u)_c^- = 0$ and $\Delta_p(u^+) = \chi_{[u\geq 0]}(\Delta_p u)_d = \alpha|\alpha|^{p-2}c_N\delta \geq 0$.*

   *Consequently*

$$\Delta_p(u^+) = \chi_{[u\geq 0]}(\Delta_p u)_d - (\Delta_p u)_c^- \qquad inD'(\Omega).$$

## 7.5 Inverse maximum principle

**Theorem 4** *[15, 16]* **(Inverse maximum principle )** *Assume $u \in W^{1,p^*}_{\text{loc}}(\Omega)$, $u \geq 0$ and $u$ is admissible. Then we have*

$$(-\Delta_p u)_c \geq 0 \quad on\ \Omega. \tag{21}$$

**Corollary 2** *Assume $u \in W^{1,p^*}_{\text{loc}}(\Omega)$ and $u$ is admissible. Then we have*

$$(-\Delta_p(u^+))_c = (-\Delta_p u)_c^+ \quad on\ \Omega. \tag{22}$$

### 7.6 A quick sketch of the proof of Theorem 4:

Recall:

$T_k(u) \in W^{1,p}_{\text{loc}}(\Omega)$, $\Delta_p(T_k(u)) \in M(\Omega)$ for $\forall k > 0$.
Moreover we have

$$\Delta_p(T_k(u)) \le (\Delta_p u)^+ \quad \text{in} \quad D'(\Omega).$$

Set $\Delta_p u = \mu \in M(\Omega)$.
For some compact set $K$, s.t. $|\mu_c|(\Omega \setminus K) = 0$, $C_p(K, \Omega) = 0$.

*Then* $\quad \Delta_p T_k(u) \le \mu^+ \quad$ in $D'(\Omega \setminus K)$.

As $k \to \infty$, $\quad \mu = \Delta_p u \le \chi_{\Omega \setminus K} \mu^+ \quad$ in $D'(\Omega)$.

*Then* $\quad \mu_c|_K = \mu_K \le 0 \quad$ in $D'(\Omega)$.

$$\Downarrow$$

$$\mu_c \le 0 \quad \text{in } D'(\Omega). \tag{23}$$

□

### 7.7 Application of IMP

**Theorem 5** *[17]Suppose that $u$ is admissible. Then* $\operatorname{supp} \mu_c^{\pm} \subset \{x : u = \pm\infty\}$ *for* $\mu = \Delta_p u$.

**Remark 7.2** *From this fact,*
*if $u \in W^{1,p^*}_{\text{loc}}(\Omega)$ is an admissible solution of of $-\Delta_p u = \mu \in M(\Omega)$,*
*then $u$ is also a (local) renormalized solution of $-\Delta_p u = \mu$.*

### 7.8 A quick sketch of proof of Theorem 5

Suppose that $u$ is admissible.

$$\Downarrow$$

$$\Delta_p u = \Delta_p(T_k u) + \Delta_p(u-k)^+ - \Delta_p(u+k)^-$$

$$-\Delta_p \mu = \mu_d + \mu_c^+ - \mu_c^-$$

$$\Downarrow$$

$$\Delta_p(u-k)^+, \Delta_p(u+k)^- \le 0 \qquad (IMP)$$

$$\Downarrow$$

Note that $\Delta_p(T_k u)$ is diffuse and $k$ is an arbitrary number.

$$\Downarrow$$

$$\operatorname{supp}\mu_c^{\pm} \subset \{x : u = \pm\infty\}$$

□

## 8 Existence of admissible solution

**Theorem 6** *[17]Assumethat $\mu \in M(\Omega)$ and $|\mu|(\Omega) < \infty$. Then*

$$\begin{cases} -\Delta_p u = \mu, & \text{in } \Omega, \\ u = 0, & \text{on } \Omega. \end{cases} \tag{24}$$

*has an admissible solution in $W_0^{1,p^*}(\Omega)$.*

The proof relies on the following lemma.

**Lemma 3** *Let $\{\mu_n\}$ satisfy $\sup_n |\mu_n|(\Omega) < \infty$ and $\{u_n\}$ be admissible. Assume that*

$$\begin{cases} -\Delta_p u_n = \mu_n, & \text{in } \Omega, \\ u_n = 0, & \text{on } \Omega. \end{cases} \tag{25}$$

*holds for $n = 1, 2, \ldots$.*

*Then , up to a subsequence, $u_n \to {}^{\exists}u \in W_0^{1,p^*}(\Omega)$ s.t. $u$ is admissible and satisfy (24) for ${}^{\exists}\mu$.*

## 9 Problems

1. (Nonlinear version of Good measure problem)
   Let $g(s)$ be continuous, nonnegative and nondecreasing on $[0, \infty)$. When does the next equation have an admissible solution? $-\Delta_p u + g(u) = \mu, \quad u|_{\partial\Omega} = 0$

2. (Nonlinear version of boundary Kato's inequality)
   If $u, \Delta_p u \in L^1$, then $\Delta_p u^+$ is a finite measure?

   Ex: Even if $p = 2$, there is a $u \in H^1(\Omega)$, s.t. $\Delta u = 0$, but $\int_\Omega |\Delta u^+| = \infty$

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