数理解析研究所講究録
第2052巻 2017年 1-7

A full-twist inequality for the ν^+ invariant

Kouki Sato [1]
School of Science, Tokyo Institute of Technology

1 Back ground

The ν^+-invariant is a non-negative integer valued knot concordance invariant defined by Hom and Wu [2]. The ν^+-invariant dominates many concordance invariants derived from Heegaard Floer homology, in terms of obstructions to sliceness, and hence it plays a special role among such knot concordance invariants.

In this section, we give a short review of knot concordance theory and its relationship to Heegaard Floer theory.

1.1 Knot concordance

For two knots K and J in S^3, let $-K$ denote the orientation reversed mirror image of J and $K\#J$ the connected sum of K and J. We say that K is *concordant* to J if there exists a smooth disk in B^4 with boundary $K\#(-J)$, and we denote the relation by $K \underset{\text{conc.}}{\sim} J$. It is well-known that the relation $\underset{\text{conc.}}{\sim}$ is an equivalence relation on the set of knots in S^3, and connected sum endows the quotient set $\mathcal{C} := \{\text{knots in } S^3\}/\underset{\text{conc.}}{\sim}$ with an abelian group structure. We often call this group $\mathcal{C}$ the *knot concordance group.*

While the knot concordance group has been studied intensively for more than 50 years, the following fundamental problems are still open.

Problem 1. *Which two knots are concordant?*

Problem 2. *Which knots are concordant to the unknot? (Such knots are called slice knots.) Find an algorithm or algebraic criteria for detecting the sliceness.*

Problem 3. *Determine the group structure of* $\mathcal{C}$. *(It is known that* $\mathcal{C}$ *has* $\mathbb{Z}^\infty \oplus (\mathbb{Z}/2\mathbb{Z})^\infty$ *as a summand.)*

[1] The author was supported by JSPS KAKENHI Grant Number 15J10597.

To attack these problems, we use many kinds of *knot concordance invariants*, i.e. well-defined maps

$$\mathcal{C} \to S$$

for some set S.

1.2 Knot concordance invariants from Heegaard Floer theory

Heegaard Floer thoery is a Floer homology theory for 3-manifolds established by Ozsváth and Szabó [6, 7]. From Heegaard Floer theory, many knot concordance invariants have been introduced and used to resolve many problems on knot concordance theory. Here we show several such invariants.

- The *correction terms* $d(S^3_{p/q}(-), i) : \mathcal{C} \to \mathbb{Q}$ ($p/q \in \mathbb{Q}$, $i \in \mathbb{Z}/p\mathbb{Z}$) defined by Ozsváth and Szabó [8]. Originally, these are invariants of Dehn surgeries along a knot.
- The *τ-invariant* $\tau : \mathcal{C} \to \mathbb{Z}$ defined by Ozsváth and Szabó [9]. This is famous as a group homomorphism.
- The *V_k-sequence* $V_k : \mathcal{C} \to \mathbb{Z}_{\geq 0}$ ($k \in \mathbb{Z}_{\geq 0}$) defined by Ni and Wu [4]. It is known that all correction terms $d(S^3_{p/q}(-), i)$ are determined by the V_k-sequence.
- The *ν^+-invariant* $\nu^+ : \mathcal{C} \to \mathbb{Z}_{\geq 0}$ defined by Hom and Wu [2]. This represents a complexity of the V_k-sequence.
- The *Upsilon invariant* $\Upsilon : \mathcal{C} \to \mathrm{Cont}([0,2], \mathbb{R})$ defined by Ozsváth, Stipsicz and Szabó [5]. Here $\mathrm{Cont}([0,2], \mathbb{R})$ denotes the set of continuous functions on the closed interval $[0,2]$. This invariant is a group homomorphism whose image contains $\mathbb{Z}^\infty$ as a subgroup.

Then, how strong are these concordance invariants? Actually, they are invariant under a weaker equivalence relation than $\underset{\text{conc.}}{\sim}$, which is defined as follows.

Definition 1. For two elements $x, y \in \mathcal{C}$, we say that x is *ν^+-equivalent* to y (and denote the relation by $x \underset{\nu^+}{\sim} y$) if the equalities $\nu^+(x - y) = \nu^+(y - x) = 0$ hold.

We can verify that the relation $\underset{\nu^+}{\sim}$ is an equivalence relation, and Hom proves the following theorem.

Theorem 1.1 (Hom [1]). *The quotient $\mathcal{C}_{\nu^+} := \mathcal{C}/\underset{\nu^+}{\sim}$ becomes a quotient group of $\mathcal{C}$. Moreover, the invariants $d(S^3_{p/q}(-), i), \tau, V_k, \nu^+$ and Υ are invariant under $\underset{\nu^+}{\sim}$. In other words, these invariants can be seen as maps on $\mathcal{C}_{\nu^+}$.*

Theorem 1.1 implies that all the above invariants are determined by the ν^+-equivalence class of knots. Hence, it is an important problem to understand $\underset{\nu^+}{\sim}$ and $\mathcal{C}_{\nu^+}$.

Furthermore, ν^+-equivalence is meaningful for Heegaard Floer theory, too. In [6], Ozsváth and Szabó associated to a knot a "$\mathbb{Z} \oplus \mathbb{Z}$-filtered" chain complex CFK^∞. The $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain homotopy equivalence class of CFK^∞ is a knot invariant, and we can compute various kinds of Floer homology groups from CFK^∞; indeed, we can compute

- the knot Floer homology $\widehat{HFK}$ (and hence we can detect the knot genus and fiberedness as a result), and
- the (all original) Heegaard Floer homology groups $\widehat{HF}, HF^\infty, HF^\pm$ of ALL Dehn surgeries.

In [1], Hom also proves that the ν^+-equivalence can be translated into an equivalence relation with respect to CFK^∞. Let $[K]$ denote the concordance class of a knot K.

Theorem 1.2 (Hom [1]). *Two knot concordance classes $[K]$ and $[J]$ are ν^+-equivalent if and only if there exists a $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain homotopy equivalence*

$$CFK^\infty(K) \oplus A_1 \simeq CFK^\infty(J) \oplus A_2,$$

where A_1 and A_2 are $\mathbb{Z} \oplus \mathbb{Z}$-filtered chain complexes with $H_*(A_1) = H_*(A_2) = 0$.

Now, it seems natural to ask the following problems.

Problem 4. *Determine the group structure of $\mathcal{C}_{\nu^+}$.*

Problem 5. *Find geometrical meaning of $\underset{\nu^+}{\sim}$.*

In contrast to the case of $\mathcal{C}$, we can conclude whether a given knot is ν^+-equivalent to the unknot by using ν^+. In this work, we mainly consider Problem 5. In particular, we study effects of *full-twist operations* on ν^+-invariant.

2 Full-twist inequalities for the ν^+-invariant

As main results of this work, we obtained full-twist inequalities for the ν^+-invariant. To state the inequalities, we first describe *full-twist operations*. Let K be a knot in S^3 and D a disk in S^3 which intersects K in its interior. By performing (-1)-surgery along ∂D, we obtain a new knot J in S^3 from K. Let $n = \operatorname{lk}(K, \partial D)$. Since reversing the orientation of D does not affect the result, we may assume that $n \geq 0$. Then we say that K is deformed into J by a *positive full-twist with n-linking*, and call such an operation a *full-twist operation*. The main theorem of this paper is stated as follows.

Theorem 2.1. *Suppose that a knot K is deformed into a knot J by a positive full-twist with n-linking. If $n = 0$, then $\nu^+(J\#(-K)) = 0$. Otherwise, we have*

$$\frac{(n-1)(n-2)}{2} \leq \nu^+(J\#(-K)) \leq \frac{n(n-1)}{2}.$$

Remark 1. For any coprime $p, q > 0$, let $T_{p,q}$ denote the (p,q)-torus knot. Then we note that $\nu^+(T_{p,q}) = (p-1)(q-1)/2$ [2, 9], and hence the inequality in Theorem 2.1 implies

$$\nu^+(T_{n,n-1}\#K\#(-K)) \leq \nu^+(J\#(-K)) \leq \nu^+(T_{n,n+1}\#K\#(-K)).$$

Since both $T_{n,n-1}\#K$ and $T_{n,n+1}\#K$ are obtained from K by a positive full-twist with n-linking, the inequalities are best possible for any K.

Here we note that Theorem 2.1 gives an inequality for $J\#(-K)$ rather than J and K. However, by subadditivity of ν^+, we also have the following result for J and K.

Theorem 2.2. *Suppose that K is deformed into J by a positive full-twist with n-linking. If $n = 0$, then $\nu^+(J) \leq \nu^+(K)$. Otherwise, we have*

$$\frac{(n-1)(n-2)}{2} - \nu^+(-K) \leq \nu^+(J) \leq \frac{n(n-1)}{2} + \nu^+(K).$$

3 Applications

In this section, we show two applications of our full-twist inequalities.

3.1 ν^+-invariant for cable knots

As an application of Theorem 2.2, we gave a lower bound for the ν^+-invariant of all cable knots.

Theorem 3.1. *For any knot K and coprime integers p, q with $p > 0$, we have*

$$\nu^+(K_{p,q}) \geq p\nu^+(K) + \frac{(p-1)(q-1)}{2},$$

where $K_{p,q}$ denotes the the (p,q)-cable of K.

Note that Wu proves in [10] that the equality holds in the case where $p, q > 0$ and $q \geq (2\nu^+(K) - 1)p - 1$. Hence Theorem 3.1 extends his result to arbitrary cables in the form of inequality. Furthermore, Theorem 3.1 also enables us to extend Wu's 4-ball genus bound for particular positive cable knots to all positive cable knots.

Corollary 3.2. *If $\nu^+(K) = g_4(K)$, then for any coprime $p, q > 0$, we have*

$$\nu^+(K_{p,q}) = g_4(K_{p,q}) = pg_4(K) + \frac{(p-1)(q-1)}{2}.$$

As an application of Corollary 3.2, for instance, we can determine the 4-ball genus for all positive cables of the knot $T_{2,5}\#T_{2,3}\#T_{2,3}\#(-(T_{2,3})_{2,5})$. This example is used in [2] to show that $\nu^+ \neq \tau$. Remark that the τ-invariant cannot determine the 4-ball genus for any positive cable of the knot. Also note that this generalizes [2, Proposition 3.5] and Wu's result in the introduction of [10].

3.2 A partial order on ν^+-equivalence classes

As another application, we introduced a partial order on $\mathcal{C}_{\nu^+}$ and studied its relationship to full-twists by using Theorem 2.1. Our partial order is defined as follows.

Definition 2. For two elements $x, y \in \mathcal{C}_{\nu^+}$, we write $x \leq y$ if $\nu^+(x-y)=0$.

Note that the equality in the above definition is one of the equalities in the definition of ν^+-equivalence, and so this partial order seems to be very natural. In fact, we can prove the following proposition.

Proposition 3.3. *The relation $\leq$ is a partial order on $\mathcal{C}_{\nu^+}$ with the following properties;*

1. *For elements $x, y, z \in \mathcal{C}_{\nu^+}$, if $x \leq y$, then $x+z \leq y+z$.*
2. *For elements $x, y \in \mathcal{C}_{\nu^+}$, if $x \leq y$, then $-y \leq -x$.*
3. *For coprime integers $p, q > 0$, $k \in \mathbb{Z}_{\geq 0}$ and $0 \leq i \leq p-1$, all of $-d(S^3_{p/q}(\cdot), i)$, τ, V_k, ν^+ and $-\Upsilon$ preserve the partial order.*

Here the third assertion in Proposition 3.3 implies that there are many algebraic obstructions to one element of $\mathcal{C}_{\nu^+}$ being less than another. On the other hand, the following theorem establishes similar obstructions in terms of geometric deformations.

Theorem 3.4. *Suppose that K is deformed into J by a positive full-twist with n-linking.*

1. *If $n = 0$ or 1, then $[J]_{\nu^+} \leq [K]_{\nu^+}$.*
2. *If $n \geq 3$, then $[J]_{\nu^+} \not\leq [K]_{\nu^+}$. In particular, if the geometric intersection number between K and D is equal to n, then $[J]_{\nu^+} > [K]_{\nu^+}$.*

Here $[K]_{\nu^+}$ denotes the ν^+-equivalence class of a knot K, and the symbol $>$ means $x \geq y$ and $x \neq y$ for elements $x, y \in \mathcal{C}_{\nu^+}$.

In the above theorem, we can see that only the case of $n = 2$ tells us nothing about the partial order. This follows from the fact that Theorem 2.1 gives $0 \leq \nu^+(x-y) \leq 1$ for $n = 2$ and hence we can show neither $\nu^+(x-y) = 0$ nor $\nu^+(x-y) \neq 0$.

We also mention the relationship between our partial order and satellite knots. Let P be a knot in a standard solid torus $V \subset S^3$ with the longitude l, and K a knot in S^3. For $n \in \mathbb{Z}$, Let $e_n : V \to S^3$ be an embedding so that $e(V)$ is a tubular neighborhood of K and $\mathrm{lk}(K, e_n(l)) = n$. Then we call $e_n(P)$ the *n-twisted satellite knot of K with pattern P*, and denote it by $P(K, n)$. Furthermore, if P represents m times generators of $H_1(V; \mathbb{Z})$ for $m \geq 0$, then we denote $w(P) := m$. It is proved in [3, Theorem B] that the map $[K]_{\nu^+} \mapsto [P(K, n)]_{\nu^+}$ for any pattern P with $w(P) \neq 0$. We extend their theorem to all satellite knots, and show that those maps preserve our partial order.

Proposition 3.5. *For any pattern P and $n \in \mathbb{Z}$, the map $P_n : \mathcal{C}_{\nu^+} \to \mathcal{C}_{\nu^+}$ defined by $P_n([K]_{\nu^+}) := [P(K, n)]_{\nu^+}$ is well-defined and preserve the partial order $\leq$.*

By Proposition 3.5, we obtain infinitely many order-preserving maps on $\mathcal{C}_{\nu^+}$ which have geometric meaning. Now it is an interesting problem to compare these satellite maps. Theorem 3.4 tells us the relationship among the maps $\{P_n\}_{n \in \mathbb{Z}}$ for some particular patterns.

Corollary 3.6. *Let P be a pattern.*

1. *If $w(P) = 0$ or 1, then the inequality $P_m(x) \geq P_n(x)$ holds for any integers $m < n$ and $x \in \mathcal{C}_{\nu^+}$.*
2. *If the geometric intersection number between P and the meridian disk of V is equal to $w(P)$ and $w(P) \geq 3$, then $P_m(x) < P_n(x)$ for any $m < n$ and $x \in \mathcal{C}_{\nu^+}$.*

References

[1] J. Hom, *A survey on Heegaard Floer homology and concordance.* J. Knot Theory Ramifications 26 (2017), no. 2, 1740015, 24 pp.

[2] J. Hom and Z. Wu, *Four-ball genus bounds and a refinement of the Ozsváth-Szabó tau-invariant.* J. Symplectic Geom. 14 (2016), no. 1, 305–323.

[3] M. H. Kim and K. Park, *An infinite-rank summand of knots with trivial Alexander polynomial.* To appear in J. Symplectic Geom. arXiv:1604.04037 (2016).

[4] Y. Ni and Z. Wu, *Cosmetic surgeries on knots in S^3.* J. Reine Angew. Math. 706 (2015), 1–17.

[5] P. Ozsváth, A. Stipsicz and Z. Szabó, *Concordance homomorphisms from knot Floer homology.* Adv. Math. 315 (2017), 366–426.

[6] P. Ozsváth and Z. Szabó, *Holomorphic disks and knot invariants.* Adv. Math. 186 (2004), no. 1, 58–116.

[7] P. Ozsváth and Z. Szabó, *Holomorphic disks and topological invariants for closed three-manifolds.* Ann. of Math. (2) 159 (2004), no. 3, 1027–1158.

[8] P. Ozsváth and Z. Szabó, *Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary.* Adv. Math. 173 (2003), 179–261.

[9] P. Ozsváth and Z. Szabó, *Knot Floer homology and the four-ball genus.* Geom. Topol. 7 (2003), 615–639.

[10] Z. Wu, *A cabling formula for ν^+ invariant.* Proc. Amer. Math. Soc. 144 (2016), no. 9, 4089–4098.

School of Science
Tokyo Institute of Technology
2-12-1 Ookayama, Meguro-ku, Tokyo
JAPAN
E-mail address: `sato.k.bs@m.titech.ac.jp`

東京工業大学・理学院　佐藤　光樹