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Some conjectures about the colored Jones polynomial

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1 Introduction

The Jones polynomial was discovered by Jones in 1984 and has made a revolution in knot theory. For a knot K in S^3, the colored Jones function $J_K(n)$ is a sequence

$$J_K : \mathbb{Z} \to \mathbb{C}[t^{\pm 1}]$$

of Laurent polynomials in t, which essentially measures the Jones polynomials of K and its parallels. Technically, $J_K(n)$ is the quantum invariant using the n-dimensional representation of sl_2, normalized so that

$$J_{\text{unknot}}(n) = (t^{2n} - t^{-2n})/(t^2 - t^{-2})$$

and extended to integers n by $J_K(n) = -J_K(-n)$.

In this note, we give a short survey about some old and new conjectures about the geometry and topology of the colored Jones polynomial. These includes:

- The volume conjecture of Kashaev [Ka] and Murakami–Murakami [MuMu] that relates the colored Jones polynomial of a knot in the 3-sphere and the hyperbolic volume of the knot complement.
- The AJ conjecture of Garoufalidis [Ga1] that relates the colored Jones polynomial and the A-polynomial of a knot. This conjecture is based on the theory of noncommutative A-polynomial of Frohman–Gelca–LoFaro [FGL] and the theory of q-holonomicity of quantum invariants of Garoufalidis–Le [GL].
- The slope conjecture of Garoufalidis [Ga3] and strong slope conjecture of Kalfagianni–Tran [KaTr] that relate the colored Jones polynomial of a knot and the boundary slopes and Euler characteristic of incompressible surfaces in the knot complement.

2 The volume conjecture

According to Thurston's theory, by cutting the knot complement $S^3 \setminus K$ along appropriate disjoint tori one gets a collection of pieces, each is either Seifert fibered or hyperbolic; and $\mathrm{Vol}(K)$ is defined as the sum of the hyperbolic volume of the hyperbolic pieces.

Then the volume conjecture of Kashaev [Ka] and Murakami–Murakami [MuMu] is stated as follows.

Volume conjecture. Suppose K is a knot in S^3. Then

$$\lim_{n\to\infty} \frac{\log|J_K(n; t = e^{\pi i/2n})|}{n} = \frac{\mathrm{Vol}(K)}{2\pi}.$$

The volume conjecture was confirmed for the following knots:

- 4_1 (by Ekholm),
- 5_2 (by Kashaev–Yokota [KY], Ohtsuki [Oh1]),
- torus knots (by Kashaev–Tirkkonen [KaTi]),
- Whitehead doubles of torus knots of type $(2, b)$ (by Zheng [Zh]),
- knots with up to 8 crossings (by Ohtsuki–Yokota [OY], Ohtsuki [Oh2], Takata [Ta]).

3 The AJ conjecture

Garoufalidis and Le [GL] proved that, for a fixed knot K, the colored Jones function $J_K(n)$ satisfies a non-trivial linear recurrence relation of the form

$$a_d(t^{2n}, t)J_K(n+d) + \cdots + a_0(t^{2n}, t)J_K(n) = 0$$

for all n, where $a_j(u, v) \in \mathbb{C}[u, v]$. For example, the colored Jones polynomial of the trefoil knot is

$$J_K(n) = t^{-6(n^2-1)} \sum_{j=-(n-1)/2}^{(n-1)/2} t^{24j^2+12j} \frac{t^{8j+2} - t^{-(8j+2)}}{t^2 - t^{-2}}.$$

It satisfies the following linear recurrence relation

$$(t^{8n+12} - 1)J_K(n+2) + (t^{-4n-6} - t^{-12n-10} - t^{8n+10} + t^{-2})J_K(n+1)$$
$$-(t^{-4n+4} - t^{-12n-8})J_K(n) = 0.$$

Consider a discrete function $f : \mathbb{Z} \to \mathbb{C}[t^{\pm 1}]$, and define the linear operators M and L acting on such functions by:

$$(Mf)(n) = t^{2n} f(n), \quad (Lf)(n) = f(n+1).$$

Then $LM = t^2 ML$. Moreover $M^{\pm 1}, L^{\pm 1}$ generate the *quantum torus* $\mathcal{T}$, a noncommutative ring with presentation

$$\mathcal{T} = \mathbb{C}[t^{\pm 1}]\langle M^{\pm 1}, L^{\pm 1}\rangle / (LM = t^2 ML).$$

The *recurrence ideal* of the discrete function f is the left-ideal $\mathcal{A}$ in $\mathcal{T}$ annihilates f:

$$\mathcal{A} = \{P \in \mathcal{T} : Pf = 0\}.$$

Denote by $\mathcal{A}_K$ the recurrence ideal of $J_K(n)$. Since $J_K(n)$ satisfies a non-trivial linear recurrence relation, $\mathcal{A}_K \neq 0$. The ring $\mathcal{T}$ is not a principal left-ideal domain. However, by adding the inverses of polynomials in t, M to $\mathcal{T}$, we get a principal left-ideal domain $\widetilde{\mathcal{T}}$, and a generator α_K of the extension $\widetilde{\mathcal{A}}_K = \widetilde{\mathcal{T}}\mathcal{A}_K$. We call $\alpha_K \in \mathbb{C}[t, L, M]$ the *recurrence polynomial* of K. For example, the recurrence polynomial of the trefoil knot is

$$\begin{aligned}\alpha_K &= (t^{12}M^4 - t^{-4}M^{-2})L^2 \\ &\quad + (t^{-6}M^{-2} - t^{-10}M^{-6}) - (t^{10}M^4 - t^{-2})L \\ &\quad - t^4M^{-2} + t^{-8}M^{-6}.\end{aligned}$$

The AJ conjecture of Garoufalidis [Ga1] is stated as follows.

AJ conjecture. For any knot K in S^3, $\alpha_K \mid_{t=-1}$ is equal to the A-polynomial of K, up to a factor depending on M only.

For example, for the trefoil knot

$$\alpha_K \mid_{t=-1} = (M^4 - 1)(L - 1)(LM^6 + 1)$$

is equal to the A-polynomial up to the polynomial $M^4 - 1$.

The A-polynomial of a knot in S^3, defined by Cooper-Culler-Gillett-Long-Shalen [CCGLS], describes the $SL_2(\mathbb{C})$-character variety of the knot complement as viewed from the boundary. It carries a lot of information about the knot. For example, the Newton polygon of the A-polynomial give rise to essential surfaces in the knot complement.

The AJ conjecture was confirmed for the following knots:

- $3_1, 4_1, 7_4$ (by Garoufalidis [Ga1], Garoufalidis–Koutschan [GK]),
- torus knots (by Hikami [Hi], Tran [Tr4]),
- some classes of two-bridge knots and pretzel knots (by Le [Le], Le–Tran [LT1], Le–Zhang [LZ]).

4 The slope conjecture

For a knot $K \subset S^3$, let X_K be its complement. Let $\langle \mu, \lambda \rangle$ be the canonical meridian–longitude basis of $H_1(\partial X_K)$. An element $p/q \in \mathbb{Q} \cup \{1/0\}$ is called a *boundary slope* of K if there is an essential surface $(S, \partial S) \subset (X_K, \partial X_K)$, such that ∂S represents $p\mu + q\lambda \in H_1(\partial X_K)$. We use bs_K to denote the set of boundary slopes of K. Hatcher [Ha] proved that bs_K is always a finite set.

Let $\hbar[J_K(n)]$ denote the highest degree of $J_K(n)$ in t. For every knot $K \subset S^3$, Garoufalidis [Ga3] proved that $\hbar[J_K(n)]$ is a *quasi-quadratic polynomial* in n. Namely, there exist periodic functions $a_K(n), b_K(n), c_K(n)$ such that

$$\hbar[J_K(n)] = a_K(n)\, n^2 + b_K(n)n + c_K(n).$$

Elements of $js_K := \{a(n)\}_n$ are called *Jones slopes* of K. Then the slope conjecture of Garoufalidis [Ga3] is stated as follows.

Slope Conjecture. For every knot $K \subset S^3$ we have

$$js_K \subset bs_K.$$

A similar conjecture can be stated for $\ell[J_K(n)]$, the lowest degree of $J_K(n)$ in t.

Example. For the $(-2,3,7)$-pretzel knot we have

$$\begin{aligned} \hbar[J_K(n)] &= 37n^2/2 + 34n + e(n), \\ \ell[J_K(n)] &= 5n, \end{aligned}$$

where $e(n)$ is a periodic sequence of period 4. On the other hand, the $(-2,3,7)$-pretzel knot is a Montesinos knot and its boundary slopes are given by $\{0, 16, 37/2, 20\}$. This confirms the slope conjecture for the $(-2,3,7)$-pretzel knot.

The slope conjecture was confirmed for the following knots:

- alternating knots, knots with up to nine crossings, torus knots, $(-2,3,2n+1)$-pretzel knots (by Garoufalidis [Ga3]),
- adequate knots (by Futer–Kalfagianni–Purcell [FKP]),
- 2-fusion knots (by Dunfield–Garoufalidis [DG], Garoufalidis–v.d.Veen [GV]),
- iterated cables of adequate knots (Kalfagianni–Tran [KaTr]),
- graph knots (Motegi–Takata [MT]),
- some classes of 3-stranded pretzel knots (Lee–v.d.Veen [LV]).

5 New conjectures about the colored Jones polynomial

Recall that for every knot $K \subset S^3$, there exist periodic functions $a_K(n), b_K(n), c_K(n)$ such that $\hbar[J_K(n)] = a_K(n)\, n^2 + b_K(n) n + c_K(n)$.

Then the following conjectures were proposed in [KaTr].

Conjecture 1 (Kalfagianni–Tran) For every non-trivial knot $K \subset S^3$, we have

$$b_K(n) \leq 0.$$

Note that $b_U(n) = 1/2$ for the trivial knot U.

Conjecture 2 [Strong slope conjecture] (Kalfagianni–Tran) Let K be a non-trivial knot and $r/s \in js_K$, with $s > 0$ and $\gcd(r,s) = 1$, a Jones slope of K. Then there is an essential surface $S \subset M_K$ with boundary slope r/s, and such that

$$\frac{\chi(S)}{|\partial S| s} \in \{b_K(n)\}_n.$$

Example. Consider the pretzel knot $K_p = (-2,3,p)$, where $p \geq 5$ is an odd integer. Then, by [Ga3], we have

$$a_{K_p}(n) = \frac{p^2 - p - 5}{(p-3)/2} \quad \text{and} \quad b_{K_p}(n) = -\frac{(p-5)/2}{(p-3)/2}.$$

It is known [Ca] that K_p has an essential surface with boundary slope $\frac{p^2-p-5}{(p-3)/2}$, with 2 boundary components, and Euler characteristic

$$-(p-5) = 2(\frac{p-3}{2})b_{K_p}(n).$$

This confirms the strong slope conjecture for the $(-2,3,p)$-pretzel knot.

Lee–v.d.Veen [LV] have recently verified the strong slope conjecture for some classes of 3-stranded pretzel knots.

6 Cable knots

In this last section, we discuss the above conjectures for cable knots. Suppose K is a knot, and p, q are coprime integers. The (p,q)-cable $K_{p,q}$ of K is the satellite of K with pattern (p,q)-torus knot. Then Morton [Mo] (see also [Ve]) proved that for $n > 0$ we have

$$J_{K_{p,q}}(n) = t^{-pq(n^2-1)/4} \sum_{k=-(n-1)/2}^{(n-1)/2} t^{4pk(qk+1)} J_K(2qk+1).$$

6.1 The volume conjecture for cable knots

Let E be the figure eight knot. By Habiro and Le, we have

$$J_E(n) = \frac{t^{2n}-t^{-2n}}{t^2-t^{-2}} \sum_{k=0}^{n-1} \prod_{l=1}^{k} (t^{4n}+t^{-4n}-t^{4l}-t^{-4l}).$$

In this case we have the following result in [LT2].

Theorem. (Le–Tran) The volume conjecture holds true for the cable knot $E_{r,2}$ for every odd number r.

6.2 The AJ conjecture for cable knots

A formula for the A-polynomial of cable knots has recently been given by Ni–Zhang [NZ]. For an odd integer r we have

$$A_{K_{r,2}}(L,M) = \mathrm{Res}_\lambda \left(A_K(\lambda, M^2), \lambda^2 - L\right)(M^{2r}L+1).$$

On the other hand, from the cabling formula of Morton we have

$$J_{K_{r,2}}(n+1) = -t^{-2r(2n+1)} J_{K_{r,2}}(n) + t^{-2rn} J_K(2n+1).$$

Let $\mathbb{J}_K(n) := J_K(2n+1)$. Then, under certain conditions, we have

$$\alpha_{K_{r,2}} = \alpha_{\mathbb{J}_K} M^r (L + t^{-2r} M^{-2r}).$$

Using this, the AJ conjecture has been shown to hold true for

- most cable knots over torus knots (by Ruppe–Zhang [RZ]),
- most $(r, 2)$-cable knots over the figure eight knot (by Ruppe [Ru], Tran [Tr2]), over some two-bridge knots (by Druivenga [Dr]).
- most $(r, 2)$-cables of some classes of two-bridge knots and pretzel knots (by Tran [Tr3], [Tr1]).

6.3 The slope conjecture for cable knots

The following result relates the boundary slopes of a knot and its cable.

Theorem A (Motegi–Takata [MT]; Kalfagianni–Tran [KaTr]) For every knot $K \subset S^3$ and (p, q) coprime integers we have

$$\left(q^2 bs_K \cup \{pq\}\right) \subset bs_{K_{p,q}}.$$

By combining Theorem A and results about the degree of the colored Jones polynomial of adequate knots in [FKP], we have the following result in [KaTr].

Theorem B (Kalfagianni–Tran) Suppose K' is an iterated cable of an adequate knot K. Then K' satisfies the slope conjecture.

An iterated torus knot is an iterated cable of the trivial knot. Hence iterated torus knots satisfy the slope conjecture.

Motegi–Takata [MT] showed that the slope conjecture is preserved under connected sums. This, together with a modified version of Theorem B, implies that graph knots (knots whose hyperbolic volume is 0) satisfies the slope conjecture.

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