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ON GANGL-ZAGIER'S ENHANCED ZETA VALUE FOR IMAGINARY QUADRATIC FIELDS

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1. INTRODUCTION

1.1. Main theme. The main theme of this article is a certain refinement of Zagier's polylogarithm conjecture. The Zagier conjecture is one of the conjectures about special values of arithmetic zeta functions. In 2000, Gangl-Zagier gave a certain refinement of the Zagier conjecture for imaginary quadratic base fields, which they called an *enhancement* of the Zagier conjecture.

Gangl-Zagier's conjecture only deals with unramified abelian extension of imaginary quadratic fields (i.e. partial zeta values of ideal classes). In this article, we generalize their conjecture to arbitrary abelian extension of imaginary quadratic fields (i.e. partial zeta values of ray classes).

1.2. Gangl-Zagier's method VS our method. Gangl-Zagier's method and our method seems quite different at first glance. Gangl-Zagier's method is based on the Eichler integral of a holomorphic Eisenstein series, while our method is based on the theory of partial derivative of Shintani L-functions of two variable. The advantage of our method is that it is applicable to higher number fields with exactly one complex place.

One of our main result is an equality between Gangl-Zagier's method and our method (in unramified extension case). We also give several numerical examples of our generalized conjecture.

1.3. General notations. Throughout this article, we specify ourselves to imaginary quadratic base fields. Therefore, we fix an imaginary quadratic field F, an embedding

$$\rho : F \hookrightarrow \mathbb{C},$$

and use the following notations:

- $\mathrm{Cl}_{\mathfrak{m}}$: the ray class group of modulus $\mathfrak{m}$, $\mathrm{Cl} := \mathrm{Cl}_{\mathbf{1}}$: the ideal class group of F.
- $U(\mathfrak{m})$: the subgroup of $F^{\times}$ formed by the elements congruent to 1 mod $\mathfrak{m}$.
- $w_{\mathfrak{m}} := \left|U(\mathfrak{m}) \cap \mathcal{O}_F^{\times}\right|, w := w_{\mathbf{1}}$.
- rec: Artin's reciprocity map.
- $\mathcal{P}^{(k)} := \{P \in \mathbb{Q}[\tau] \,|\, \deg P \leq k\}$, $\mathcal{P}_*^{(k)} := \{P \in \mathbb{Q}[\tau, \overline{\tau}] \,|\, \overline{P} = P, \deg P \leq k\}$.
- $\mathbb{Q}(k) := (2\pi i)^k \mathbb{Q}$.
- $\Re_k : \mathbb{C}/\mathbb{Q}(k) \to \mathbb{R}$: the map defined by $\Re_k(z) = \Re(i^{k+1}z)$.
- $\widetilde{\mathbb{C}^{\times}}$: the universal covering group of $\mathbb{C}^{\times}$.
- $\mathfrak{H}$: the upper-half plane.

2. ZAGIER'S POLYLOGARITHM CONJECTURE

Very roughly speaking, the Zagier conjecture is an equality of the form

$$\text{(zeta value)} = \text{(polylog value)}.$$

In the most classical sense, the Zagier conjecture refers to a conjecture about the special values of the Dedekind zeta function of a number field. For each $m \in \mathbb{Z}_{\geq 2}$ and a number field F, the conjecture predicts that the value $\zeta_F(m)$ of the Dedekind zeta function is expressible as a multiple of a power of $2\pi i$ and a determinant whose entries are linear combinations of special values of a certain real-valued

m-logarithm evaluated at the numbers in F. This is an apparent analog of the Dedekind's analytic class number formula, where $\zeta_F(m)$ is replaced by the residue (the leading coefficient in the Laurent expansion) of $\zeta_F(s)$ at $s=1$, and the polylogarithm by $\log|z|$.

In this article, we use the term "Zagier conjecture" for a more general conjecture about special values of partial zeta functions of number fields. Firstly, let us define the partial zeta function and a certain real-valued polylogarithm which we need in the formulation of the Zagier conjecture.

Definition 1. For a ray class $\mathcal{A}$, we define the partial zeta function associated to $\mathcal{A}$ by

$$\zeta(s,\mathcal{A}) = \sum_{\mathfrak{a}\in\mathcal{A}} \frac{1}{N(\mathfrak{a})^s}.$$

Here, the sum runs through all integral ideals in $\mathcal{A}$.

It is known that the Dirichlet series on the right-hand side is absolutely convergent for $\Re(s)>1$ and analytically continued to $\mathbb{C}$ except for a simple pole at $s=1$.

Definition 2. For $k\in\mathbb{Z}_{\geq 2}$, we define a real-valued k-logarithm $\mathcal{L}_k(z)$ by

$$\mathcal{L}_k(z) := \Re_k\left\{\sum_{j=0}^{k-1}\frac{B_j}{j!}\left(2\log|z|\right)^j \mathrm{Li}_{k-j}(z)\right\}$$

where B_j is the j-th Bernoulli number, and $\mathrm{Li}_j(z)$ is the j-logarithm function defined by the analytic continuation of the power series

$$\sum_{m=1}^{\infty}\frac{z^m}{m^j}\quad (|z|<1).$$

Note that despite the monodromy of $\mathrm{Li}_k(z)$ around $0,1$, $\mathcal{L}_k(z)$ is well-defined on $\mathbb{C}\setminus\{0,1\}$. Thus we can linearly extend its domain of definition to $\mathbb{Z}\left[\mathbb{C}\setminus\{0,1\}\right]$.

For a subfield H of $\mathbb{C}$, we define the k-th Bloch group $\mathcal{B}_k(H)$, which is a quotient

$$\mathcal{A}_k(H)/\mathcal{C}_k(H)$$

of certain submodules $\mathcal{C}_k(H)\subset\mathcal{A}_k(H)\subset\mathbb{Z}[H\setminus\{0,1\}]$ (see [2] for the definition of $\mathcal{A}_k(H)$ and $\mathcal{C}_k(H)$). It is known that $\mathcal{L}_k$ vanishes on $\mathcal{C}_k(H)$. So, $\mathcal{L}_k(z)$ is well-defined on $\mathcal{B}_k(H)$.

Using these notions, the Zagier conjecture for an imaginary quadratic field is stated as follows.

Conjecture (Zagier [2]). *Fix $k\in\mathbb{Z}_{\geq 2}$, an abelian extension H of F, and a complex embedding ρ_H of H lying over ρ. Then, there exists unique $\xi\in\mathcal{B}_k(H)\otimes\mathbb{Q}$ such that*

$$|D_F|^{k-\frac{1}{2}}\frac{(k-1)!}{2(2\pi)^k}\zeta(k,\mathrm{rec}^{-1}(\sigma)) = \mathcal{L}_k(\rho_H\circ\sigma(\xi))$$

for $\sigma\in\mathrm{Gal}(H/F)$.

Here, $\zeta(k,\mathrm{rec}^{-1}(\sigma))$ is a (finite sum of) partial zeta values of an ideal class. This conjecture is a higher analog of the abelian Stark conjecture.

3. Gangl-Zagier's enhanced conjecture

3.1. **The enhanced conjecture.** To formulate Gangl-Zagier's enhanced conjecture, one needs two objects: the *enhanced polylogarithm* map and the *enhanced zeta value.*

- Gangl-Zagier constructed the *enhanced* k-logarithm

  $$\widehat{\mathcal{L}}_k : \mathcal{B}_k(\mathbb{C})\to\mathbb{C}/\mathbb{Q}(k)$$

  with the property

  $$\Re_k\circ\widehat{\mathcal{L}}_k = \mathcal{L}_k.$$

  We remark that $\widehat{\mathcal{L}}_k$ is a more delicate object than $\mathcal{L}_k$, since it is defined only on a submodule $\mathcal{A}_k(\mathbb{C})$ of $\mathbb{Z}[\mathbb{C}\setminus\{0,1\}]$.

- For an ideal class $\mathcal{A}$, they also constructed the *enhanced* partial zeta value
$$I_k(\mathcal{A}) \in \mathbb{C}/\mathbb{Q}(k)$$
with the property
$$\Re_k\left(I_k(\mathcal{A})\right) = |D_F|^{k-\frac{1}{2}} \frac{(k-1)!}{2(2\pi)^k}\zeta(k,\mathcal{A}).$$

Thus they formulated the following conjecture.

Conjecture (Gangl-Zagier [2]). *Let H be an unramified abelian extension of F. Fix $k \in \mathbb{Z}_{\geq 2}$ and a complex embedding ρ_H of H lying over ρ. Then there exists unique $\xi \in \mathcal{B}_k(H) \otimes \mathbb{Q}$ such that*
$$I_k(\mathrm{rec}^{-1}(\sigma)) = \widehat{\mathcal{L}}_k(\rho_H \circ \sigma(\xi))$$
for $\sigma \in \mathrm{Gal}(H/F)$.

By taking $\Re_k$ of both-sides, this conjecture reduces to the polylogarithm conjecture for the partial zeta value $\zeta(k, \mathrm{rec}^{-1}(\sigma))$.

3.2. **Gangl-Zagier's construction of the enhanced zeta values.** In this section, we review Gangl-Zagier's construction of the enhanced zeta values. Fix $k \in \mathbb{Z}_{\geq 1}$. Let us start from the specific *Eichler integral*
$$\widetilde{E}_{-2k}(\tau) := \zeta(2k+1) + \zeta(-2k-1)\frac{(2\pi i\tau)^{2k+1}}{(2k+1)!} + 2\sum_{m=1}^{\infty} \frac{e^{2\pi i m\tau}}{m^{2k+1}\left(1-e^{2\pi i m\tau}\right)} \tag{3.1}$$
of holomorphic Eisenstein series $E_{2k+2}(\tau)$. It is known that

Lemma. *For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have*
$$(c\tau+d)^{2k}\widetilde{E}_{-2k}\left(\frac{a\tau+b}{c\tau+d}\right) - \widetilde{E}_{-2k}(\tau) \in (2\pi i)^{2k+1}\mathcal{P}^{(2k)}.$$

Set $\psi_s := (\overline{\tau}-\tau)\frac{\partial}{\partial\tau} - s$, $d_k := \psi_{-k-1} \circ \psi_{-k-2} \circ \cdots \circ \psi_{-2k}$ and
$$\mathcal{E}_k(\tau) := d_k\widetilde{E}_{-2k}(\tau).$$
Since $d_k\mathcal{P}^{(2k)} \subset \mathcal{P}_*^{(2k)}$, it holds that
$$|c\tau+d|^{2k}\mathcal{E}_k\left(\frac{a\tau+b}{c\tau+d}\right) - \mathcal{E}_k(\tau) \in (2\pi i)^{2k+1}\mathcal{P}_*^{(2k)}.$$

Let $\mathcal{A} \in \mathrm{Cl}$, and $\omega_1, \omega_2 \in F^\times$ be a $\mathbb{Z}$-basis of any fractional ideal $\mathfrak{a}$ in $\mathcal{A}$. Let $\mathbb{B}$ denote the set of $SL_2(\mathbb{Z})$-equivalence classes of positive definite integral binary quadratic forms. Then we can associate $\mathcal{A}_* := [Q_\omega] \in \mathbb{B}$ with $\mathcal{A}$, where
$$Q_\omega(X,Y) := \frac{(\omega_1X-\omega_2Y)(\overline{\omega_1}X-\overline{\omega_2}Y)}{N(\mathfrak{a})}.$$
For $Q(X,Y) \in \mathbb{B}$, we define $\lambda_Q \in \mathbb{Z}_{>0}$, $\tau_Q \in \mathfrak{H}$ by
$$Q(X,Y) = \lambda_Q(X-\tau_QY)(X-\overline{\tau_Q}Y).$$
Then we have the following theorem.

Theorem (Gangl-Zagier [2]). *The value $I_k(Q) := 2w^{-1}(2\pi i)^{1-k}\lambda_Q^{k-1}\mathcal{E}_{k-1}(\tau_Q)$ modulo $\mathbb{Q}(k)$ is independent of the choice of $Q \in \mathcal{A}_*^{-1}$. Thus the element*
$$I_k(\mathcal{A}) := \left(I_k(Q) \bmod \mathbb{Q}(k)\right) \in \mathbb{C}/\mathbb{Q}(k)$$
is well-defined. Moreover, $I_k(\mathcal{A})$ satisfies
$$\Re_k\left(I_k(\mathcal{A})\right) = |D_F|^{k-\frac{1}{2}} \frac{(k-1)!}{2(2\pi)^k}\zeta(k,\mathcal{A})$$
where $\zeta(s,\mathcal{A})$ is the partial zeta function of $\mathcal{A}$.

4. The Shintani L-function of two variables for a lattice in $\mathbb{C}$.

4.1. The definition of the Shintani L-function. Let $\mathbb{L} \subset \mathbb{C}$ be a lattice. We assume that $\mathbb{L}$ is of the form $\mathbb{Z} + \tau\mathbb{Z}$ with some $\tau \in \mathfrak{H}$. We study L-functions of the form

$$\sum_{\omega \in \mathbb{L} \setminus \{0\}} \frac{\phi(\omega)}{\omega^{s_1} \overline{\omega}^{s_2}}.$$

Such L-functions are well-studied for $s_1 - s_2 \in \mathbb{Z}$, where they reduce to real analytic Eisenstein series. To deal with more delicate case, where these L-functions have ambiguity that comes from the branch of the map $z \mapsto z^s$ for complex values of z, it is necessary to consider z as an element of $\widetilde{\mathbb{C}^\times}$. Let $p : \widetilde{\mathbb{C}^\times} \to \mathbb{C}^\times$ be the covering group hom and $\mathscr{X}(\mathbb{L})$ the set of periodic functions on $\mathbb{L}$, with respect to some sublattice of $\mathbb{L}$. Set $\widetilde{\mathbb{L}} := p^{-1}(\mathbb{L} \setminus \{0\})$ and define $A(\mathbb{L})$ as the $\mathbb{Z}$-module generated by characteristic functions of rational open cones in $\widetilde{\mathbb{C}^\times}$. Here, a rational open cone means a cone spanned by the elements of $\widetilde{\mathbb{L}}$ all of which lie in the set $\left\{ z \in \widetilde{\mathbb{C}^\times} \,\middle|\, \arg z \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right\}$ if rotated by some angle simultaneously.

Definition 3. For $\boldsymbol{s} = (s_1, s_2) \in \mathbb{C}^2$ with $\Re(s_1 + s_2) > 2$, $\phi \in \mathscr{X}(\mathbb{L})$, and $f \in A(\mathbb{L})$ we define the Shintani L-function $L(\boldsymbol{s}, \phi, f)$ by

$$L(\boldsymbol{s}, \phi, f) := \sum_{\omega \in \widetilde{\mathbb{L}}} \frac{\phi(\omega) f(\omega)}{\omega^{s_1} \overline{\omega}^{s_2}}.$$

Roughly speaking, $f \in A(\mathbb{L})$ chooses the branch of ω^s.

4.2. Basic notions and properties of the Shintani L-function. We define an action of $\varepsilon \in \ker p$ on $A(\mathbb{L})$ by $(\varepsilon f)(z) := f(\varepsilon^{-1} z)$, and we set $\widetilde{f} := \sum_{\varepsilon \in \ker p} \varepsilon f$.

Definition 4. We say $D \in A(\mathbb{L})$ is a fundamental domain, if $\widetilde{D} = 1_{\widetilde{\mathbb{C}^\times}}$.

That is to say, a fundamental domain D is characterized by $pD = 1_{\mathbb{C}^\times}$ where $(pD)(z) := \sum_{w \in p^{-1}(z)} D(w)$.

We denote by $\varepsilon \in \widetilde{\mathbb{C}^\times}$ the unique element with $\varepsilon^s = e^{2\pi i s}$. Thus $\ker p$ is generated by ε.

Lemma 5 ([1]). *If $\widetilde{f} = 0$, then there exists $g \in A(\mathbb{L})$ such that $f = (1 - \varepsilon)g$. In other words, if D_1, D_2 are fundamental domains, there exists $g \in A(\mathbb{L})$ such that $D_1 - D_2 = (1 - \varepsilon)g$.*

We fix $\phi \in \mathscr{X}(\mathbb{L})$ and denote by $L(\mathbf{s}, f)$ for $L(\boldsymbol{s}, \phi, f)$, if there is no risk of confusion.

Lemma 6 ([1]). *For $f \in A(\mathbb{L})$, $Z(\boldsymbol{s}, f) := \left(e^{2\pi i(s_1+s_2)} - 1\right) L(\boldsymbol{s}, f)$ is an entire function on $\mathbb{C}^2$. Moreover,*

(1) $Z(\boldsymbol{s} - \boldsymbol{k}, f) = O(s_1, s_2)$ *for* $\boldsymbol{k} \in \mathbb{Z}^2_{\geq 0}$,

(2) $Z(\boldsymbol{s} - \boldsymbol{k}, D) = O(s_1^2, s_1 s_1, s_2^2)$ *for* $\boldsymbol{k} \in \mathbb{Z}^2_{\geq 0} \setminus \{(0, 0)\}$ *and a fundamental domain* D.

Put $L_\star(s, f) := L((s, s), f)$ for $f \in A(\mathbb{L})$. Then from the integral representation for $Z(\boldsymbol{s}, f)$, we can prove the following lemma.

Lemma 7 ([1]). *For $k \in \mathbb{Z}_{\geq 0}$ and $f \in A(\mathbb{L})$,*

$$L_\star(-k, f) \in \mathbb{Q}.$$

Fix a modulus $\mathfrak{m}$, $\mathcal{A} \in \mathrm{Cl}_\mathfrak{m}$, and consider the case where the lattice $\mathbb{L}$ is a fractional ideal $\mathfrak{a} \in \mathcal{A}^{-1}$ coprime to $\mathfrak{m}$ and ϕ is the characteristic function of $\mathfrak{a} \cap U(\mathfrak{m})$. Then the partial zeta function is expressed by the Shintani L-function as

$$\zeta(s, \mathcal{A}) = w_\mathfrak{m}^{-1} N(\mathfrak{a})^s L_\star(s, D)$$

where $w_\mathfrak{m} := \left| U(\mathfrak{m}) \cap \mathcal{O}_F^\times \right|$. Put

$$L_\star^{(i)}(-\boldsymbol{k}, D) := \left. \frac{\partial L}{\partial s_i}(-\boldsymbol{k} + (s, s), D) \right|_{s=0} \quad \text{for } i \in \{1, 2\}$$

for a fundamental domain D. These values are well-defined, thanks to the property (2) of Lemma 6. We thus define the "partial derivatives at $-k$" of $\zeta(s, \mathcal{A})$ as follows.

Definition 8. For $i \in \{1,2\}$, $k \in \mathbb{Z}_{\geq 1}$ and $\mathfrak{a} \in \mathcal{A}^{-1}$, we define $\Lambda^{(i)}(-k, D, \mathfrak{a})$.

$$\Lambda^{(i)}(-k, D, \mathfrak{a}) := w_{\mathfrak{m}}^{-1} N(\mathfrak{a})^{-k} L_{\star}^{(i)}((-k,-k), D).$$

Then, using Lemma 6, one can show the following theorem.

Theorem 9 ([1]). *For $i \in \{1,2\}$ and $k \in \mathbb{Z}_{\geq 1}$, the values $\Lambda^{(i)}(-k, D, \mathfrak{a})$ modulo $\mathbb{Q}(1)$ does not depend on the choice of D and $\mathfrak{a} \in \mathcal{A}^{-1}$. Hence the ray class invariant*

$$\Lambda^{(i)}(-k, \mathcal{A}) := \left(\Lambda^{(i)}(-k, D, \mathfrak{a}) \bmod \mathbb{Q}(1)\right) \in \mathbb{C}/\mathbb{Q}(1)$$

is well-defined. Moreover, they satisfy $\Lambda^{(2)}(-k, \mathcal{A}) = \overline{\Lambda^{(1)}(-k, \mathcal{A})}$ and

$$\begin{aligned}\zeta'(-k, \mathcal{A}) &= \sum_{i=1}^{2} \Lambda^{(i)}(-k, \mathcal{A}) \\ &= 2\Re\left(\Lambda^{(1)}(-k, \mathcal{A})\right).\end{aligned}$$

We will see later that, for an ideal class $\mathcal{A}$ (i.e. $\mathfrak{m} = \mathbf{1}$), our $\Lambda^{(1)}(-k, \mathcal{A})$ is equal to $I_{k+1}(\mathcal{A}^{-1})$ up to a simple factor of $2\pi i$.

4.3. **The equivalence of the $I_{k+1}(\mathcal{A})$ and $\Lambda^{(1)}(-k, \mathcal{A})$.** In this section, we basically fix a fundamental domain D. Therefore, we use simpler notation $L(\boldsymbol{s})$ for $L(\boldsymbol{s}, D)$, if there is no risk of confusion.

Recall the operators $\psi_s := (\overline{\tau} - \tau)\frac{\partial}{\partial \tau} - s$ and $d_k := \psi_{-k-1} \circ \psi_{-k-2} \circ \cdots \circ \psi_{-2k}$. Next lemma shows that ψ_k effects on the value $L_{\star}^{(1)}(-\boldsymbol{k})$ as a shifting operator.

Lemma 10 ([1]). *For $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 0}$,*

$$\psi_{-k_1}\left(L_{\star}^{(1)}(-\boldsymbol{k})\right) = k_1 L_{\star}^{(1)}(-\boldsymbol{k} + \delta)$$

where $\delta := (1, -1)$. In particular,

$$d_k\left(L_{\star}^{(1)}((-2k, 0))\right) = \frac{(2k)!}{k!} L_{\star}^{(1)}((-k, -k))$$

for $k \in \mathbb{Z}_{\geq 1}$.

Now define $R_{\boldsymbol{k}}$ by the coefficient of s^2 in the Taylor expansion of $Z((-k_1 - s, -k_2 + s))$ at $s = 0$. For $\boldsymbol{k} = (k_1, k_2) \in \mathbb{Z}_{\geq 0}^2$, we put

$$\begin{aligned}L^{(1)}(-\boldsymbol{k}) &:= \frac{dL}{ds}(-\boldsymbol{k} + (s, 0))\bigg|_{s=0}, \\ L^{(2)}(-\boldsymbol{k}) &:= \frac{dL}{ds}(-\boldsymbol{k} + (0, s))\bigg|_{s=0}.\end{aligned}$$

Then $L_{\star}^{(i)}(-\boldsymbol{k})$ and $L^{(i)}(-\boldsymbol{k})$ are related by $R_{\boldsymbol{k}}$ as follows.

Lemma 11 ([1]). *For $i \in \{1,2\}$ and $\boldsymbol{k} \in \mathbb{Z}_{\geq 0}^2$,*

$$L_{\star}^{(i)}(-\boldsymbol{k}) = L^{(i)}(-\boldsymbol{k}) - \frac{1}{8\pi i} R_{\boldsymbol{k}}.$$

The proof of Lemma 11 uses an analysis of the singularity at $\boldsymbol{s} = -\boldsymbol{k}$ of $L(\boldsymbol{s})$.

Since both $L^{(1)}((-2k, 0))$ and $R_{\boldsymbol{k}}$ have closed contour integral representations, they can be expressed in terms of residues. We avoid writing down the general formula, and only note the special case where $\phi = 1_{\mathbb{L}}$ and $D = 1_X$ with $X := \left\{ z \in \widetilde{\mathbb{C}^{\times}} \,\middle|\, \arg z \in [0, 2\pi) \right\}$.

Lemma 12 ([1]). *For $k \in \mathbb{Z}_{\geq 1}$,*

$$L^{(1)}((-2k,0),1_X) = \frac{(2k)!}{(2\pi i)^{2k}} \left\{ \zeta(2k+1) + 2\sum_{m=1}^{\infty} \frac{e^{2\pi i m\tau}}{m^{2k+1}\left(1-e^{2\pi i m\tau}\right)} \right\}$$

and

$$R_{(2k,0)}(1_X) = -2(2\pi i)^2 \frac{\zeta(-2k-1)}{2k+1}(\tau-\overline{\tau})^{2k+1}.$$

From Lemma 11 and 12, we see that

$$\begin{aligned}&\frac{(2\pi i)^{2k}}{(2k)!} L_{\star}^{(1)}((-2k,0),1_X) \\ &= \zeta(2k+1) + \frac{\zeta(-2k-1)}{2(2k+1)!}(2\pi i(\tau-\overline{\tau}))^{2k+1} + 2\sum_{m=1}^{\infty} \frac{e^{2\pi i m\tau}}{m^{2k+1}\left(1-e^{2\pi i m\tau}\right)}\end{aligned}$$

and $\widetilde{E}_{2k}(\tau)$ defined by (3.1) are mostly equal, except that τ^{2k+1} is replaced by $\frac{1}{2}(\tau-\overline{\tau})^{2k+1}$.

Lemma 13 ([1]). We have $d_k\left(\tau^{2k+1} - \frac{1}{2}(\tau-\overline{\tau})^{2k+1}\right) \in \mathcal{P}_{\star}^{(2k+1)}$.

Proof. The lemma follows from the equalities

$$d_k\left(\tau^s\right) = k! \sum_{j=0}^{k} \binom{s}{j}\binom{2k-s}{k-j} \tau^{s-j}\overline{\tau}^j \tag{4.1}$$

$$d_k\left((\tau-\overline{\tau})^{2k+1}\right) = (-1)^k k! (\tau-\overline{\tau})^{2k+1}, \tag{4.2}$$

for $k \in \mathbb{Z}_{\geq 0}, s \in \mathbb{Z}$. (4.1) can be shown by using an induction, while (4.2) follows from the relation $\psi_{-j}\left((\tau-\overline{\tau})^s\right) = (j-s)(\tau-\overline{\tau})^s$. □

Theorem 14 ([1]). *For $k \in \mathbb{Z}_{\geq 1}$* and an ideal class $\mathcal{A}$ of F, we have

$$I_{k+1}(\mathcal{A}^{-1}) = \frac{(2\pi i)^k}{k!} \Lambda^{(1)}(-k, \mathcal{A})$$

as an element of $\mathbb{C}/\mathbb{Q}(k+1)$.

Proof. Choose any ideal $\mathfrak{a} \in \mathcal{A}^{-1}$ and let $\omega_1, \omega_2 \in F^\times$ be any $\mathbb{Z}$-basis of $\mathfrak{a}$ such that $\iota(\omega_1^{-1}\omega_2) \in \mathfrak{H}$. Recall that, in Zagier's construction, the direct correspondence from ω to $I_{k+1}(\mathcal{A}^{-1})$ is given by

$$I_{k+1}(\mathcal{A}^{-1}) := \frac{\mathcal{E}_k(\omega_1^{-1}\omega_2)}{w\left(2\pi i N(\omega_1^{-1}\mathfrak{a})\right)^k} \bmod \mathbb{Q}(k+1).$$

On the other hand, by setting $\mathbb{L} = \omega_1^{-1}\mathfrak{a}$ and $\phi = 1_{\mathbb{L}}$, we have

$$\begin{aligned}\Lambda^{(1)}(-k,\mathcal{A}) &= \Lambda^{(i)}(-k, D, \omega_1^{-1}\mathfrak{a}) \bmod \mathbb{Q}(1) \\ &= \frac{L_{\star}^{(1)}\left((-k,-k),1_X\right)}{w\, N\left(\omega_1^{-1}\mathfrak{a}\right)^k} \bmod \mathbb{Q}(1).\end{aligned}$$

From Lemma 11, 12 and 13, we have

$$\begin{aligned}&\mathcal{E}_k(\tau) - \frac{(2\pi i)^{2k}}{k!} L_{\star}^{(1)}\left((-k,-k),1_X\right) \\ =&d_k\left(\widetilde{E}_{-2k}(\tau) - \frac{(2\pi i)^{2k}}{(2k)!} L_{\star}^{(1)}\left((-2k,0),1_X\right)\right) \\ =&d_k\left(\zeta(-2k-1)\frac{(2\pi i)^{2k+1}}{(2k+1)!}\left(\tau^{2k+1} - \frac{1}{2}(\tau-\overline{\tau})^{2k+1}\right)\right) \in (2\pi i)^{2k+1}\mathcal{P}_{\star}^{(2k+1)}.\end{aligned} \tag{4.3}$$

Since $\omega_1^{-1}\omega_2 \in F$, (4.3) yields

$$\mathcal{E}_k(\omega_1^{-1}\omega_2) - \frac{(2\pi i)^{2k}}{k!} L_*^{(1)}\left((-k,-k), 1_X\right) \in \mathbb{Q}(2k+1).$$

That is to say

$$I_{k+1}(\mathcal{A}^{-1}) - \frac{(2\pi i)^k}{k!} \Lambda^{(1)}(-k, \mathcal{A}) \in \mathbb{Q}(k+1).$$

Hence we obtain the theorem. □

Led by Theorem 14, we naturally expand Conjecture 3.1 as follows.

Conjecture. *Let $k \in \mathbb{Z}_{\geq 1}$, H/F an abelian extension of modulus $\mathfrak{m}$. Fix an embedding ρ_H of H lying on ρ. Then, there exists an element $\xi \in \mathcal{B}_{k+1}(H) \otimes \mathbb{Q}$ such that*

$$(2\pi i)^k \Lambda^{(1)}(-k, \mathrm{rec}^{-1}(\sigma)) = \widehat{\mathcal{L}}_{k+1}(\rho(\sigma^{-1}\xi))$$

for $\sigma \in \mathrm{Gal}(H/F)$.

Remark 15. One theoretical support of this conjecture is that one can show that the value $(2\pi i)^k \Lambda^{(1)}(-k, \mathrm{rec}^{-1}(\sigma))$ is a period in the sense of Konstevich and Zagier, by using its relation to Eichler integrals of Eisenstein series.

5. Numerical examples

Here, we give several numerical examples in support of our conjecture. We shall check our conjecture in the form

$$(2\pi i)^{-1} \Lambda^{(1)}(1-k, \mathrm{rec}^{-1}(\sigma)) = (2\pi i)^{-k} \widehat{\mathcal{L}}_k(\rho(\sigma^{-1}\xi)) \pmod{\mathbb{Q}}.$$

To calculate the values $\Lambda^{(1)}(1-k, \mathrm{rec}^{-1}(\sigma)) \in \mathbb{C}/\mathbb{Q}(1)$ and $\widehat{\mathcal{L}}_k(\xi, \mathrm{L}) \in \mathbb{C}/\mathbb{Q}(k)$, we defined certain quantities $\Lambda^{(1)}(1-k, \mathcal{A}, \mathfrak{a}) \in \mathbb{C}$ and $\widehat{\mathcal{L}}_k(\xi, \mathrm{L}) \in \mathbb{C}$ such that

$$\Lambda^{(1)}(1-k, \mathcal{A}, \mathfrak{a}) \bmod \mathbb{Q}(1) = \Lambda^{(1)}(1-k, \mathcal{A}),$$
$$\widehat{\mathcal{L}}_k(\xi, \mathrm{L}) \bmod \mathbb{Q}(k) = \widehat{\mathcal{L}}_k(\xi).$$

For the definitions of them and more detail of the calculation, see [1]. The calculations were performed using the open-source mathematical software **Sage**. In each example, we have verified the equalities up to 60-digits precision.

5.0.1. *Example 1.* Let $F = \mathbb{Q}(\sqrt{-1})$, $k = 2$ and $H = F[a]/(a^2 + (\sqrt{-1} - 1)\,a + 1)$. The conductor of H/F is $(4 + 2\sqrt{-1})$, and $\mathrm{Gal}(H/F) = \{\mathrm{id}, \sigma\} \simeq \mathbb{Z}/2\mathbb{Z}$ where $\sigma(a) = a^{-1}$. We embed H into $\mathbb{C}$ by

$$a \mapsto 0.25706586412167716\ldots + 0.52908551363574612\ldots i.$$

Then, we have

$$\begin{aligned}
&(2\pi i)^{-1}\Lambda_1(-1, \mathrm{id}, (1, \sqrt{-1})) \\
&= -0.82800611728595416434095573102 6\ldots + 0.11681140596039366778369017241 5\ldots i, \\
&(2\pi i)^{-1}\Lambda_1(-1, \sigma^{-1}, (3, 3\sqrt{-1})) \\
&= +0.742172783952620831007622397693\ldots - 0.101343619597444827660608339898\ldots i.
\end{aligned}$$

Put $x_0 = \sqrt{-1}$, $x_1 = a$, $a + \sqrt{-1}$ and $g(k_0, k_1, k_2) = [x_0^{k_0} x_1^{k_1} x_2^{k_2}]$. Then $\mathcal{B}_2(H) \otimes \mathbb{Q}$ is generated by ξ_1 and ξ_2, where

$$\xi_1 = 4g(0, -2, -3) + g(-1, 5, 6),$$
$$\xi_2 = 24g(0, -1, 0) - g(-1, 5, 6).$$

Put $\xi = \frac{1}{6}\xi_1 + \frac{1}{5}\xi_2$. and take any homomorphism $\mathrm{L} : H^\times/\mu_H \to \mathbb{C}$ such that

$$\begin{cases} \exp(\mathrm{L}(x)) = x \\ -\pi < \Im(\mathrm{L}(x)) \leq \pi \end{cases}$$

for $x \in \{x_1, x_2\}$. Then we have

$$(2\pi i)^{-2}\widehat{\mathcal{L}}_2(\xi, \mathrm{L})$$
$$= -0.410436672841509719896511286582\ldots + 0.116811405960393667783690172415\ldots i,$$
$$(2\pi i)^{-2}\widehat{\mathcal{L}}_2(\sigma(\xi), \mathrm{L})$$
$$= -0.648591104936268057881266491195\ldots - 0.101343619597444827660608339898\ldots i.$$

Therefore, the following equalities hold in high accuracy.

$$(2\pi i)^{-1}\Lambda_1(-1, \mathrm{id}, (1, \sqrt{-1})) - (2\pi i)^{-2}\widehat{\mathcal{L}}_2(\xi, \mathrm{L}) = -\frac{1}{2^6 3^2 5^2} \times 6013,$$
$$(2\pi i)^{-1}\Lambda_1(-1, \sigma^{-1}, (3, 3\sqrt{-1})) - (2\pi i)^{-2}\widehat{\mathcal{L}}_2(\sigma(\xi), \mathrm{L}) = \frac{1}{2^6 3^2 5^2} \times 20027.$$

5.0.2. *Example 2.* Let F,H, g be as in Example 1 and $k = 3$. Then

$$(2\pi i)^{-1}\Lambda_1(-2, \mathrm{id}, (1, \sqrt{-1}))$$
$$= +22.551395299623280967293134627914\ldots - 0.965351077820958862319864732841127\ldots i,$$
$$(2\pi i)^{-1}\Lambda_1(-2, \sigma^{-1}, (3, 3\sqrt{-1}))$$
$$= -25.076395299623280967293134627914\ldots + 0.878122673714396064702344979624005\ldots i.$$

$\mathcal{B}_3(H) \otimes \mathbb{Q}$ is generated by ξ_1 and ξ_2, where

$$\xi_1 = g(1, 0, 0)$$
$$\xi_2 = 2g(0, -1, 0) + 2g(-1, -1, 0) - g(-1, 2, 0) + g(2, -2, 0).$$

Put $\xi := 1168\xi_1 + 96\xi_2$. Then we have

$$(2\pi i)^{-3}\widehat{\mathcal{L}}_3(\xi, \mathrm{L})$$
$$= -4.366104700376719032706865372085\ldots - 0.965351077820958862319864732841\ldots i,$$
$$(2\pi i)^{-3}\widehat{\mathcal{L}}_3(\sigma(\xi), \mathrm{L})$$
$$= -3.633895299623280967293134627914\ldots + 0.878122673714396064702344979624\ldots i.$$

Therefore, the following equalities hold in high accuracy.

$$(2\pi i)^{-1}\Lambda_1(-2, \mathrm{id}, (1, \sqrt{-1})) - (2\pi i)^{-3}\widehat{\mathcal{L}}_3(\xi, \mathrm{L}) = \frac{1}{2^4 5^2} \times 10767,$$
$$(2\pi i)^{-1}\Lambda_1(-2, \sigma^{-1}, (3, 3\sqrt{-1})) - (2\pi i)^{-3}\widehat{\mathcal{L}}_3(\sigma(\xi), \mathrm{L}) = -\frac{1}{2^4 5^2} \times 8577.$$

5.0.3. *Example 3.* Let F,H, g be as in Example 1 and $k = 4$. Then we have

$$(2\pi i)^{-1}\Lambda_1(-3, \mathrm{id}, (1, \sqrt{-1}))$$
$$= -1649.956754323663658721263100094474\ldots + 17.378488635838497309162077981703\ldots i,$$
$$(2\pi i)^{-1}\Lambda_1(-3, \sigma^{-1}, (3, 3\sqrt{-1}))$$
$$= +1955.960837656996992054596433427807\ldots - 16.497252745225743566155151381111\ldots i.$$

On the other hand, $\mathcal{B}_4(H) \otimes \mathbb{Q}$ is generated by ξ_1 and ξ_2, where

$$\xi_1 = g(-1, 0, 0),$$
$$\xi_2 = 6030g(1, -2, -4) - 38592g(-1, -1, -1) + 6300g(2, -2, 0) - 4288g(-1, 3, 3)$$
$$+ 38592g(2, 1, 1) + 14472g(1, 2, 2) - 315g(2, -4, 0) - 20160g(0, 1, 0).$$

Put

$$\xi := \frac{1523024}{215}\xi_1 - \frac{24}{43}\xi_2.$$

Then we have

$$(2\pi i)^{-4}\widehat{\mathcal{L}}_4(\xi, \mathrm{L})$$
$$= 572.65913319248621207977049163679 1\ldots + 17.37848863583849730916207798170 3\ldots i,$$
$$(2\pi i)^{-4}\widehat{\mathcal{L}}_4(\sigma(\xi), \mathrm{L})$$
$$= 394.96277168477476983237421120558 5\ldots - 16.49725274522574356615515138111 1\ldots i.$$

Therefore, the following equalities hold in high accuracy.

$$(2\pi i)^{-1}\Lambda_1(-3, \mathrm{id}, (1, \sqrt{-1})) - (2\pi i)^{-4}\widehat{\mathcal{L}}_4(\xi, \mathrm{L}) = -\frac{1}{2^8 3^2 5^3 43} \times 27524875151,$$
$$(2\pi i)^{-1}\Lambda_1(-3, \sigma^{-1}, (3, 3\sqrt{-1})) - (2\pi i)^{-4}\widehat{\mathcal{L}}_4(\sigma(\xi), \mathrm{L}) = \frac{1}{2^8 3^2 5^3} \times 449567443.$$

5.0.4. *Example 4.* Consider the case where $F = \mathbb{Q}(\sqrt{-1})$, $k = 2$ and

$$H = F[a]/(a^3 + a^2 + (-\sqrt{-1} + 1)\, a + 1)$$
$$= \mathbb{Q}[a]/(a^6 + 2a^5 + 3a^4 + 4a^3 + 4a^2 + 2a + 1).$$

The conductor $\mathfrak{m}$ of the abelian extension H/F is $(3 + 2\sqrt{-1})$, and the Galois group of the extension is given by $\mathrm{Gal}(H/F) = \{\mathrm{id}, \sigma, \sigma^2\} \simeq \mathbb{Z}/3\mathbb{Z}$, where

$$\sigma(a) = a^3 + \sqrt{-1}.$$

We embed H into $\mathbb{C}$ by

$$a \mapsto -1.049136453746963\ldots - 0.552653068016644\ldots i.$$

Then we have

$$(2\pi i)^{-1}\Lambda_1(-1, \mathrm{id}, (1, \sqrt{-1}))$$
$$= -0.47522356925671454695125646730 3\ldots + 0.03988718473520052711999965834 9\ldots i,$$
$$(2\pi i)^{-1}\Lambda_1(-1, \sigma^{-1}, (3, 3\sqrt{-1}))$$
$$= +1.18707039635350232820725881363 5\ldots - 0.01084165321417145393292239336 2\ldots i.$$
$$(2\pi i)^{-1}\Lambda_1(-1, \sigma^{-2}, (7, 7\sqrt{-1}))$$
$$= -0.94421862196858265305087414120 3\ldots - 0.07544889060987559355632276253 6\ldots i$$

We put

$$\begin{cases} x_0 = \sqrt{-1} \\ x_1 = a^4 + a^3 + 2a^2 + 2a + 1 \\ x_2 = a^4 + a^3 + a^2 + a + 1 \end{cases}$$

and $g(k_0, k_1, k_2) = [x_0^{k_0} x_1^{k_1} x_2^{k_2}]$. Then $\mathcal{B}_2(H) \otimes \mathbb{Q}$ is generated by ξ_1, ξ_2 and ξ_3, where

$$\xi_1 = 3g(0, -4, -3) + g(0, -3, -1) + g(-1, 4, 2),$$
$$\xi_2 = g(0, -4, -3) + 3g(0, -1, -2),$$
$$\xi_3 = g(0, -3, -1) + 7g(0, -1, -2).$$

Put

$$\xi = -\frac{23}{18}\xi_1 + \frac{23}{9}\xi_2 - \frac{5}{3}\xi_3,$$

and take any homomorphism $\mathrm{L} : H^\times/\mu_H \to \mathbb{C}$ such that

$$\begin{cases} \exp(\mathrm{L}(x)) = x \\ -\pi < \Im(\mathrm{L}(x)) \leq \pi \end{cases}$$

for $x \in \{x_1, x_2\}$. Then we have

$$
\begin{aligned}
&(2\pi i)^{-2}\widehat{\mathcal{L}}_2(\xi, \mathrm{L}) \\
&= 6.5799313452731999829632734472\,26\ldots + 0.0398871847352005271199996583\,49\ldots i, \\
&(2\pi i)^{-2}\widehat{\mathcal{L}}_2(\sigma(\xi), \mathrm{L}) \\
&= 2.4959646271227330974380280444\,04\ldots - 0.0108416532141714539329223933\,62\ldots i, \\
&(2\pi i)^{-2}\widehat{\mathcal{L}}_2(\sigma^2(\xi), \mathrm{L}) \\
&= 1.8668123609374002529320318417\,02\ldots - 0.0754488906098755935563227625\,36\ldots i.
\end{aligned}
$$

Therefore, the following equalities hold in high accuracy.

$$
\begin{aligned}
(2\pi i)^{-1}\Lambda_1(-1, \mathrm{id}, (1, \sqrt{-1})) - (2\pi i)^{-2}\widehat{\mathcal{L}}(\xi, \mathrm{L}) &= -\frac{1}{2^6 3^2 13} \times 52829, \\
(2\pi i)^{-1}\Lambda_1(-1, \sigma^{-1}, (3, 3\sqrt{-1})) - (2\pi i)^{-2}\widehat{\mathcal{L}}(\sigma(\xi), \mathrm{L}) &= -\frac{1}{2^6 13} \times 1089, \\
(2\pi i)^{-1}\Lambda_1(-1, \sigma^{-2}, (7, 7\sqrt{-1})) - (2\pi i)^{-2}\widehat{\mathcal{L}}(\sigma^2(\xi), \mathrm{L}) &= -\frac{1}{2^6 3^2 13} \times 21049.
\end{aligned}
$$

5.0.5. *Example 5.* Consider the case where $F = \mathbb{Q}(\alpha)$, $\alpha = \frac{1+\sqrt{-15}}{2}$, $k = 2$ and

$$
\begin{aligned}
H &= F[a]/(a^4 + \alpha a^3 + (\alpha - 2)\, a^2 - \alpha a + 1) \\
&= \mathbb{Q}[a]/(a^8 + a^7 + a^6 + 5a^5 - 5a^3 + a^2 - a + 1).
\end{aligned}
$$

The conductor $\mathfrak{m}$ of the abelian extension H/F is (2α), and the Galois group of the extension is given by $\mathrm{Gal}(H/F) = \{\mathrm{id}, \sigma, \sigma^2, \sigma^3\} \simeq \mathbb{Z}/4\mathbb{Z}$, where

$$
\sigma(a) = -\frac{5}{4}a^7 - 2a^6 - \frac{11}{4}a^5 - 8a^4 - 5a^3 + \frac{9}{4}a^2 + \frac{1}{2}a + \frac{9}{4}.
$$

We embed H into $\mathbb{C}$ by

$$
a \mapsto -1.472308583487351\ldots + 0.228052190401739\ldots i.
$$

Then we have

$$
\begin{aligned}
&(2\pi i)^{-1}\Lambda_1(-1, \mathrm{id}, (1, \alpha)) \\
&= -0.5494164885957489658397803382\,75\ldots + 0.4817598174579106040931989747\,24\ldots i, \\
&(2\pi i)^{-1}\Lambda_1(-1, \sigma^{-1}, (3, 1+\alpha)) \\
&= -0.4251924496055124555152105487\,57\ldots - 0.0798687503428147628771922270\,70\ldots i, \\
&(2\pi i)^{-1}\Lambda_1(-1, \sigma^{-2}, (3, 3\alpha)) \\
&= -0.2005835114042510341602196617\,24\ldots - 0.4156085245837687359142919123\,55\ldots i, \\
&(2\pi i)^{-1}\Lambda_1(-1, \sigma^{-3}, (9, 3+3\alpha)) \\
&= -1.8248075503944875444847894512\,42\ldots + 0.0534684668166029952932881488\,07\ldots i.
\end{aligned}
$$

We put

$$
\begin{cases}
x_0 = \frac{7}{8}a^7 + \frac{9}{8}a^6 + \frac{3}{2}a^5 + 5a^4 + 2a^3 - \frac{19}{8}a^2 + \frac{5}{8}a - \frac{1}{2} \\
x_1 = \frac{1}{8}a^6 + \frac{3}{8}a^5 + a^3 + a^2 - \frac{13}{8}a - \frac{1}{8} \\
x_2 = x_0 - a \\
x_3 = x_0 - x_1 - 1
\end{cases}
$$

and $g(k_0, k_1, k_2, k_3) = [x_0^{k_0} x_1^{k_1} x_2^{k_2} x_3^{k_3}]$. Note that, despite its appearance, $x_0^3 = -1$. Then $\mathcal{B}_2(H) \otimes \mathbb{Q}$ is generated by ξ_1, ξ_2, ξ_3 and ξ_4, where

$$\begin{aligned}
\xi_1 &= g(-1,0,0,0),\\
\xi_2 &= 2g(3,-1,1,1) - g(-1,1,1,0) - 2g(2,1,-1,-1),\\
\xi_3 &= 6g(3,-1,1,1) - 2g(-2,1,0,0) - g(2,1,-1,-1) + g(0,-1,-2,-1),\\
\xi_4 &= 2g(3,-1,1,1) + 2g(-2,1,0,0) - g(-2,1,1,2).
\end{aligned}$$

Put

$$\xi := -\frac{23}{5}\xi_1 - \frac{17}{2}\xi_2 + \frac{11}{4}\xi_3 + 6\xi_4,$$

and take any homomorphism $\mathrm{L} : H^\times/\mu_H \to \mathbb{C}$ such that

$$\begin{cases} \exp(\mathrm{L}(x)) = x \\ -\pi < \Im(\mathrm{L}(x)) \leq \pi \end{cases}$$

for $x \in \{x_1, x_2, x_3\}$. Then we have

$$\begin{aligned}
&(2\pi i)^{-2}\widehat{\mathcal{L}}_2(\xi, \mathrm{L})\\
&= -0.30167343304019341028422478271 9\ldots + 0.48175981745791060409319897472 4\ldots i\\
&(2\pi i)^{-2}\widehat{\mathcal{L}}_2(\sigma(\xi), \mathrm{L})\\
&= -4.26008828293884578884854388209 1\ldots - 0.07986875034281476287719222707 0\ldots i,\\
&(2\pi i)^{-2}\widehat{\mathcal{L}}_2(\sigma^2(\xi), \mathrm{L})\\
&= +5.67910398859574896583978033827 5\ldots - 0.41560852458376873591429191235 5\ldots i,\\
&(2\pi i)^{-2}\widehat{\mathcal{L}}_2(\sigma^3(\xi), \mathrm{L})\\
&= -6.98262005039448754448478945124 2\ldots + 0.05346846681660299529328814880 7\ldots i.
\end{aligned}$$

Therefore, the following equalities hold in high accuracy.

$$\begin{aligned}
(2\pi i)^{-1}\Lambda_1(-1, \mathrm{id}, (1,\alpha)) - (2\pi i)^{-2}\widehat{\mathcal{L}}_2(\xi, \mathrm{L}) &= -\frac{1}{2^7 3^2 5} \times 1427,\\
(2\pi i)^{-1}\Lambda_1(-1, \sigma^{-1}, (3, 1+\alpha)) - (2\pi i)^{-2}\widehat{\mathcal{L}}_2(\sigma(\xi), \mathrm{L}) &= \frac{1}{2^7 3^1 5} \times 7363,\\
(2\pi i)^{-1}\Lambda_1(-1, \sigma^{-2}, (3, 3\alpha)) - (2\pi i)^{-2}\widehat{\mathcal{L}}_2(\sigma^2(\xi), \mathrm{L}) &= -\frac{1}{2^7 5} \times 3763,\\
(2\pi i)^{-1}\Lambda_1(-1, \sigma^{-3}, (9, 3+3\alpha)) - (2\pi i)^{-2}\widehat{\mathcal{L}}_2(\sigma^3(\xi), \mathrm{L}) &= \frac{1}{2^7 5} \times 3301.
\end{aligned}$$

References

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