NEW FORMS IN THE KOHNEN PLUS SPACE

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1. INTRODUCTION

Let \( k \geq 2 \) be an odd integer, \( \chi \) a Dirichlet character \( \text{mod} \ 4N \) where \( N \) is a natural number. By \( S_{k+1/2}(4N, \chi) \) and \( S_{2k}(2N, \chi^2) \) we denote the spaces of cusp forms of weight \( k + 1/2 \) and \( 2k \) with respect to the congruence subgroups \( \Gamma_0(4N) \) and \( \Gamma_0(2N) \), respectively. For any \( f \in S_{k+1/2}(4N, \chi) \) and square-free integer \( t \), Shimura showed that there exists \( Sh_t(f) \in S_{2k}(2N, \chi^2) \) which can be described exactly by the Fourier coefficients. If \( f \) is an eigenform, then so does \( Sh_t(f) \) and they share the same eigenvalues for all Hecke operators \( T_p \) and \( T_{p^2} \), respectively, where \( p \) is an odd prime number. Note that the above is also true for \( k = 1 \) if \( f \) is in the complement of the subspace of \( S_{3/2}(4N, \chi) \) spanned by all single variable theta functions (otherwise \( Sh_t(f) \) may not be a cusp form). By taking linear combinations of such correspondences for square-free \( t \)'s, one gets various liftings from \( S_{k+1/2}(4N, \chi) \) to \( S_{2k}(2N, \chi^2) \). However, in general, one cannot get a bijective lifting in such a way. A natural problem is to identify the image of such liftings, or the subspace of \( S_{k+1/2}(4N, \chi) \) by restricting some lifting to which one can get a injective lifting. A partial answer to this question comes from the Kohnen plus space.

Definition 1.1. For \( N \) odd and square-free and \( \chi \) quadratic, the plus space \( S_{k+1/2}^+(4N, \chi) \) is the subspace of \( S_{k+1/2}(4N, \chi) \) consisting of those forms whose \( n \)-th Fourier coefficients vanish for all natural number \( n \) such that \((-1)^k\chi(-1)n \equiv 2 \) or \( 3 \) \( \text{mod} \ 4 \).

Kohnen initially introduced the plus space in 1980 [3] for the classical case and generalized it to the version as the definition above in 1982 [4]. He showed that there exists a one-to-one correspondence, which is a lifting introduced above, between \( S_{k+1/2}^+(4N, \chi) \) and \( S_{2k}(2N, \chi^2) \). From now we want to consider the case for general totally real number field, that is, the Hilbert case.
2. Definitions

Let $F$ be a totally real number field with degree $n$ over $\mathbb{Q}$. As usual, $\mathfrak{o}$ and $\mathfrak{d}$ denote its ring of integers and different over $\mathbb{Q}$, respectively. We fix an odd square-free ideal $\mathfrak{J}$ of $\mathfrak{o}$ and a primitive quadratic character $\chi$ of $(\mathfrak{o})$ with conductor $(\mathfrak{f})$, a principal ideal generated by some $\mathfrak{f} \in \mathfrak{o}$. Thus explicitly, we can write $\chi$ in the form

$$\chi(d) = \prod_{v \mid 2} (\mathfrak{f}, d)_v \prod_{v \mid \mathfrak{f}} (\mathfrak{f}, d)_v$$

where $v$ runs over places of $F$ and $(\cdot, \cdot)_v$ is the Hilbert symbol of the local field $F_v$ corresponding to $v$.

For ideals $\mathfrak{b}$ and $\mathfrak{c}$ of $F$ such that $\mathfrak{bc} \subset \mathfrak{o}$, we put

$$\Gamma[\mathfrak{b}, \mathfrak{c}] = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(F) \mid a, d \in \mathfrak{o}, b \in \mathfrak{b}, c \in \mathfrak{c} \right\}$$

and

$$\Gamma_0(\mathfrak{a}) = \Gamma[\mathfrak{o}^{-1}, \mathfrak{a}\mathfrak{o}]$$

for ideal $\mathfrak{a}$ of $\mathfrak{o}$.

For simplicity, we let $k \in \mathbb{N}^n$ be parallel and $\mathfrak{f}$ be with the sign $(-1)^k$, that is, the norm of $\mathfrak{f}$ over $\mathfrak{Q}$ has the same sign with $(-1)^k$.

We define the theta function $\theta$ on $\mathfrak{h}^n$, where $\mathfrak{h}$ is the upper-half part of the complex plane, by

$$\theta(z) = \sum_{\xi \in \mathfrak{o}} \exp(2\pi i \text{tr}(\xi^2 z)) .$$

Applying $\theta$, we can define the factor of automorphy of weight $1/2$ by

$$j(\gamma, z) = \theta(\gamma z) / \theta(z)$$

where $\gamma \in \Gamma_0(4)$ and $\gamma z$ denotes the image of $z$ under the Möbius transformation by $\gamma$.

Putting $S_{k+1/2}(4\mathfrak{J}, \chi)$ to be the space consisting of Hilbert cusp forms with respect to the factor of automorphy given by $j(\gamma, z)^{2k+1} \chi(\gamma)$ where

$$\chi \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) := \chi(d) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4\mathfrak{J}),$$

we give the definition of the plus space.

**Definition 2.1.** With the notations stated above, the Kohnen plus space $S_{k+1/2}^+(4\mathfrak{J}, \chi)$ of weight $k + 1/2$, level $4\mathfrak{J}$ and character $\chi$ is defined to be the subspace of $S_{k+1/2}(4\mathfrak{J}, \chi)$ such that $h \in S_{k+1/2}^+(4\mathfrak{J}, \chi)$ if and only if the $\xi$-th Fourier coefficient of $h$ vanishes unless there exists $\lambda \in \mathfrak{o}$ such that $\xi - \mathfrak{f}\lambda^2 \in 4\mathfrak{o}$. 
Let $\Lambda$ be the adele ring of $F$ with finite part $\Lambda_f$ and $\text{Mp}_2(\Lambda_f)$ be the metaplectic double covering of $\text{SL}_2(\Lambda_f)$.

An eigenform $h \in S_{k+1/2}^+(4\sigma_x)$ generates an irreducible representation $\pi_f = \prod_{v \in \infty} \pi_v$ of $\text{Mp}_2(\Lambda_f)$ where $\pi_v$ is an irreducible representation of $\text{Mp}_2(F_v)$. An eigenform $h$ is called a Hecke new form if for any finite place $v$ dividing $\mathfrak{f}$, $\pi_v$ is equivalent to a Steinberg representation, which is a certain ramified subrepresentation of some principal series representation. We let $S_{k+1/2}^{+,\text{NEW}}(4, \chi)$ be the $\mathbb{C}$-space spanned by Hecke new forms given above. Any form in $S_{k+1/2}^{+,\text{NEW}}(4, \chi)$ is called a new form. Note that the definition of new forms coincides with the one given by Kohnen.

3. AN IF−AND−ONLY−IF CONDITION FOR THE HECKE NEW FORMS

In this section, for simplicity, we set $\chi = 1$.

For $v | \mathfrak{f}$, we let

$$\Gamma_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(F_v) \mid a, d \in \mathfrak{o}_v, b \in \mathfrak{o}_v^{-1}, c \in \mathfrak{w}_v \mathfrak{o}_v \right\}$$

where $\mathfrak{w}_v \in \mathfrak{o}_v$ is the uniformizer corresponding to the place $v$. We denote the inverse image of $\Gamma_v$ in $\text{Mp}_2(F_v)$ by $\tilde{\Gamma}_v$.

Let $\tilde{\mathcal{H}}_v = \tilde{\mathcal{H}}_v(\tilde{\Gamma}_v \backslash \text{Mp}_2(F_v) / \tilde{\Gamma}_v, \epsilon_v)$ be the Hecke algebra with respect to the genuine character $\epsilon_v$ of $\tilde{\Gamma}_v$ which comes from some Weil representation of $\text{Mp}_2(F_v)$.

**Definition 3.1.** Let $\tilde{T}_v$ and $\tilde{U}_v$ be the Hecke operators in $\tilde{\mathcal{H}}_v$ which are supported on $\tilde{\Gamma}_v \begin{pmatrix} \mathfrak{w}_v & 0 \\ 0 & \mathfrak{w}_v^{-1} \end{pmatrix}$ $\tilde{\Gamma}_v$ and $\tilde{\Gamma}_v \begin{pmatrix} 0 & -\delta_v^{-1} \mathfrak{w}_v^{-1} \\ \delta_v \mathfrak{w}_v & 0 \end{pmatrix}$ $\tilde{\Gamma}_v$, respectively, such that

$$\tilde{T}_v \left( \begin{pmatrix} \mathfrak{w}_v & 0 \\ 0 & \mathfrak{w}_v^{-1} \end{pmatrix} \right) = q_v^{-1/2} \alpha_v(\mathfrak{w}_v) \alpha_v(1)$$

and

$$\tilde{U}_v \left( \begin{pmatrix} 0 & -\delta_v^{-1} \mathfrak{w}_v^{-1} \\ \delta_v \mathfrak{w}_v & 0 \end{pmatrix} \right) = \alpha_v(\delta_v \mathfrak{w}_v).$$

Here $\delta_v \in \mathfrak{o}_v$ is one which generates the local principal ideal $\mathfrak{o}_v$, $\alpha_v$ denotes the Weil constant and $q_v$ is the index of the local residue field with respect to $v$.

In the definition above, $\tilde{T}_v$ is the usual Hecke operator and $\tilde{U}_v$ is the Atkin-Lehner operator.
Theorem 3.1. An eigenform \( h \in S_{k+1/2}^{+}(4\mathcal{I}, 1) \) is a Hecke new form if and only if
\[
\overline{T_v} \tilde{U_v} h = -h = \overline{\mathcal{U}_v} T_v h
\]
for all finite \( v|\mathcal{I} \).

This theorem is motivated by a result from [1]. They treated the case for integral weight, \( F = \mathbb{Q}, \chi = 1 \) and general level.

4. APPLICATION OF WALDSPURGER’S THEORY

Theorem 4.1. The plus space \( S_{k+1/2}^{+}(4\mathcal{I}, \chi) \) is the \( E^K \)-fixed subspace of \( S_{k+1/2}^{+}(4\mathcal{I}, \chi) \) for some Hecke operator \( E^K = \otimes_{v<\infty} E_v^K \in \otimes_{v<\infty} \tilde{\mathcal{H}}_v \) where for each \( \tilde{\mathcal{H}}_v = (\Gamma_v \backslash \text{Mp}_2(F_v)/\Gamma_v, \epsilon_v) \) we set
\[
\Gamma_v = \begin{cases} 
\Gamma_0(1)_v & \text{if } v \nmid 2\mathcal{I}, \\
\Gamma_0(4)_v & \text{if } v \mid 2 \\
\Gamma_0(\varpi_v)_v & \text{if } v \mid \mathcal{I}.
\end{cases}
\]

The Hecke operator \( E^K \) is an idempotent and can be written down explicitly, but we omit its definition here. The following proposition was given by Hiraga and Ikeda [2].

Proposition 4.1. Let \( v \) be a finite place of \( F \) not dividing \( \mathcal{I} \) and \( B \) be the Borel subgroup of \( \text{SL}_2(F_v) \) consisting of upper-triangular matrices. For \( s \in \mathbb{C} \), if the principal series \( \text{Ind}^\text{MP}_2(F_v)_{B}((a b \ x \ y) \mapsto \frac{a\epsilon(a)}{a_v(a)} |a|_v^{s+1}) \) is irreducible, then its \( E_v^K \)-fixed subspace is of one dimension.

Proposition 4.2. The \( E_v^K \)-fixed subspace of a Steinberg representation is of one dimension for \( v \mid \mathcal{I} \).

Now let \( k \geq 2 \). By Waldspurger’s results, each irreducible representation \( \pi \) of \( \text{Mp}_2(\mathbb{A}) \) from an eigenform \( h \in S_{k+1/2}^{+}(4\mathcal{I}, \chi) \) corresponds to an irreducible cuspidal automorphic representation of \( \text{PGL}_2(\mathbb{A}) \), which gives a non-zero unique-up-to-non-zero-scalar-multiplications eigenform in the space of cuspidal automorphic forms
\[
\mathcal{A}^{\text{Cusp}}_{2k}(\mathcal{I}) = \mathcal{A}^{\text{Cusp}}_{2k}(\text{PGL}_2(F) \backslash \text{PGL}_2(\mathbb{A})/\prod_{v<\infty} \Gamma'_v(\mathcal{I}))
\]
where \( \Gamma'_v(\mathcal{I}) \) is a congruence subgroup which is maximal compact if \( v \nmid \mathcal{I} \) and Iwahori if \( v \mid \mathcal{I} \). We put \( \mathcal{A}^{\text{Cusp,NEW}}_{2k}(\mathcal{I}) \) to be the subspace of \( \mathcal{A}^{\text{Cusp}}_{2k}(\mathcal{I}) \) spanned by \( g \) such that its corresponding representation of \( \text{PGL}_2(\mathbb{A}) \) is locally a Steinberg representation at any finite \( v \mid \mathcal{I} \).
Theorem 4.2. The plusspace $S_{k+1/2}^{+\text{NEW}}(4\mathcal{I}, \chi)$ is Hecke isomorphic to $A_{2k}^{\text{CUSP,NEW}}(3)$.

Note that for the case $\mathcal{I} = 1$ the theorem was treated by Hiraga and Ikeda in [2].

Using Theorem 3.1 and Theorem 4.2 we can get an analogue of the result from Baruch and Purkait in [1] for the Hilbert modular forms of integral weight.

REFERENCES


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