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<tr>
<th>Title</th>
<th>Eigenvalue Problem of Anti-Wick (Toeplitz) Operators in Bargmann-Fock Space and Applications to Daubechies Operators (Wavelet analysis and signal processing)</th>
</tr>
</thead>
<tbody>
<tr>
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Kyoto University
**Eigenvalue Problem of Anti - Wick (Toeplitz) Operators in Bargmann - Fock Space and Applications to Daubechies Operators**

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**Abstract:** In this paper we will consider algebraic background of Gabor analysis and eigenvalue problem of anti - Wick (Toeplitz) operators in Bargmann - Fock space. We will clarify the relationship between anti - Wick (Toeplitz) operators and Daubechie (localization) operators. We apply our results to eigenvalue problem of Daubechie operators.

1 Gabor transform

In this section we will recall the definition and properties of Gabor transform([5], [6]). Gabor transform $W_{\phi}(f)(p,q)$ is defined as follows:

$$W_{\phi}(f)(p,q) = \int_{\mathbb{R}^n} \overline{\phi_{p,q}(x)} f(x) dx, \quad (f(x) \in L^2(\mathbb{R}^n), x, p, q \in \mathbb{R}^n)$$

$\phi(x) = \pi^{-n/4}e^{-x^2/2}$ is Gaussian and $\phi_{p,q}(x) = \pi^{-n/4}e^{ipx}e^{-(x-q)^2/2}$ is Gabor function. We have following inversion formula(resolution of identity)

**Proposition 1** (Inversion formula of Gabor transform)

$$f(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} \phi_{p,q}(x) W_{\phi}(f)(p,q) dp dq$$

(Proof) \[
\int_{\mathbb{R}^{2n}} \phi_{p,q}(x) W_{\phi}(f)(p,q) dp dq \\
= \int_{\mathbb{R}^n} \phi_{p,q}(x) \int_{\mathbb{R}^n} e^{ipy} \phi(y-q) f(y) dy dp dq \\
= \int_{\mathbb{R}^n} e^{-ipx} \phi(x-q) e^{ipy} \phi(y-q) f(y) dy dp dq \\
= \int_{\mathbb{R}^n} \left\{ \int_{\mathbb{R}^n} e^{-ip(x-y)} dp \right\} \phi(x-q) \phi(y-q) f(y) dy dq \\
= (2\pi)^n \int_{\mathbb{R}^n} \delta(x-y) \phi(x-q) \phi(y-q) f(y) dy dq \\
= (2\pi)^n \int_{\mathbb{R}^n} \phi(x-q) \phi(x-q) f(x) dq = (2\pi)^n <\phi, \phi> f(x) = (2\pi)^n f(x)
\]

**Proposition 2** (Unitarity of Gabor Transform)
\[ \langle W_\phi(f), W_\phi(g) \rangle = (2\pi)^{-n} \langle f, g \rangle \]

1.1 The relationship between FBI transform, Bargmann transform and Gabor transform

Gabor transform is closely related to FBI (Fourier - Bros - Iagolnitzer) transform and Bargmann transform ([7]).
FBI transform \( P^t(f)(p, q) \) is defined by

\[ P^t(f)(p, q) = \int_{\mathbb{R}^n} e^{-ipx} e^{-t(x-q)^2} f(x) dx \]

1. FBI transform is related to Gabor transform as follows:

\[ P^{1/2}(f)(p, q) = \int_{\mathbb{R}^n} e^{-ipx} e^{-(x-q)^2/2} f(x) dx \]

2. Bargmann transform is related to Gabor transform as follows:

\[ B(f)(z) = \pi^{-n/4} e^{1/4(p^2 + q^2 + 2ipq)} \int_{\mathbb{R}^n} e^{-ipx} e^{-(x-q)^2/2} f(x) dx, \]

\( z = \frac{q + ip}{\sqrt{2}}, p, q \in \mathbb{R}^n \).

Remark.
1. Gabor transform is used for iris identification and signal analysis of human voice. It is also used for the definition of Feichtinger(Segal) algebra and modulation space([9]).
2. Recently the relationship between Gabor analysis and operator algebra is studied by several mathematicians([8],[13],[14],[15],[17]).

2 Projective representation of time frequency plane(phrase space)

In Gabor analysis the function \( e^{ipx}g(x-q) \) frequently appears. We already saw this type of function(Gabor function) in the Gabor transform. Another example is Zak transform: \( Z(g)(s, t) = \sum_{n\in \mathbb{Z}} e^{int} g(s-n) \).

And here is celebrated Balian - Low Theorem.

\[ \int_{\mathbb{R}^n} x^2 |g(x)|^2 dx = \infty \text{ or } \int_{\mathbb{R}^n} \xi^2 |\hat{g}(\xi)|^2 d\xi = \infty. \]
2.1 Modulation operator and translation operator

In this section we will consider the meaning of the function $e^{ipx}g(x - q)$. For $g(x) \in L^2(\mathbb{R}^n)$, we define modulation operator $M_p g(x) = e^{ipx} g(x)$ and translation operator $T_q g(x) = g(x - q)$.

Both are unitary operators and satisfy $M_a M_b = M_{a+b}$ and $T_a T_b = T_{a+b}$. Namely $M_p$ and $T_q$ are unitary representations of additive group $\mathbb{R}^n$.

We have the following commutative diagram:

$$
\begin{array}{c}
L^2(\mathbb{R}^n) \xrightarrow{F} L^2(\mathbb{R}^n) \\
\downarrow T_q & \downarrow M_a \\
L^2(\mathbb{R}^n) \xrightarrow{F} L^2(\mathbb{R}^n)
\end{array}
$$

$F$ is the Fourier transform (intertwining operator).

$M_p$ and $T_q$ satisfy $M_p T_q = e^{-ipq} T_q M_p$.

2.2 An interpretation of $e^{ipx}g(x - q)$ by projective representation of time frequency plane

For $g(x) \in L^2(\mathbb{R}^n)$, we put

$$
\pi(p, q) g(x) = M_p T_q g(x) = e^{ipx} g(x - q), \quad (p, q) \in \mathbb{R}^n \times \mathbb{R}^n.
$$

$\pi(p, q)$ satisfies $\pi(p_1, q_1) \pi(p_2, q_2) = e^{-ip_2q_1} \pi(p_1 + p_2, q_1 + q_2)$. Although $\pi(p, q)$ is unitary operator, it is not unitary representation because of factor $e^{-ip_2q_1}$. So $\pi(p, q)$ is called projective representation (ray representation, Weyl - Heisenberg operator) of $\mathbb{R}^n \times \mathbb{R}^n$. To make projective representation $\pi(p, q)$ to unitary representation, we will introduce Heisenberg group.

2.3 Heisenberg Group

We identify phase space (time frequency plane for $n = 1$) $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$.

Remark that $\mathbb{C}^n$ has symplectic structure. i.e. $\mathbb{C}^n$ is symplectic vector space. We have the following exact sequence.

$$
0 \rightarrow \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{C}^n \rightarrow \mathbb{C}^n \rightarrow 0
$$

$\mathbb{R} \times \mathbb{C}^n = H_n$ is called the Heisenberg group (polarized).

We put $\pi(t, p, q) g(x) = e^{it} e^{ipx} g(x - q), \quad (g \in L^2(\mathbb{R}^n), p, q \in \mathbb{R}^n, t \in \mathbb{R})$

$\pi(t, p, q)$ is unitary representation (Schrödinger representation) of the Heisenberg group and $\pi(0, p, q) = \pi(p, q)$. 

Example \( H_1 \) is realized as the group of matrix.
\[
H_1 \ni (t, p, q) \mapsto \begin{pmatrix} 1 & p & t \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}, \quad H_1 \cong \left\{ \begin{pmatrix} 1 & p & t \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} : t, p, q, \in \mathbb{R} \right\}
\]

Remark
1. For the details of Heisenberg group, we refer the reader to [7], [10], [12], [16] and [18].
2. Projective representation (ray representation) of continuous group is studied by V. Bargmann ([1]).
3. To construct irreducible unitary representation of the Heisenberg group, we use \( L^2(\mathbb{R}^n) \) (Schrödinger representation) or Bargmann - Fock space \( BF(\mathbb{C}^n) \) (Fock representation).

\[
L^2(\mathbb{R}^n) \xrightarrow{\pi(t,p,q)} BF(\mathbb{C}^n) \xrightarrow{B \circ (t,p,q) \circ B^{-1}} \]

\( B \) is the Bargmann transform (intertwining operator).

3 Bargmann transform and Bargmann - Fock space

3.1 Bargmann transform

We recall the definition of Bargmann transform and its properties([2]). We put \( A_n(z, x) \) as follows :
\[
A_n(z, x) = \pi^{-n/4} \exp \left\{ -\frac{1}{2} (z^2 + x^2) + \sqrt{2} z \cdot x \right\}, \quad (z \in \mathbb{C}^n, x \in \mathbb{R}^n).
\]

The Bargmann transform \( B(\psi) \) is defined as follows :
\[
B(f)(z) \overset{\text{def}}{=} \int_{\mathbb{R}^n} f(x) A_n(z, x) dx, \quad (f(x) \in L^2(\mathbb{R}^n)).
\]

Example([2]) Let \( h_m(x) \) be Hermite function of degree \( m \). Then
\[
B(h_m)(z) = \frac{z^m}{\sqrt{m!}}, \quad (m \in \mathbb{N})
\]
3.2 Bargmann - Fock space $BF(C^n)$

We put

$$BF(C^n) = \{ g \in H(C^n) : \int_{C^n} |g(z)|^2 e^{-|z|^2} dz \wedge d\bar{z} < \infty \}.$$ 

$H(C^n)$ denotes the space of entire functions.

**Example ([28])**

1. Polynomials and entire functions of exponential type belong to Bargmann - Fock space. For example, sinc function $\frac{\sin z}{z}$ and prolate spheroidal function (eigenfunction of $(\tau^2 - t^2) \frac{dz}{dt} - 2t \frac{d}{dt} - \sigma^2 t^2$) are entire functions of exponential type ([20]). Hence they belong to Bargmann - Fock space.

2. $\sigma(z) = z \prod \left(1 - \frac{z}{\lambda_{m,n}}\right) \exp \left(\frac{z}{\lambda_{m,n}} + \frac{z^2}{2\lambda^2_{m,n}}\right)$, is Weierstrass $\sigma$ - function and $\lambda_{m,n}$ are lattice points in $C$.

Under suitable conditions on lattice points, $\frac{\sigma(z)}{z}$ belongs to Bargmann - Fock space ([10]).

**Theorem 1 ([2])**

Bargmann transform is a unitary mapping from $L^2(\mathbb{R}^n)$ to $BF(C^n)$.

The inverse Bargmann transform $B^{-1}$ is given by

$$B^{-1}(g)(x) \overset{def}{=} \frac{1}{(2\pi i)^n} \int_{C^n} g(z) \overline{A_n(z, x)} e^{-|z|^2} d\bar{z} \wedge dz, \quad (g \in BF(C^n)).$$

Inner product in $BF(C^n)$ is defined by following formula:

$$< f, g >_{BF} = \frac{1}{(2\pi i)^n} \int_{C^n} \overline{f(z)} g(z) e^{-|z|^2} d\bar{z} \wedge dz$$

$BF(C^n)$ is Hilbert space with this inner product.
3.3 Projection, Bergman Kernel and Reproducing Formula

Since $BF(C^n)$ is a closed subspace of

$L^2(C^n, e^{-|z|^2}) = \{ g(z) : \int_{C^n} |g(z)|^2 e^{-|z|^2} d\bar{z} \wedge dz < \infty \}$,

we have the following orthogonal decomposition:

$L^2(C^n, e^{-|z|^2}) = BF(C^n) \oplus BF(C^n)^\perp$

Proposition 3([29])

Projection $P : L^2(C^n, e^{-|z|^2}) \rightarrow BF(C^n)$ is the following integral operator:

$(Pg)(z) = \frac{1}{(2i \pi)^n} \int_{C^n} e^{z\bar{w}} g(w)e^{-|w|^2} d\bar{w} \wedge dw, \quad (g \in L^2(C^n, e^{-|z|^2}))$.

Proposition 4

Following statements are equivalent:

1. $g(z) \in BF(C^n)$
2. $P(g)(z) = g(z)$
3. (Reproducing formula)

$g(z) = \frac{1}{(2i \pi)^n} \int_{C^n} e^{z\bar{w}} g(w)e^{-|w|^2} d\bar{w} \wedge dw$

Remark

$e^{z\bar{w}}$ is Bergman (reproducing) kernel with respect to Gaussian measure $(2\pi i)^{-n} e^{-|w|^2} d\bar{w} \wedge dw$.

4 Anti-Wick(Toeplitz) Operator

4.1 Toeplitz operator

In this subsection we will recall the definition of Toeplitz operators. For a region $D$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$), we put $L^2(D : d\mu) = \{ f(z) : \int_D |f(z)|^2 d\mu(z) < \infty \}$.

Suppose that $H$ is a closed subspace of $L^2(D : d\mu)$ and $P_H : L^2(D : d\mu) \rightarrow H$ is projection operator. If $h(z)$ is a bounded function in $\mathbb{R}^n$ (or $\mathbb{C}^n$), then we can define multiplication operator $m_h(f)(z) = h(z)f(z)$.

We put $T = P_H \circ m_h$. i.e. $T(f)(z) = P_H(h(z)f(z))$.

$T : L^2(D : d\mu) \xrightarrow{m_h} L^2(D : d\mu) \xrightarrow{P_H} H$,

$T$ is called Toeplitz operator.
4.2 Toeplitz operator on Bargmann - Fock space

Since Toeplitz operator $T_F$ with symbol $F$ is a composition of multiplication operator and projection operator, we have

$$(T_F f)(z) = \int_{\mathbb{C}^n} e^{z\overline{w}} F(w, \overline{w}) f(w) d\mu(w), \quad (\forall f \in L^2(\mathbb{C}^n, d\mu)), $$

where $F(w, \overline{w})$ is a bounded function on $\mathbb{C}^n$ and $d\mu(w) = (2\pi i)^{-n} e^{-|w|^2} d\overline{w} \wedge dw$.

**Remark** For the recent development of the theory of Toeplitz operators on Bargmann - Fock space, we refer the reader to [3], [4], [11], [19], [28] and [29].

4.3 Wick Operator and Anti - Wick Operator

According to ([7]), we will recall the definition of Wick Operator and anti-Wick Operator. For $f \in BF(\mathbb{C}^n)$, we define Wick operator $T_F^W$ as follows:

$$T_F^W f(z) = \sum a_{\alpha, \beta} z^\alpha \frac{d^\beta}{dz^\beta} f(z).$$

If we employ reproducing formula (3 in Proposition 4), then we obtain following integral representation of Wick operator $T_F^W$:

$$T_F^W f(z) = \int_{\mathbb{C}^n} e^{z\overline{w}} F(z, \overline{w}) f(w) d\mu(w).$$

$F(z, \overline{w})$ is an entire function of $(z, \overline{w})$ with some estimate.

We define anti-Wick operator as follows:

$$T_F^{AW} f(z) = \sum a_{\alpha, \beta} \frac{d^\beta}{dz^\beta} z^\alpha f(z).$$

If we employ reproducing formula (3 in Proposition 4), then we obtain following integral representation of anti-Wick operator $T_F^{AW}$:

$$T_F^{AW} f(z) = \int_{\mathbb{C}^n} e^{z\overline{w}} F(w, \overline{w}) f(w) d\mu(w).$$

$F(w, \overline{w})$ is measurable function with some estimate.

**Remark**

If $F(w, \overline{w})$ is bounded function, then $T_F^{AW}$ is Toeplitz operator.

**Example**

1. If we consider harmonic oscillator operator in Bargmann - Fock space, then it is Wick operator.
If \( T = -\frac{d^2}{dx^2} + x^2 - 1 : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \), then

\[
(B \circ T \circ B^{-1})f(z) = z \frac{d}{dz} f(z) : BF(\mathbb{C}) \rightarrow BF(\mathbb{C}) \quad ([2])
\]

\[
z \frac{d}{dz} f(z) = z \frac{d}{dz} \int_{\mathbb{C}} e^{z\overline{w}} f(w) d\mu(w) = \int_{\mathbb{C}} z\overline{w} e^{z\overline{w}} f(w) d\mu(w),
\]

So we have \( F(z, \overline{w}) = z\overline{w} \).

2. \( \frac{d}{dz} z : BF(\mathbb{C}) \rightarrow BF(\mathbb{C}) \) is anti-Wick operator.

\[
\frac{d}{dz} z f(z) = \frac{d}{dz} \int_{\mathbb{C}} e^{z\overline{w}} w f(w) d\mu(w) = \int_{\mathbb{C}} \overline{w} e^{z\overline{w}} f(w) d\mu(w),
\]

Hence we have \( F(w, \overline{w}) = \overline{w} = |w|^2 \).

### 4.4 Eigenvalue problem of Anti-Wick (Toeplitz) Operator on Bargmann-Fock Space

In this subsection we will consider the eigenvalue problem of anti-Wick (Toeplitz) operator \( T_F(f)(z) = \int_{\mathbb{C}^n} e^{z\overline{w}} F(w, \overline{w}) f(w) d\mu(w) \).

**Theorem 2**\([28]\) Suppose that \( F(w, \overline{w}) \) is bounded integrable and polyradial function. i.e. \( F(w, \overline{w}) = F(|w_1|^2, \cdots, |w_n|^2) \). Then

1. \( z^m \) is eigenfunction of \( T_F \).

2. Eigenvalue \( \lambda_m \) of \( T_F \) is given by

\[
\lambda_m = \frac{1}{m!} \int_0^\infty \cdots \int_0^\infty \tilde{F}(s_1, \cdots s_n) \prod_{i=1}^n e^{-s_i} s_i^{m_i} ds_i, \ m = (m_1, \cdots, m_n) \in \mathbb{N}^n.
\]

**Proof** For brevity's sake, we put \( n = 1 \).

\[
(T_F)(w^m)(z) = \int_{\mathbb{C}} \tilde{F}(|w|^2) e^{z\overline{w}} w^m d\mu(w) = \frac{1}{\pi} \int_{\mathbb{C}} \tilde{F}(|w|^2) e^{z\overline{w}} w^m e^{-|w|^2} dm(w)
\]

\[
= \frac{1}{\pi} \int_{\mathbb{C}} \tilde{F}(|w|^2) \left( \sum_{n=0}^\infty \frac{(z\overline{w})^n}{n!} \right) w^m e^{-|w|^2} dm(w)
\]

\[
= \sum_{n=0}^\infty \frac{z^n}{n!} \frac{1}{\pi} \int_{\mathbb{C}} \tilde{F}(|w|^2) \overline{w}^n w^m e^{-|w|^2} dm(w).
\]

By using the polar coordinate \( w = re^{i\theta} \),

\[
= \frac{1}{\pi} \sum_{n=0}^\infty \frac{z^n}{n!} \int_0^{2\pi} \int_0^{\infty} \tilde{F}(r^2) e^{i(m-n)\theta} r^n r^m e^{-r^2} r dr d\theta
\]
or

\[ \frac{1}{m!} \int_{0}^\infty e^{-r^2} \overline{F}(r^2) r^{2m} 2rdr = \frac{1}{m!} \int_{0}^\infty e^{-s} s^{m} \tilde{F}(s) ds. \]

Hence we obtain

\[ (T_F)(w^m)(z) = \frac{z^m}{m!} \int_{0}^\infty e^{-s} s^{m} \tilde{F}(s) ds. \]

**Example ([24], [28])**

1. \( F(w, \overline{w}) = \exp\left(\frac{a-1}{a} |w|^2\right), \quad (0 < a < 1) \)

\( \tilde{F}(s) = \exp\left(\frac{a-1}{a} s\right) \)

\( \lambda_m = a^{m+1} \)

## 5. Daubechies (Localization) Operator

### 5.1 Daubechies (Localization) Operator

Daubechies operator was introduced by Ingrid Daubechies in ([5], [6]). Daubechies operator \( P_F \) is defined as follows:

\[ P_F(f)(x) = (2\pi)^{-n} \int \int_{\mathbb{R}^{2n}} F(p, q) \phi_{p,q}(x) \overline{W_\phi(f)(p, q)} dpdq, \]

\( f(x) \in L^2(\mathbb{R}^n) \). \( \phi_{p,q}(x) = \pi^{-n/4} e^{-ipx} e^{-(x-q)^2/2} \).

\( W_\phi(f)(p, q) = \int_{\mathbb{R}^n} \overline{\phi_{p,q}(y)} f(y) dy \) is Gabor transform of \( f(x) \) and \( F(p, q) \) is symbol function of \( P_F \).

**Remark** If \( F(p, q) \) is 1, then \( P_F \) is identity operator. i.e. We have

**Resolution of identity** (Inversion formula of Gabor transform)

\[ f(x) = \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{R}^{2n}} \phi_{p,q}(x) W_\phi(f)(p, q) dpdq \]

### 5.2 Daubechies Operator in Bargmann - Fock space

If we consider Daubechies operator in Bargmann - Fock space, then we have following theorem ([28]).

**Theorem 3** For \( g(z) \in BF(\mathbb{C}^n) \), we have

\[ (B \circ P_F \circ B^{-1})(g)(z) = (2\pi i)^{-n} \int \int_{\mathbb{C}^n} F(w, \overline{w}) e^{z\overline{w}} g(w) e^{-|w|^2} dw \wedge \overline{dw}. \]

Especially if \( F(w, \overline{w}) = 1 \), then we obtain
Corollary (Relationship between resolution of identity and reproducing formula)

\[ f(x) = \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{R}^{2n}} \phi_{p,q}(x) W_{\phi}(f)(p,q) dp dq, \quad f(x) \in L^2(\mathbb{R}^n) \]

is equivalent to

\[ g(z) = \int_{\mathbb{C}^n} e^{z\overline{w}} g(w) d\mu(w), \quad (\forall g(z) \in BF). \]

6 Application to Daubechies Localization Operator

6.1 Hermite Functions

Hermite functions \( h_m(x) \) of one variable is defined by

\[ h_m(x) = (-1)^m (2^m m! \sqrt{\pi})^{-1/2} \exp(x^2/2) \frac{d^m}{dx^m} \exp(-x^2). \]

Generating function of Hermite functions is the kernel function of Bargmann transform.

\[ \pi^{-1/4} \exp \left\{ -\frac{1}{2} (z^2 + x^2) + \sqrt{2} z \cdot x \right\} = \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} h_m(x), \quad (z \in \mathbb{C}^1, x \in \mathbb{R}^1). \]

We also have the following expression.

\[ h_m(x) = \frac{1}{\sqrt{2^m m!}} \left( \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} \right) \right)^m h_0(x). \]

Hermite functions \( h_m(x) \) of several variables is defined by

\[ h_m(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} h_{m_i}(x_i), \quad m = (m_1, \ldots, m_n) \in N^n \]

Example

1. \( h_0(x) = \pi^{-1/4} \exp(-x^2/2), \) (coherent state)
2. \( h_2(x) = \pi^{-1/4} \frac{2x^2 - 1}{\sqrt{2}} \exp(-x^2/2), \) (Mexican hat wavelet)
6.2 Daubechies' result

As an application of our result, we will give a new proof of following Daubechies' result.

**Theorem 4([5])** Suppose that $F(p, q)$ is integrable polyradial function. Then we have

1. $P_F(h_m)(x) = \lambda_m h_m(x)$

2. $\lambda_m = \frac{1}{m!} \int_0^\infty \cdots \int_0^\infty \mathcal{F}(s_1, \cdots s_n) \prod_{i=1}^n e^{-s_i} s_i^{m_i} ds_i,$
   \[ m = (m_1, \cdots, m_n) \in \mathbb{N}^n. \]

**(Proof)** For the simplicity we put $n=1$. Let $P_F$ be Daubechies operator with integrable polyradial symbol $F$. Then $T_F = B \circ P_F \circ B^{-1}$ is Toeplitz operator with integrable polyradial symbol $F$. So we can apply Theorem 2 to $T_F$. Hence we have

$\lambda_m = \frac{1}{m!} \int_0^\infty \mathcal{F}(s) e^{-s} s^m ds$, \hspace{1cm} $T_F(\frac{z^m}{\sqrt{m!}}) = \lambda_m \frac{z^m}{\sqrt{m!}}.$

By inverse Bargmann transform, $h_m(x) = B^{-1}(\frac{z^m}{\sqrt{m!}})(x).$

So we obtained following Daubechies' results.

$P_F(h_m)(x) = \lambda_m h_m(x)$, \hspace{1cm} $\lambda_m = \frac{1}{m!} \int_0^\infty e^{-s} s^m \mathcal{F}(s) ds.$

7 Reconstruction of symbol function from eigenvalues

7.1 The first reconstruction formula

We consider the analytic continuation of eigenvalues $\lambda_m$ of $T_F$. It is given by

$\lambda(z) = \frac{1}{\Gamma(z + 1)} \int_0^\infty e^{-s} s^z \mathcal{F}(s) ds,$

where $\Gamma(z)$ is Euler Gamma function. We have $\lambda(m) = \lambda_m$ by Theorem 2.

**Theorem 5([21])**

$\mathcal{F}(s) = \frac{e^s}{s} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \lambda(z) \Gamma(z + 1) s^{-z} dz.$
Integral representation \( \lambda(z) = \frac{1}{\Gamma(z+1)} \int_0^\infty e^{-s}s\tilde{F}(s)s^{z-1}ds \), means that \( \lambda(z)\Gamma(z+1) \) is Mellin transform of \( e^{-s}\tilde{F}(s) \). Hence we obtain above formula by inverse Mellin transform.

### 7.2 The second reconstruction formula

For eigenvalues \( \{\lambda_m\} \) of anti-Wick(Toeplitz) operator \( T_F \), we put
\[
\Lambda(w) = \sum_{m=0}^{\infty} \lambda_m w^m.
\]
\( \Lambda(w) \) is generating function (of eigenvalues) of anti-Wick(Toeplitz) operator \( T_F \). In signal analysis \( \Lambda(w) \) is called \( z \)-transform instead of generating function. In what follows we assume that \( F(p,q) \) is integrable and polyradial function.

**Proposition 5 ([23])** Suppose that \( \lambda_m \) are eigenvalues of \( T_F \). Then we have

(i) \( \exists C > 0 \) s.t. \( |\lambda_m| \leq \frac{C}{\sqrt{|m|}}, \quad (m \in \mathbb{N}^n) \).

(ii) \( \Lambda(w) \) is holomorphic in \( \prod_{i=1}^{n}\{w \in \mathbb{C}^n : |w_i| < 1\} \).

(iii) \( \Lambda(w) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n} e^{-s_i(1-w_i)}\tilde{F}(s_1, \ldots, s_n)ds_1\ldots ds_n \).

(iv) \( \Lambda(w) \) is holomorphic in \( \prod_{i=1}^{n}\{w \in \mathbb{C}^n : Re(w_i) < 1\} \) and bounded in its closure.

(v) \( \Lambda(iv) \in C_0(\mathbb{R}^n), (v \in \mathbb{R}^n) \). i.e. \( \Lambda(iv) \in C(\mathbb{R}^n) \) and \( \lim_{|v| \to \infty} \Lambda(iv) = 0 \).

**Proof** Without loss of genelarity, we can assume that \( n = 1 \).

(i) By Theorem 2, \( \lambda_m = \frac{1}{m!} \int_0^\infty e^{-s}\tilde{F}(s)s^m ds \).

Since \( e^{-s}s^m \leq e^{-m}m^m \), we have
\[
|\lambda_m| \leq \frac{1}{m!} e^{-m}m^m \int_0^\infty |\tilde{F}(s)|ds.
\]
By Stirling's formula \( m! \sim \sqrt{2\pi m}e^{-m}m^m \), for sufficiently large \( m \),
\[
|\lambda_m| \leq C\frac{1}{\sqrt{m}} \text{ valids.}
\]

(iii) \( \Lambda(w) = \sum_{m=0}^{\infty} \lambda_m w^m = \sum_{m=0}^{\infty} \frac{w^m}{m!} \int_0^\infty e^{-s}s^m \tilde{F}(s)ds = \)
\[
\int_{0}^{\infty} e^{-s} \tilde{F}(s) \sum_{m=0}^{\infty} \frac{(ws)^{m}}{m!} ds = \int_{0}^{\infty} e^{-s(1-w)} \tilde{F}(s) ds.
\]

(iv) For \(\text{Re}(w) \leq 1\), we have
\[
|\Lambda(w)| \leq \int_{0}^{\infty} |e^{-s(1-w)}||\tilde{F}(s)|ds \leq ||\tilde{F}||_{L^{1}}.
\]

(v) Since \(\Lambda(iv)\) is Fourier transform of \(L^1\) function \(e^{-s}\tilde{F}(s)\), it is in \(C_0(\mathbb{R}^n)\) by Riemann - Lebesgue theorem.

**Theorem 6([21])**

\[
\tilde{F}(s) = (2\pi)^{-1}e^{s}\int_{-\infty}^{+\infty} e^{-isv}\Lambda(iv)dv,
\]

valids in distribution sense.

**Proof** For the simplicity, we put \(n = 1\).

By (iii) in Proposition 5, we have
\[
\Lambda(iv) = \int_{0}^{\infty} e^{-s(1-iv)} \tilde{F}(s) ds = \int_{0}^{\infty} e^{isv} e^{-s} \tilde{F}(s) ds, \quad (v \in \mathbb{R}).
\]

This means that \(\Lambda(iv)\) is the inverse Fourier transform of integrable function \(e^{-s}\tilde{F}(s)\). Since \(\Lambda(iv)\) is continuous bounded function, \(\Lambda(iv)\) is tempered distribution. Hence as tempered distribution we have
\[
\tilde{F}(s) = e^{s} F(\Lambda(iv))(s).
\]

**Example([24])** \(F(w, \bar{w}) = e^{\frac{a-1}{2a}|w|^2} \quad (0 < a < 1)\).

\[
\lambda_m = a^{m+1}, \quad \lambda(z) = a^{z+1}, \quad \Lambda(w) = \frac{a}{1-aw},
\]

### 7.3 Conclusion

1. Daubechies operator in Bargmann - Fock space \(B \circ P_F \circ B^{-1}\) is anti-Wick(Toeplitz) operator.

2. Applying the results of the eigenvalue problem of anti-Wick(Toeplitz) operator in Bargmann - Fock space, we can derive Daubechies' results more easily.

3. For anti-Wick operator \(T_F\) with polyradial symbols, we can reconstruct polyradial symbol function \(F(w, \bar{w})\) from eigenvalues of \(T_F\).

**Remark** For the details of our study, we refer the reader to [21], [22], [23], [24], [25], [26], [27], [28].
References


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