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Eigenvalue Problem of Anti-Wick (Toeplitz) Operators in Bargmann-Fock Space and Applications to Daubechies Operators

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Abstract: In this paper we will consider algebraic background of Gabor analysis and eigenvalue problem of anti-Wick (Toeplitz) operators in Bargmann-Fock space. We will clarify the relationship between anti-Wick (Toeplitz) operators and Daubechie (localization) operators. We apply our results to eigenvalue problem of Daubechie operators.

1 Gabor transform

In this section we will recall the definition and properties of Gabor transform ([5], [6]). Gabor transform $W_\phi(f)(p, q)$ is defined as follows:

$$W_\phi(f)(p, q) = \int_{\mathbb{R}^n} \overline{\phi_{p,q}(x)} f(x) dx, \quad (f(x) \in L^2(\mathbb{R}^n), x, p, q \in \mathbb{R}^n)$$

$\phi(x) = \pi^{-n/4}e^{-x^2/2}$ is Gaussian and $\phi_{p,q}(x) = \pi^{-n/4}e^{ipx}e^{-(x-q)^2/2}$ is Gabor function. We have following inversion formula (resolution of identity)

Proposition 1 (Inversion formula of Gabor transform)

$$f(x) = (\frac{1}{2\pi})^n \int_{\mathbb{R}^{2n}} \phi_{p,q}(x) W_\phi(f)(p, q) dpdq$$

(Proof)

$$= \int_{\mathbb{R}^{2n}} \phi_{p,q}(x) \int_{\mathbb{R}^n} e^{ipy} \phi(y-q) f(y) dy dpdq$$

$$= \int_{\mathbb{R}^{3n}} e^{-ipx} \phi(x-q)e^{ipy} \phi(y-q) f(y) dy dpdq$$

$$= \int_{\mathbb{R}^{2n}} \left\{ \int_{\mathbb{R}^n} e^{-ip(x-y)} dp \right\} \phi(x-q) \phi(y-q) f(y) dy dq$$

$$= (2\pi)^n \int_{\mathbb{R}^{2n}} \delta(x-y) \phi(x-q) \phi(y-q) f(y) dy dq$$

$$= (2\pi)^n \int_{\mathbb{R}^n} \phi(x-q) \phi(x-q) f(x) dq = (2\pi)^n < \phi, \phi > f(x) = (2\pi)^n f(x)$$

Proposition 2 (Unitarity of Gabor Transform)
1.1 The relationship between FBI transform, Bargmann transform and Gabor transform

Gabor transform is closely related to FBI (Fourier - Bros - Iagolnitzer) transform and Bargmann transform ([7]). FBI transform \( P^t(f)(p, q) \) is defined by

\[
P^t(f)(p, q) = \int_{\mathbb{R}^n} e^{-ipx} e^{-t(x-q)^2} f(x) dx
\]

1. FBI transform is related to Gabor transform as follows:

\[
P^{1/2}(f)(p, q) = \int_{\mathbb{R}^n} e^{-ipx} e^{-(x-q)^2/2} f(x) dx
\]

2. Bargmann transform is related to Gabor transform as follows:

\[
B(f)(z) = \pi^{-n/4} e^{1/4(p^2+q^2+2ipq)} \int_{\mathbb{R}^n} e^{-ipx} e^{-(x-q)^2/2} f(x) dx,
\]

\( z = \frac{q+ip}{\sqrt{2}}, p, q \in \mathbb{R}^n \).

Remark.

1. Gabor transform is used for iris identification and signal analysis of human voice. It is also used for the definition of Feichtinger(Segal) algebra and modulation space([9]).
2. Recently the relationship between Gabor analysis and operator algebra is studied by several mathematicians([8], [13], [14], [15], [17]).

2 Projective representation of time frequency plane(phase space)

In Gabor analysis the function \( e^{ipx}g(x-q) \) frequently appears. We already saw this type of function(Gabor function) in the Gabor transform. Another example is Zak transform: \( Z(g)(s, t) = \sum_{n \in \mathbb{Z}} e^{int} g(s-n) \).

And here is celebrated Balian - Low Theorem.

Balian - Low Theorem([6]). If \( \{e^{i2\pi mx}g(x-n)\}_{n,m \in \mathbb{Z}} \) is a Frame, then

\[
\int_{\mathbb{R}^n} x^2|g(x)|^2 dx = \infty \quad \text{or} \quad \int_{\mathbb{R}^n} |\xi|^2|\hat{g}(\xi)|^2 d\xi = \infty.
\]
2.1 Modulation operator and translation operator

In this section we will consider the meaning of the function $e^{ipx}g(x - q)$. For $g(x) \in L^2(\mathbb{R}^n)$, we define modulation operator $M_p g(x) = e^{ipx}g(x)$ and translation operator $T_q g(x) = g(x - q)$. Both are unitary operators and satisfy $M_a M_b = M_{a+b}$ and $T_a T_b = T_{a+b}$. Namely $M_p$ and $T_q$ are unitary representations of additive group $\mathbb{R}^n$.

We have the following commutative diagram:

\[
\begin{array}{c}
L^2(\mathbb{R}^n) \xrightarrow{F} L^2(\mathbb{R}^n) \\
\downarrow T_q \quad \quad \downarrow M_a \\
L^2(\mathbb{R}^n) \xrightarrow{F} L^2(\mathbb{R}^n)
\end{array}
\]

$F$ is the Fourier transform (intertwining operator).

$M_p$ and $T_q$ satisfy $M_p T_q = e^{-ipq}T_q M_p$.

2.2 An interpretation of $e^{ipx}g(x - q)$ by projective representation of time frequency plane

For $g(x) \in L^2(\mathbb{R}^n)$, we put

\[\pi(p, q)g(x) = M_p T_q g(x) = e^{ipx}g(x - q), \quad (p, q) \in \mathbb{R}^n \times \mathbb{R}^n.\]

$\pi(p, q)$ satisfies $\pi(p_1, q_1)\pi(p_2, q_2) = e^{-ip_2q_1}\pi(p_1 + p_2, q_1 + q_2)$. Although $\pi(p, q)$ is unitary operator, it is not unitary representation because of factor $e^{-ip_2q_1}$. So $\pi(p, q)$ is called projective representation (ray representation, Weyl - Heisenberg operator) of $\mathbb{R}^n \times \mathbb{R}^n$. To make projective representation $\pi(p, q)$ to unitary representation, we will introduce Heisenberg group.

2.3 Heisenberg Group

We identify phase space (time frequency plane for $n = 1$) $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$. Remark that $\mathbb{C}^n$ has symplectic structure. i.e. $\mathbb{C}^n$ is symplectic vector space. We have the following exact sequence.

\[
0 \rightarrow \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{C}^n \rightarrow \mathbb{C}^n \rightarrow 0
\]

$\mathbb{R} \times \mathbb{C}^n = H_n$ is called the Heisenberg group (polarized).

We put $\pi(t, p, q)g(x) = e^{it}e^{ipx}g(x - q), \quad (g \in L^2(\mathbb{R}^n), p, q \in \mathbb{R}^n, t \in \mathbb{R})$ 

$\pi(t, p, q)$ is unitary representation (Schrödinger representation) of the Heisenberg group and $\pi(0, p, q) = \pi(p, q)$. 

**Example**  
$H_1$ is realized as the group of matrix.

\[ H_1 \ni (t,p,q) \rightarrow \begin{pmatrix} 1 & p & t \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}, \quad H_1 \cong \left\{ \begin{pmatrix} 1 & p & t \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} : t,p,q, \in \mathbb{R} \right\} \]

**Remark**

1. For the details of Heisenberg group, we refer the reader to [7], [10], [12], [16] and [18].
2. Projective representation (ray representation) of continuous group is studied by V. Bargmann ([1]).
3. To construct irreducible unitary representation of the Heisenberg group, we use $L^2(\mathbb{R}^n)$ (Schrödinger representation) or Bargmann - Fock space $BF(\mathbb{C}^n)$ (Fock representation).

\[
\begin{align*}
L^2(\mathbb{R}^n) & \xrightarrow{B} BF(\mathbb{C}^n) \\
\pi(t,p,q) & \downarrow \downarrow B_{\pi(t,p,q)}B^{-1} \\
L^2(\mathbb{R}^n) & \xrightarrow{B} BF(\mathbb{C}^n)
\end{align*}
\]

$B$ is the Bargmann transform (intertwining operator).

### 3 Bargmann transform and Bargmann - Fock space

#### 3.1 Bargmann transform

We recall the definition of Bargmann transform and its properties([2]). We put $A_n(z, x)$ as follows:

\[
A_n(z, x) = \pi^{-n/4} \exp \left\{ -\frac{1}{2} (z^2 + x^2) + \sqrt{2}z \cdot x \right\}, \quad (z \in \mathbb{C}^n, x \in \mathbb{R}^n).
\]

The Bargmann transform $B(\psi)$ is defined as follows:

\[
B(f)(z) \overset{\text{def}}{=} \int_{\mathbb{R}^n} f(x)A_n(z, x)dx, \quad (f(x) \in L^2(\mathbb{R}^n)).
\]

**Example**([2])  
Let $h_m(x)$ be Hermite function of degree $m$. Then

\[
B(h_m)(z) = \frac{z^m}{\sqrt{m!}}, \quad (m \in \mathbb{N})
\]
3.2 Bargmann - Fock space $BF(\mathbb{C}^n)$

We put

$$BF(\mathbb{C}^n) = \{ g \in H(\mathbb{C}^n) : \int_{\mathbb{C}^n} |g(z)|^2 e^{-|z|^2} \, dz \wedge d\overline{z} < \infty \}. $$

$H(\mathbb{C}^n)$ denotes the space of entire functions.

**Example** ([28])

1. Polynomials and entire functions of exponential type belong to Bargmann - Fock space. For example, sinc function $\frac{\sin z}{z}$ and prolate spheroidal function (eigenfunction of $(\tau^2 - t^2) \frac{d}{dt} - 2t \frac{d}{dt} - \sigma^2 t^2$) are entire functions of exponential type ([20]). Hence they belong to Bargmann - Fock space.

2. $\sigma(z) = z \prod \left(1 - \frac{z}{\lambda_{m,n}}\right) \exp \left(\frac{z}{\lambda_{m,n}} + \frac{z^2}{2\lambda_{m,n}^2}\right)$, is Weierstrass $\sigma$ - function and $\lambda_{m,n}$ are lattice points in $\mathbb{C}$. Under suitable conditions on lattice points, $\frac{\sigma(z)}{z}$ belongs to Bargmann - Fock space ([10]).

**Theorem 1** ([2])

Bargmann transform is a unitary mapping from $L^2(\mathbb{R}^n)$ to $BF(\mathbb{C}^n)$.

The inverse Bargmann transform $B^{-1}$ is given by

$$B^{-1}(g)(x) \overset{def}{=} \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n} g(z) A_n(z, x) e^{-|z|^2} \, dz \wedge d\overline{z}, \quad (g \in BF(\mathbb{C}^n)). $$

Inner product in $BF(\mathbb{C}^n)$ is defined by following formula:

$$< f, g >_{BF} = \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n} \overline{f(z)} g(z) e^{-|z|^2} \, d\overline{z} \wedge dz$$

$BF(\mathbb{C}^n)$ is Hilbert space with this inner product.
3.3 Projection, Bergman Kernel and Reproducing Formula

Since $BF(\mathbb{C}^n)$ is a closed subspace of
\[ L^2(\mathbb{C}^n, e^{-|z|^2}) = \{g(z) : \int_{\mathbb{C}^n} |g(z)|^2 e^{-|z|^2} d\bar{z} \wedge dz < \infty\}, \]
we have the following orthogonal decomposition:
\[ L^2(\mathbb{C}^n, e^{-|z|^2}) = BF(\mathbb{C}^n) \oplus BF(\mathbb{C}^n)^\perp \]

**Proposition 3([29])**

Projection $P : L^2(\mathbb{C}^n, e^{-|z|^2}) \rightarrow BF(\mathbb{C}^n)$ is the following integral operator:
\[ (Pg)(z) = \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n} e^{z\bar{w}} g(w) e^{-|w|^2} d\bar{w} \wedge dw, \quad (g \in L^2(\mathbb{C}^n, e^{-|z|^2})). \]

**Proposition 4** Following statements are equivalent:
1. $g(z) \in BF(\mathbb{C}^n)$
2. $P(g)(z) = g(z)$
3. (Reproducing formula)
\[ g(z) = \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n} e^{z\bar{w}} g(w) e^{-|w|^2} d\bar{w} \wedge dw \]

**Remark**

$e^{z\bar{w}}$ is Bergman (reproducing) kernel with respect to Gaussian measure $(2\pi i)^{-n} e^{-|w|^2} d\bar{w} \wedge dw$.

4 Anti-Wick(Toeplitz) Operator

4.1 Toeplitz operator

In this subsection we will recall the definition of Toeplitz operators. For a region $D$ in $\mathbb{R}^n$ (or $\mathbb{C}^n$), we put $L^2(D : d\mu) = \{f(z) : \int_D |f(z)|^2 d\mu(z) < \infty\}$.

Suppose that $H$ is a closed subspace of $L^2(D : d\mu)$ and $P_H : L^2(D : d\mu) \rightarrow H$ is projection operator. If $h(z)$ is a bounded function in $\mathbb{R}^n$ (or $\mathbb{C}^n$), then we can define multiplication operator $m_h(f)(z) = h(z)f(z)$.

We put $T = P_H \circ m_h$, i.e. $T(f)(z) = P_H(h(z)f(z))$.

$T : L^2(D : d\mu) \xrightarrow{m_h} L^2(D : d\mu) \xrightarrow{P_H} H$,

$T$ is called Toeplitz operator.
4.2 Toeplitz operator on Bargmann - Fock space

Since Toeplitz operator $T_F$ with symbol $F$ is a composition of multiplication operator and projection operator, we have

$$(T_Ff)(z) = \int_{\mathbb{C}^n} e^{\bar{w}z} F(w, \bar{w}) f(w) d\mu(w), \quad (\forall f \in L^2(\mathbb{C}^n, d\mu)),$$

where $F(w, \bar{w})$ is a bounded function on $\mathbb{C}^n$ and $d\mu(w) = (2\pi i)^{-n} e^{-|w|^2} d\bar{w} \wedge dw$.

**Remark** For the recent development of the theory of Toeplitz operators on Bargmann - Fock space, we refer the reader to [3], [4], [11], [19], [28] and [29].

4.3 Wick Operator and Anti - Wick Operator

According to ([7]), we will recall the definition of Wick Operator and anti-Wick Operator. For $f \in BF(\mathbb{C}^n)$, we define Wick operator $T_F^W$ as follows:

$$T_F^W f(z) = \sum a_{\alpha, \beta} z^\alpha \frac{d^\beta}{dz^\beta} f(z).$$

If we employ reproducing formula (3 in Proposition 4), then we obtain following integral representation of Wick operator $T_F^W$:

$$T_F^W f(z) = \int_{\mathbb{C}^n} e^{\bar{w}z} F(z, \bar{w}) f(w) d\mu(w).$$

$F(z, \bar{w})$ is an entire function of $(z, \bar{w})$ with some estimate.

We define anti-Wick operator as follows:

$$T_F^{AW} f(z) = \sum a_{\alpha, \beta} z^\alpha \frac{d^\beta}{dz^\beta} f(z).$$

If we employ reproducing formula (3 in Proposition 4), then we obtain following integral representation of anti-Wick operator $T_F^{AW}$:

$$T_F^{AW} f(z) = \int_{\mathbb{C}^n} e^{\bar{w}z} F(w, \bar{w}) f(w) d\mu(w).$$

$F(w, \bar{w})$ is measurable function with some estimate.

**Remark**

If $F(w, \bar{w})$ is bounded function, then $T_F^{AW}$ is Toeplitz operator.

**Example**

1. If we consider harmonic oscillator operator in Bargmann - Fock space, then it is Wick operator.
If \( T = -\frac{d^2}{dx^2} + x^2 - 1 : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \), then
\[
(B \circ T \circ B^{-1})f(z) = z \frac{d}{dz} f(z) : BF(\mathbb{C}) \rightarrow BF(\mathbb{C})([2]).
\]
\[
z \frac{d}{dz} f(z) = z \frac{d}{dz} \int e^{zw} f(w) d\mu(w) = \int z\overline{w} e^{zw} f(w) d\mu(w),
\]
So we have \( F(z, \overline{w}) = z\overline{w} \).

2. \( \frac{d}{dz}z : BF(\mathbb{C}) \rightarrow BF(\mathbb{C}) \) is anti-Wick operator.
\[
\frac{d}{dz}zf(z) = \frac{d}{dz} \int e^{zw} w f(w) d\mu(w) = \int \overline{w}\overline{w} e^{zw} f(w) d\mu(w),
\]
Hence we have \( F(w, \overline{w}) = \overline{w}\overline{w} = |w|^2 \).

### 4.4 Eigenvalue problem of Anti-Wick(Toeplitz) Operator on Bargmann-Fock Space

In this subsection we will consider the eigenvalue problem of anti-Wick(Toeplitz) operator \( T_F(f)(z) = \int_{\mathbb{C}^n} e^{z\overline{w}} F(w, \overline{w}) f(w) d\mu(w) \).

**Theorem 2([28]):** Suppose that \( F(w, \overline{w}) \) is bounded integrable and polyradial function. i.e. \( F(w, \overline{w}) = F(|w_1|^2, \cdots, |w_n|^2) \). Then

1. \( z^m \) is eigenfunction of \( T_F \).

2. Eigenvalue \( \lambda_m \) of \( T_F \) is given by
\[
\lambda_m = \frac{1}{m!} \int_0^\infty \cdots \int_0^\infty \tilde{F}(|s_1|^2, \cdots, |s_n|^2) \prod_{i=1}^n e^{-s_i} s_i^{m_i} ds_i, \ m = (m_1, \cdots, m_n) \in \mathbb{N}^n.
\]

**(Proof)** For brevity’s sake, we put \( n = 1 \).
\[
(T_F)(w^m)(z) = \int_{\mathbb{C}} \tilde{F}(|w|^2) e^{z\overline{w}} w^m d\mu(w) = \frac{1}{\pi} \int_{\mathbb{C}} \tilde{F}(|w|^2) e^{z\overline{w}} w^m e^{-|w|^2} d\mu(w)
\]
\[
= \frac{1}{\pi} \int_{\mathbb{C}} \tilde{F}(|w|^2) \left( \sum_{n=0}^\infty \frac{(z\overline{w})^n}{n!} \right) w^m e^{-|w|^2} d\mu(w)
\]
\[
= \sum_{n=0}^\infty \frac{z^n}{n!} \frac{1}{\pi} \int_{\mathbb{C}} \tilde{F}(|w|^2) \overline{w}^n w^m e^{-|w|^2} d\mu(w).
\]
By using the polar coordinate \( w = re^{i\theta} \),
\[
= \frac{1}{\pi} \sum_{n=0}^\infty \frac{z^n}{n!} \int_0^{2\pi} \int_0^\infty \tilde{F}(r^2) e^{i(m-n)\theta} r^n r^m e^{-r^2} r dr d\theta
\]
\[
= z^m \frac{1}{m!} \int_0^\infty e^{-r^2} \tilde{F}(r^2) r^{2m} 2r dr = z^m \frac{1}{m!} \int_0^\infty e^{-s} s^m \tilde{F}(s) ds.
\]

Hence we obtain

\[
(T_F)(w^m)(z) = z^m \frac{1}{m!} \int_0^\infty e^{-s} s^m \tilde{F}(s) ds.
\]

Example ([24], [28])

1. \(F(w, \overline{w}) = \exp\left(\frac{a-1}{a}|w|^2\right), \quad (0 < a < 1)\)
   \[
   \tilde{F}(s) = \exp\left(\frac{a-1}{a}s\right), \quad \lambda_m = a^{m+1}
   \]

5 Daubechies (Localization) Operator

5.1 Daubechies (Localization) Operator

Daubechies operator was introduced by Ingrid Daubechies in ([5], [6]). Daubechies operator \(P_F\) is defined as follows:

\[
P_F(f)(x) = (2\pi)^{-n} \int \int_{\mathbb{R}^{2n}} F(p, q) \phi_{p,q}(x) W_{\phi}(f)(p, q) dp dq,
\]

\(f(x) \in L^2(\mathbb{R}^n)\). \(\phi_{p,q}(x) = \pi^{-n/4} e^{-ipx} e^{-(x-q)^2/2}\).

\(W_{\phi}(f)(p, q) = \int_{\mathbb{R}^n} \overline{\phi_{p,q}(y)} f(y) dy\) is Gabor transform of \(f(x)\) and \(F(p, q)\) is symbol function of \(P_F\).

**Remark** If \(F(p, q)\) is 1, then \(P_F\) is identity operator. i.e. We have

Resolution of identity (Inversion formula of Gabor transform)

\[
f(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} \phi_{p,q}(x) W_{\phi}(f)(p, q) dp dq
\]

5.2 Daubechies Operator in Bargmann - Fock space

If we consider Daubechies operator in Bargmann - Fock space, then we have following theorem ([28]).

**Theorem 3** For \(g(z) \in BF(\mathbb{C}^n)\), we have

\[
(B \circ P_F \circ B^{-1})(g)(z) = (2\pi i)^{-n} \int \int_{\mathbb{C}^n} F(w, \overline{w}) e^{z\overline{w}} g(w) e^{-|w|^2} d\overline{w} \wedge dw.
\]

Especially if \(F(w, \overline{w}) = 1\), then we obtain
**Corollary** (Relationship between resolution of identity and reproducing formula)

\[ f(x) = \left( \frac{1}{2\pi} \right)^n \int_{\mathbb{R}^{2n}} \phi_{p,q}(x) W_{\phi}(f)(p, q) dpdq, \quad f(x) \in L^2(\mathbb{R}^n) \]

is equivalent to

\[ g(z) = \int_{\mathbb{C}^n} e^{z \overline{w}} g(w) d\mu(w), \quad (\forall g(z) \in BF). \]

\[ L^2(\mathbb{R}^n) \xrightarrow{B} BF \]

\[ P_F \downarrow \quad \downarrow B \circ P_F \circ B^{-1} \]

\[ L^2(\mathbb{R}^n) \xrightarrow{B} BF \]

### 6 Application to Daubechies Localization Operator

#### 6.1 Hermite Functions

Hermite functions \( h_m(x) \) of one variable is defined by

\[ h_m(x) = (-1)^m (2^m m! \sqrt{\pi})^{-1/2} \exp(x^2/2) \frac{d^m}{dx^m} \exp(-x^2). \]

Generating function of Hermite functions is the kernel function of Bargmann transform.

\[ \pi^{-1/4} \exp \left\{-\frac{1}{2} (z^2 + x^2) + \sqrt{2}z \cdot x \right\} = \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} h_m(x), \quad (z \in \mathbb{C}^1, x \in \mathbb{R}^1). \]

We also have the following expression.

\[ h_m(x) = \frac{1}{\sqrt{2^m m!}} \left( \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} \right) \right)^m h_0(x). \]

Hermite functions \( h_m(x) \) of several variables is defined by

\[ h_m(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} h_{m_i}(x_i), \quad m = (m_1, \ldots m_n) \in N^m. \]

**Example**

1. \( h_0(x) = \pi^{-1/4} \exp(-x^2/2), \quad \) (coherent state)
2. \( h_2(x) = \pi^{-1/4} \frac{2x^2 - 1}{\sqrt{2}} \exp(-x^2/2), \quad \) (Mexican hat wavelet)
6.2 Daubechies’ result

As an application of our result, we will give a new proof of following Daubechies’ result.

**Theorem 4([5])** Suppose that $F(p, q)$ is integrable polyradial function. Then we have

1. $P_F(h_m)(x) = \lambda_m h_m(x)$

2. $\lambda_m = \frac{1}{m!} \int_0^\infty \cdots \int_0^\infty \tilde{F}(s_1, \cdots, s_n) \prod_{i=1}^n e^{-s_i} s_i^{m_i} ds_i,$

$m = (m_1, \cdots, m_n) \in \mathbb{N}^n$.

**(Proof)** For the simplicity we put $n=1$. Let $P_F$ be Daubechies operaotr with integrable polyradial symbol $F$. Then $T_F = B \circ P_F \circ B^{-1}$ is Toeplitz operator with integrable polyradial symbol $F$. So we can apply Theorem 2 to $T_F$. Hence we have

$\lambda_m = \frac{1}{m!} \int_0^\infty \tilde{F}(s) e^{-s} s^m ds,$

$T_F(\frac{z^m}{\sqrt{m!}}) = \lambda_m \frac{z^m}{\sqrt{m!}}.$

By inverse Bargmann transform,

$h_m(x) = B^{-1}(\frac{z^m}{\sqrt{m!}})(x).$

So we obtained following Daubechies’ results.

$P_F(h_m)(x) = \lambda_m h_m(x), \quad \lambda_m = \frac{1}{m!} \int_0^\infty e^{-s} s^m \tilde{F}(s) ds.$

7 Reconstruction of symbol function from eigenvalues

7.1 The first reconstruction formula

We consider the analytic continuation of eigenvalues $\lambda_m$ of $T_F$. It is given by

$\lambda(z) = \frac{1}{\Gamma(z + 1)} \int_0^\infty e^{-s} s^z \tilde{F}(s) ds,$

where $\Gamma(z)$ is Euler Gamma function. We have $\lambda(m) = \lambda_m$ by Theorem 2.

**Theorem 5([21])**

$\tilde{F}(s) = \frac{e^s}{s} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \lambda(z) \Gamma(z + 1) s^{-z} dz.$
(Proof) Integral representation \( \lambda(z) = \frac{1}{\Gamma(z+1)} \int_0^\infty e^{-s}s\tilde{F}(s)s^{z-1}ds, \)
means that \( \lambda(z)\Gamma(z+1) \) is Mellin transform of \( e^{-s}\tilde{F}(s) \).
Hence we obtain above formula by inverse Mellin transform.

### 7.2 The second reconstruction formula

For eigenvalues \( \{\lambda_m\} \) of anti-Wick(Toeplitz) operator \( T_F \), we put
\[
\Lambda(w) = \sum_{m=0}^\infty \lambda_m w^m.
\]
\( \Lambda(w) \) is generating function (of eigenvalues) of anti-Wick(Toeplitz) operator \( T_F \). In signal analysis \( \Lambda(w) \) is called \( z \)-transform instead of generating function. In what follows we assume that \( F(p, q) \) is integrable and polyradial function.

**Proposition 5** ([23]) Suppose that \( \lambda_m \) are eigenvalues of \( T_F \). Then we have
(i) \( \exists C > 0 \) s.t. \( |\lambda_m| \leq \frac{C}{\sqrt{|m|}}, \quad (m \in \mathbb{N}^n) \).
(ii) \( \Lambda(w) \) is holomorphic in \( \prod_{i=1}^n \{w \in \mathbb{C}^n : |w_i| < 1\} \).
(iii) \( \Lambda(w) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^n e^{-s_i(1-w_i)}\tilde{F}(s_1, \ldots, s_n)ds_1...ds_n. \)
(iv) \( \Lambda(w) \) is holomorphic in \( \prod_{i=1}^n \{w \in \mathbb{C}^n : \Re(w_i) < 1\} \) and bounded in its closure.
(v) \( \Lambda(iv) \in C_0(\mathbb{R}^n), (v \in \mathbb{R}^n) \). i.e. \( \Lambda(iv) \in C(\mathbb{R}^n) \) and \( \lim_{|v| \rightarrow \infty} \Lambda(iv) = 0 \).

(Proof) Without loss of genelarity, we can assume that \( n = 1 \).

(i) By Theorem 2, \( \lambda_m = \frac{1}{m!} \int_0^\infty e^{-s}\tilde{F}(s)s^m ds. \)
Since \( e^{-s} s^m \leq e^{-m} m^m \), we have
\[
|\lambda_m| \leq \frac{1}{m!} e^{-m} m^m \int_0^\infty |\tilde{F}(s)|ds.
\]
By Stirling's formula \( m! \sim \sqrt{2\pi m} e^{-m} m^m \), for sufficiently large \( m \),
\[
|\lambda_m| \leq C \frac{1}{\sqrt{m}} \text{valids.}
\]

(iii) \( \Lambda(w) = \sum_{m=0}^\infty \lambda_m w^m = \sum_{m=0}^\infty \frac{w^m}{m!} \int_0^\infty e^{-s}s^m \tilde{F}(s)ds = \)
\[
\int_{0}^{\infty} e^{-s} \tilde{F}(s) \sum_{m=0}^{\infty} \frac{(ws)^{m}}{m!} ds = \int_{0}^{\infty} e^{-s(1-w)} \tilde{F}(s) ds.
\]

(iv) For \( \text{Re}(w) \leq 1 \), we have
\[
|\Lambda(w)| \leq \int_{0}^{\infty} |e^{-s(1-w)}| |\tilde{F}(s)| ds \leq ||\tilde{F}||_{L^{1}}.
\]

(v) Since \( \Lambda(iv) \) is Fourier transform of \( L^{1} \) function \( e^{-s} \tilde{F}(s) \), it is in \( C_{0}(\mathbb{R}^{n}) \) by Riemann - Lebesgue theorem.

**Theorem 6** ([21])

\[
\tilde{F}(s) = (2 \pi)^{-1} e^{s} \int_{-\infty}^{+\infty} e^{-isv} \Lambda(iv) dv,
\]
valids in distribution sense.

**Proof** For the simplicity, we put \( n = 1 \).

By (iii) in Proposition 5, we have
\[
\Lambda(iv) = \int_{0}^{\infty} e^{-s(1-iv)} \tilde{F}(s) ds = \int_{0}^{\infty} e^{isv} e^{-s} \tilde{F}(s) ds, \quad (v \in \mathbb{R}).
\]

This means that \( \Lambda(iv) \) is the inverse Fourier transform of integrable function \( e^{-s} \tilde{F}(s) \). Since \( \Lambda(iv) \) is continuous bounded function, \( \Lambda(iv) \) is tempered distribution. Hence as tempered distribution we have
\[
\tilde{F}(s) = e^{s} F(\Lambda(iv))(s).
\]

**Example** ([24]) \( F(w, \overline{w}) = e^{\frac{a-1}{2a} |w|^2} \) \( (0 < a < 1) \).

\[
\lambda_{m} = a^{m+1}, \quad \lambda(z) = a^{z+1}, \quad \Lambda(w) = \frac{a}{1-aw},
\]

### 7.3 Conclusion

1. Daubechies operator in Bargmann - Fock space \( B \circ P_{F} \circ B^{-1} \) is anti-Wick(Toeplitz) operator.

2. Applying the results of the eigenvalue problem of anti-Wick(Toeplitz) operator in Bargmann - Fock space, we can derive Daubechies' results more easily.

3. For anti-Wick operator \( T_{F} \) with polyradial symbols, we can reconstruct polyradial symbol function \( F(w, \overline{w}) \) from eigenvalues of \( T_{F} \).

**Remark** For the details of our study, we refer the reader to [21], [22], [23], [24], [25], [26], [27], [28].
References


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