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Eigenvalue Problem of Anti - Wick (Toeplitz) Operators in Bargmann - Fock Space and Applications to Daubechies Operators

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Abstract: In this paper we will consider algebraic background of Gabor analysis and eingenvalue problem of anti - Wick (Toeplitz) operators in Bargmann - Fock space. We will clarify the relationship between anti - Wick (Toeplitz) operators and Daubechie (localization) operators. We apply our results to eingenvalue problem of Daubechie operators.

1 Gabor transform

In this section we will recall the definition and properties of Gabor transform([5], [6]). Gabor transform $W_{\phi}(f)(p,q)$ is defined as follows:

$$W_{\phi}(f)(p,q)=\int_{\mathbb{R}^{n}}\overline{\phi_{p,q}(x)}f(x)dx, \quad (f(x)\in L^{2}(\mathbb{R}^{n}), x,p,q\in \mathbb{R}^{n})$$

$\phi(x)=\pi^{-n/4}e^{-x^{2}/2}$ is Gaussian and $\phi_{p,q}(x)=\pi^{-n/4}e^{ipx}e^{-(x-q)^{2}/2}$ is Gabor function. We have following inversion formula(resolution of identity)

Proposition 1(Inversion formula of Gabor transform)

$$f(x)=\left(\frac{1}{2\pi}\right)^{n}\int_{\mathbb{R}^{2n}}\phi_{p,q}(x)W_{\phi}(f)(p,q)dpdq$$

(Proof) $\int_{\mathbb{R}^{2n}}\phi_{p,q}(x)W_{\phi}(f)(p,q)dpdq$

$$=\int_{\mathbb{R}^{2n}}\phi_{p,q}(x)\int_{\mathbb{R}^{n}}e^{ipy}\phi(y-q)f(y)dydpdq$$

$$=\int_{\mathbb{R}^{3n}}e^{-ipx}\phi(x-q)e^{ipy}\phi(y-q)f(y)dydpdq$$

$$=\int_{\mathbb{R}^{2n}}\left\{\int_{\mathbb{R}^{n}}e^{-ip(x-y)}dp\right\}\phi(x-q)\phi(y-q)f(y)dydq$$

$$=(2\pi)^{n}\int_{\mathbb{R}^{2n}}\delta(x-y)\phi(x-q)\phi(y-q)f(y)dydq$$

$$=(2\pi)^{n}\int_{\mathbb{R}^{n}}\phi(x-q)\phi(x-q)f(x)dq=(2\pi)^{n}<\phi,\phi>f(x)=(2\pi)^{n}f(x)$$

Proposition 2(Unitarity of Gabor Transform)

$< W_\phi(f), W_\phi(g) >= (2\pi)^{-n} < f, g >$

1.1 The relationship between FBI transform, Bargmann transform and Gabor transform

Gabor transform is closely related to FBI (Fourier - Bros - Iagolnitzer) transform and Bargmann transform ([7]).
FBI transform $P^t(f)(p,q)$ is defined by

$$P^t(f)(p,q) = \int_{\mathbb{R}^n} e^{-ipx} e^{-t(x-q)^2} f(x)dx$$

1. FBI transform is related to Gabor transform as follows :

$$P^{1/2}(f)(p,q) = \int_{\mathbb{R}^n} e^{-ipx} e^{-(x-q)^2/2} f(x)dx$$

2. Bargmann transform is related to Gabor transform as follows:

$$B(f)(z) = \pi^{-n/4} e^{1/4(p^2+q^2+2ipq)} \int_{\mathbb{R}^n} e^{-ipx} e^{-(x-q)^2/2} f(x)dx,$$

$$(z = \frac{q+ip}{\sqrt{2}}, p, q \in \mathbb{R}^n).$$

Remark.

1. Gabor transform is used for iris identification and signal analysis of human voice. It is also used for the definition of Feichtinger(Segal) algebra and modulation space([9]).
3. Recently the relationship between Gabor analysis and operator algebra is studied by several mathematicians([8], [13], [14], [15], [17]).

2 Projective representation of time frequency plane(phase space)

In Gabor analysis the function $e^{ipx} g(x-q)$ frequently appears. We already saw this type of function(Gabor function) in the Gabor transform. Another example is Zak transform: $Z(g)(s,t) = \sum_{n\in\mathbb{Z}} e^{int} g(s-n)$.

And here is celebrated Balian - Low Theorem.
Balian - Low Theorem([6]). If $\{e^{i2\pi mx} g(x-n)\}_{n,m\in\mathbb{Z}}$ is a Frame, then

$$\int_{\mathbb{R}^n} x^2 |g(x)|^2 dx = \infty \text{ or } \int_{\mathbb{R}^n} \xi^2 |\hat{g}(\xi)|^2 d\xi = \infty.$$

2.1 Modulation operator and translation operator

In this section we will consider the meaning of the function $e^{ipx}g(x-q)$.
For $g(x) \in L^2(\mathbb{R}^n)$, we define modulation operator $M_p g(x) = e^{ipx}g(x)$ and translation operator $T_q g(x) = g(x-q)$.
Both are unitary operators and satisfy $M_a M_b = M_{a+b}$ and $T_a T_b = T_{a+b}$.
Namely M_p and T_q are unitary representations of additive group $\mathbb{R}^n$.
We have the following commutative diagram:

$$\begin{array}{ccc} L^2(\mathbb{R}^n) & \xrightarrow{F} & L^2(\mathbb{R}^n) \\ {\scriptstyle T_q}\downarrow & & \downarrow{\scriptstyle M_q} \\ L^2(\mathbb{R}^n) & \xrightarrow{F} & L^2(\mathbb{R}^n) \end{array}$$

F is the Fourier transform(intertwining operator).
M_p and T_q satisfy $M_p T_q = e^{-ipq} T_q M_p$.

2.2 An interpretation of $e^{ipx}g(x-q)$ by projective representation of time frequency plane

For $g(x) \in L^2(\mathbb{R}^n)$, we put
$\pi(p,q)g(x) = M_p T_q g(x) = e^{ipx}g(x-q), \quad (p,q) \in \mathbb{R}^n \times \mathbb{R}^n$.
$\pi(p,q)$ satisfies $\pi(p_1,q_1)\pi(p_2,q_2) = e^{-ip_2q_1}\pi(p_1+p_2, q_1+q_2)$. Although $\pi(p,q)$ is unitary operator, it is not unitary representation because of factor $e^{-ip_2q_1}$. So $\pi(p,q)$ is called projective representation(ray representation, Weyl - Heisenberg operator) of $\mathbb{R}^n \times \mathbb{R}^n$. To make projective representation $\pi(p,q)$ to unitary representation, we will introduce Heisenberg group.

2.3 Heisenberg Group

We identify phase space(time frequency plane for $n=1$) $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$.
Remark that $\mathbb{C}^n$ has symplectic structure. i.e. $\mathbb{C}^n$ is symplectic vector space. We have the following exact sequence.
$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{C}^n \longrightarrow \mathbb{C}^n \longrightarrow 0$
$\mathbb{R} \times \mathbb{C}^n = H_n$ is called the Heisenberg group(polarized).
We put $\pi(t,p,q)g(x) = e^{it}e^{ipx}g(x-q), \quad (g \in L^2(\mathbb{R}^n), p, q \in \mathbb{R}^n, t \in \mathbb{R})$
$\pi(t,p,q)$ is unitary representation (Schrödinger representation) of the Heisenberg group and $\pi(0,p,q) = \pi(p,q)$.

Example H_1 is realized as the group of matrix.

$$H_1 \ni (t,p,q) \longrightarrow \begin{pmatrix} 1 & p & t \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix}, \quad H_1 \cong \left\{ \begin{pmatrix} 1 & p & t \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} : t,p,q, \in \mathbb{R} \right\}$$

Remark

1. For the details of Heisenberg group, we refer the reader to [7], [10], [12], [16] and [18].
2. Projective representation (ray representation) of continuous group is studied by V. Bargmann ([1]).
3. To construct irreducible unitary representation of the Heisenberg group, we use $L^2(\mathbb{R}^n)$ (Schrödinger representation) or Bargmann - Fock space $BF(\mathbb{C}^n)$ (Fock representation).

$$\begin{array}{ccc} L^2(\mathbb{R}^n) & \xrightarrow{B} & BF(\mathbb{C}^n) \\ {\scriptstyle \pi(t,p,q)} \downarrow & & \downarrow {\scriptstyle B\circ\pi(t,p,q)\circ B^{-1}} \\ L^2(\mathbb{R}^n) & \xrightarrow{B} & BF(\mathbb{C}^n) \end{array}$$

B is the Bargmann transform(intertwining operator).

3 Bargmann transform and Bargmann - Fock space

3.1 Bargmann transform

We recall the definition of Bargmann transform and its properties([2]). We put $A_n(z,x)$ as follows :

$$A_n(z,x) = \pi^{-n/4} \exp\left\{ -\frac{1}{2}(z^2 + x^2) + \sqrt{2}z \cdot x \right\}, \quad (z \in \mathbb{C}^n, x \in \mathbb{R}^n).$$

The Bargmann transform $B(\psi)$ is defined as follows :

$$B(f)(z) \stackrel{def}{=} \int_{\mathbb{R}^n} f(x) A_n(z,x) dx, \quad (f(x) \in L^2(\mathbb{R}^n)).$$

Example([2]) Let $h_m(x)$ be Hermite function of degree m. Then $B(h_m)(z) = \dfrac{z^m}{\sqrt{m!}}, (m \in \mathbb{N})$

3.2 Bargmann - Fock space $BF(\mathbb{C}^n)$

We put

$$BF(\mathbb{C}^n) = \{g \in H(\mathbb{C}^n) : \int_{\mathbb{C}^n} |g(z)|^2 e^{-|z|^2} dz \wedge d\bar{z} < \infty\}.$$

$H(\mathbb{C}^n)$ denotes the space of entire functions.

Example([28])

1. Polynomials and entire functions of exponential type belong to Bargmann - Fock space. For example, sinc function $\frac{\sin z}{z}$ and prolate spheroidal function(eigenfunction of $(\tau^2 - t^2)\frac{d}{dt} - 2t\frac{d}{dt} - \sigma^2 t^2)$ are entire functions of exponential type([20]). Hence they belong to Bargmann - Fock space.

2. $\sigma(z) = z \prod \left(1 - \frac{z}{\lambda_{m,n}}\right) \exp\left(\frac{z}{\lambda_{m,n}} + \frac{z^2}{2\lambda^2{}_{m,n}}\right)$,
is Weierstrass σ - function and $\lambda_{m,n}$ are lattice points in $\mathbb{C}$.
Under suitable conditions on lattice points, $\frac{\sigma(z)}{z}$ belongs to Bargmann - Fock space([10]).

Theorem 1([2])

Bargmann transform is a unitary mapping from $L^2(\mathbb{R}^n)$ to $BF(\mathbb{C}^n)$.

The inverse Bargmann transform B^{-1} is given by

$$B^{-1}(g)(x) \overset{def}{=} \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n} g(z)\overline{A_n(z,x)} e^{-|z|^2} d\bar{z} \wedge dz, \quad (g \in BF(\mathbb{C}^n)).$$

Inner product in $BF(\mathbb{C}^n)$ is defined by following formula:

$$< f, g >_{BF} = \frac{1}{(2\pi i)^n} \int_{\mathbb{C}^n} \overline{f(z)} g(z) e^{-|z|^2} d\bar{z} \wedge dz$$

$BF(\mathbb{C}^n)$ is Hilbert space with this inner product.

3.3 Projection, Bergman Kernel and Reproducing Formula

Since $BF(\mathbb{C}^n)$ is a closed subspace of
$L^2(\mathbb{C}^n, e^{-|z|^2}) = \{g(z) : \int_{\mathbb{C}^n} |g(z)|^2 e^{-|z|^2} d\bar{z} \wedge dz < \infty\}$,
we have the following orthogonal decomposition:
$L^2(\mathbb{C}^n, e^{-|z|^2}) = BF(\mathbb{C}^n) \oplus BF(\mathbb{C}^n)^{\perp}$

Proposition 3([29])

Projection $P: \quad L^2(\mathbb{C}^n, e^{-|z|^2}) \longrightarrow BF(\mathbb{C}^n)$ is the following integral operator:

$$(Pg)(z) = \frac{1}{(2i\pi)^n} \int_{\mathbb{C}^n} e^{z\overline{w}} g(w) e^{-|w|^2} d\overline{w} \wedge dw, \ (g \in L^2(\mathbb{C}^n, e^{-|z|^2})).$$

Proposition 4 Following statements are equivalent:

1. $g(z) \in BF(\mathbb{C}^n)$
2. $P(g)(z) = g(z)$
3. (Reproducing formula)

$$g(z) = \frac{1}{(2i\pi)^n} \int_{\mathbb{C}^n} e^{z\overline{w}} g(w) e^{-|w|^2} d\overline{w} \wedge dw$$

Remark

$e^{z\overline{w}}$ is Bergman (reproducing) kernel with respect to Gaussian measure $(2\pi i)^{-n} e^{-|w|^2} d\overline{w} \wedge dw$.

4 Anti - Wick(Toeplitz) Operator

4.1 Toeplitz operator

In this subsection we will recall the definition of Toeplitz operators. For a region D in $\mathbb{R}^n$(or $\mathbb{C}^n$), we put $L^2(D : d\mu) = \{f(z) : \int_D |f(z)|^2 d\mu(z) < \infty\}$.
Suppose that H is a closed subspace of $L^2(D : d\mu)$ and
$P_H : L^2(D : d\mu) \longrightarrow H$ is projection operator. If $h(z)$ is a bounded function in $\mathbb{R}^n$(or $\mathbb{C}^n$), then we can define multiplication operator
$m_h(f)(z) = h(z)f(z)$.
We put $T = P_H \circ m_h$. i.e. $T(f)(z) = P_H(h(z)f(z))$.
$T : L^2(D : d\mu) \xrightarrow{m_h} L^2(D : d\mu) \xrightarrow{P_H} H$,
T is called Toeplitz operator.

4.2 Toeplitz operator on Bargmann - Fock space

Since Toeplitz operator T_F with symbol F is a composition of multiplication operator and projection operator, we have

$$(T_F f)(z) = \int_{\mathbb{C}^n} e^{z\overline{w}} F(w,\bar{w}) f(w) d\mu(w), \quad (\forall f \in L^2(\mathbb{C}^n, d\mu)),$$

where $F(w,\bar{w})$ is a bounded function on $\mathbb{C}^n$ and
$d\mu(w) = (2\pi i)^{-n} e^{-|w|^2} d\bar{w} \wedge dw.$

Remark For the recent development of the theory of Toeplitz operators on Bargmann - Fock space, we refer the reader to [3], [4], [11], [19], [28] and [29].

4.3 Wick Operator and Anti - Wick Operator

According to ([7]), we will recall the defintion of Wick Operator and anti - Wick Operator. For $f \in BF(\mathbb{C}^n)$, we define Wick operator T_F^W as follows:

$$T_F^W f(z) = \sum a_{\alpha,\beta} z^\alpha \frac{d^\beta}{dz^\beta} f(z).$$

If we employ reproducing formula(3 in Proposition 4), then we obtain following integral representation of Wick operator T_F^W:

$$T_F^W f(z) = \int_{\mathbb{C}^n} e^{z\overline{w}} F(z,\bar{w}) f(w) d\mu(w).$$

$F(z,\bar{w})$ is an entire function of $(z,\bar{w})$ with some estimate.

We define anti - Wick operator as follows:

$$T_F^{AW} f(z) = \sum a_{\alpha,\beta} \frac{d^\beta}{dz^\beta} z^\alpha f(z).$$

If we employ reproducing formula(3 in Proposition 4), then we obtain following integral representation of anti - Wick operator T_F^{AW}:

$$T_F^{AW} f(z) = \int_{\mathbb{C}^n} e^{z\overline{w}} F(w,\bar{w}) f(w) d\mu(w).$$

$F(w,\bar{w})$ is measurable function with some estimate.

Remark

If $F(w,\bar{w})$ is bounded function, then T_F^{AW} is Toeplitz operator.

Example

1. If we consider harmonic oscillator operator in Bargmann - Fock space, then it is Wick operator.

If $T = -\frac{d^2}{dx^2} + x^2 - 1 : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$, then

$(B \circ T \circ B^{-1})f(z) = z\frac{d}{dz}f(z) : BF(\mathbb{C}) \longrightarrow BF(\mathbb{C})([2])$.

$$z\frac{d}{dz}f(z) = z\frac{d}{dz}\int_{\mathbb{C}} e^{z\overline{w}}f(w)d\mu(w) = \int_{\mathbb{C}} z\overline{w}e^{z\overline{w}}f(w)d\mu(w),$$

So we have $F(z, \overline{w}) = z\overline{w}$.

2. $\frac{d}{dz}z : BF(\mathbb{C}) \longrightarrow BF(\mathbb{C})$ is anti - Wick operator.

$$\frac{d}{dz}zf(z) = \frac{d}{dz}\int_{\mathbb{C}} e^{z\overline{w}}wf(w)d\mu(w) = \int_{\mathbb{C}} w\overline{w}e^{z\overline{w}}f(w)d\mu(w),$$

Hence we have $F(w, \overline{w}) = w\overline{w} = |w|^2$.

4.4 Eigenvalue problem of Anti - Wick(Toeplitz) Operator on Bargmann - Fock Space

In this subsection we will consider the eigenvalue problem of anti - Wick(Toeplitz) operator $T_F(f)(z) = \int_{\mathbb{C}^n} e^{z\overline{w}}F(w, \bar{w})f(w)d\mu(w)$.

__Theorem 2__([28]) Suppose that $F(w, \bar{w})$ is bounded integrable and polyradial function. i.e. $F(w, \bar{w}) = \widetilde{F}(|w_1|^2, \cdots, |w_n|^2)$. Then

(1) z^m is eigenfunction of T_F.

(2) Eigenvalue λ_m of T_F is given by

$$\lambda_m = \frac{1}{m!}\int_0^\infty \cdots \int_0^\infty \widetilde{F}(s_1, \cdots s_n)\prod_{i=1}^n e^{-s_i}{s_i}^{m_i}ds_i,\ m = (m_1, \cdots, m_n) \in \mathbb{N}^n.$$

(Proof) For brevity's sake, we put $n = 1$.

$$(T_F)(w^m)(z) = \int_{\mathbb{C}} \widetilde{F}(|w|^2)e^{z\overline{w}}w^m d\mu(w) = \frac{1}{\pi}\int_{\mathbb{C}} \widetilde{F}(|w|^2)e^{z\overline{w}}w^m e^{-|w|^2}dm(w)$$

$$= \frac{1}{\pi}\int_{\mathbb{C}} \widetilde{F}(|w|^2)\left(\sum_{n=0}^\infty \frac{(z\overline{w})^n}{n!}\right) w^m e^{-|w|^2}dm(w)$$

$$= \sum_{n=0}^\infty \frac{z^n}{n!}\frac{1}{\pi}\int_{\mathbb{C}} \widetilde{F}(|w|^2)\overline{w}^n w^m e^{-|w|^2}dm(w).$$

By using the polar coordinate $w = re^{i\theta}$,

$$= \frac{1}{\pi}\sum_{n=0}^\infty \frac{z^n}{n!}\int_0^\infty \int_0^{2\pi} \widetilde{F}(r^2)e^{i(m-n)\theta}r^n r^m e^{-r^2}rdrd\theta$$

$= z^m \frac{1}{m!} \int_0^\infty e^{-r^2} \widetilde{F}(r^2) r^{2m} 2r dr = z^m \frac{1}{m!} \int_0^\infty e^{-s} s^m \widetilde{F}(s) ds.$

Hence we obtain

$(T_F)(w^m)(z) = z^m \frac{1}{m!} \int_0^\infty e^{-s} s^m \widetilde{F}(s) ds.$

Example([24], [28])

1. $F(w, \bar{w}) = \exp(\frac{a-1}{a}|w|^2), \quad (0 < a < 1)$

$\widetilde{F}(s) = \exp(\frac{a-1}{a}s), \quad \lambda_m = a^{m+1}$

5 Daubechies (Localization) Operator

5.1 Daubechies (Localization) Operator

Daubechies operator was introduced by Ingrid Daubechies in ([5], [6]). Daubechies operator P_F is defined as follows:

$$P_F(f)(x) = (2\pi)^{-n} \int\int_{\mathbb{R}^{2n}} F(p,q) \phi_{p,q}(x) W_\phi(f)(p,q) dp dq,$$

$f(x) \in L^2(\mathbb{R}^n)$. $\phi_{p,q}(x) = \pi^{-n/4} e^{-ipx} e^{-(x-q)^2/2}$.

$W_\phi(f)(p,q) = \int_{\mathbb{R}^n} \overline{\phi_{p,q}(y)} f(y) dy$ is Gabor transform of $f(x)$ and $F(p,q)$ is symbol function of P_F

Remark If $F(p,q)$ is 1, then P_F is identity operator. i.e. We have **Resolution of identity(Inversion formula of Gabor transform)**

$f(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} \phi_{p,q}(x) W_\phi(f)(p,q) dp dq$

5.2 Daubechies Operator in Bargmann - Fock space

If we consider Daubechies operator in Bargmann - Fock space, then we have following theorem ([28]).

Theorem 3 For $g(z) \in BF(\mathbb{C}^n)$, we have

$(B \circ P_F \circ B^{-1})(g)(z) = (2\pi i)^{-n} \int\int_{\mathbb{C}^n} F(w, \overline{w}) e^{z\overline{w}} g(w) e^{-|w|^2} d\overline{w} \wedge dw.$

Especially if $F(w, \bar{w}) = 1$, then we obtain

Corollary(Relationship between resolution of identity and reproducing formula)

$$f(x) = \left(\frac{1}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} \phi_{p,q}(x) W_\phi(f)(p,q)dpdq, \quad f(x) \in L^2(\mathbb{R}^n)$$

is equivalent to

$$g(z) = \int_{\mathbb{C}^n} e^{z\overline{w}} g(w) d\mu(w), \quad (\forall g(z) \in BF).$$

$$\begin{array}{ccc} L^2(\mathbb{R}^n) & \xrightarrow{B} & BF \\ {\scriptstyle P_F}\downarrow & & \downarrow{\scriptstyle B\circ P_F \circ B^{-1}} \\ L^2(\mathbb{R}^n) & \xrightarrow{B} & BF \end{array}$$

6 Application to Daubechies Localization Operator

6.1 Hermite Functions

Hermite functions $h_m(x)$ of one variable is defined by

$$h_m(x) = (-1)^m (2^m m! \sqrt{\pi})^{-1/2} \exp(x^2/2) \frac{d^m}{dx^m} \exp(-x^2).$$

Generating function of Hermite functions is the kernel function of Bargmann transform.

$$\pi^{-1/4} \exp\left\{-\frac{1}{2}(z^2 + x^2) + \sqrt{2} z \cdot x\right\} = \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} h_m(x), \ (z \in \mathbb{C}^1, x \in \mathbb{R}^1).$$

We also have the following expression.

$$h_m(x) = \frac{1}{\sqrt{2^m m!}} \left(\frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)\right)^m h_0(x).$$

Hermite functions $h_m(x)$ of several variables is defined by

$h_m(x_1, x_2, ... x_n) = \prod_{i=1}^n h_{m_i}(x_i)$, $m = (m_1, ... m_n) \in N^m$

Example

1. $h_0(x) = \pi^{-1/4} \exp(-x^2/2)$, (coherent state)
2. $h_2(x) = \pi^{-1/4} \dfrac{2x^2 - 1}{\sqrt{2}} \exp(-x^2/2)$, (Mexican hat wavelet)

6.2 Daubechies' result

As an application of our result, we will give a new proof of following Daubechies' result.

Theorem 4([5]) Suppose that $F(p,q)$ is integrable polyradial function. Then we have

1. $P_F(h_m)(x) = \lambda_m h_m(x)$

2. $\lambda_m = \frac{1}{m!}\int_0^\infty \cdots \int_0^\infty \widetilde{F}(s_1, \cdots s_n)\prod_{i=1}^n e^{-s_i}{s_i}^{m_i}ds_i,$

$m = (m_1, \cdots, m_n) \in \mathbb{N}^n$.

(Proof) For the simplicity we put n=1. Let P_F be Daubechies operaotr with integrable polyradial symbol F. Then $T_F = B \circ P_F \circ B^{-1}$ is Toeplitz operator with integrable polyradial symbol F. So we can apply Theorem 2 to T_F. Hence we have

$\lambda_m = \frac{1}{m!}\int_0^\infty \widetilde{F}(s)e^{-s}s^m ds, \quad T_F(\frac{z^m}{\sqrt{m!}}) = \lambda_m \frac{z^m}{\sqrt{m!}}.$

By inverse Bargnmann transform,

$h_m(x) = B^{-1}(\frac{z^m}{\sqrt{m!}})(x).$

So we obtained following Daubechies' results.

$P_F(h_m)(x) = \lambda_m h_m(x), \qquad \lambda_m = \frac{1}{m!}\int_0^\infty e^{-s}s^m\tilde{F}(s)ds.$

7 Reconstruction of symbol function from eigenvalues

7.1 The first reconstruction formula

We consider the analytic continuation of eigenvalues λ_m of T_F. It is given by

$\lambda(z) = \frac{1}{\Gamma(z+1)}\int_0^\infty e^{-s}s^z\tilde{F}(s)ds,$

where $\Gamma(z)$ is Euler Gamma function. We have $\lambda(m) = \lambda_m$ by Theorem 2.

Theorem 5([21])

$\tilde{F}(s) = \frac{e^s}{s}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \lambda(z)\Gamma(z+1)s^{-z}dz.$

(Proof) Integral representation $\lambda(z) = \dfrac{1}{\Gamma(z+1)} \displaystyle\int_0^\infty e^{-s} s\tilde{F}(s) s^{z-1} ds,$
means that $\lambda(z)\Gamma(z+1)$ is Mellin transform of $e^{-s}s\tilde{F}(s)$.
Hence we obtain above formula by inverse Mellin transform.

7.2 The second reconstruction formula

For eigenvalues $\{\lambda_m\}$ of anti - Wick(Toeplitz) operator T_F, we put
$\Lambda(w) = \displaystyle\sum_{m=0}^{\infty} \lambda_m w^m.$
$\Lambda(w)$ is generating function (of eigenvalues) of anti - Wick(Toeplitz) operator T_F. In signal analysis $\Lambda(w)$ is called z - transform instead of generating function. In what follows we assume that $F(p,q)$ is integrable and polyradial function.

Proposition 5([23]) Suppose that λ_m are eigenvalues of T_F. Then we have

(i) $\exists C > 0$ s.t. $|\lambda_m| \leq \dfrac{C}{\sqrt{|m|}}, \quad (m \in \mathbb{N}^n).$

(ii) $\Lambda(w)$ is holomorphic in $\prod_{i=1}^{n} \{w \in \mathbb{C}^n : |w_i| < 1\}$.

(iii) $\Lambda(w) = \displaystyle\int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n} e^{-s_i(1-w_i)} \tilde{F}(s_1,...,s_n) ds_1 ... ds_n.$

(iv) $\Lambda(w)$ is holomorphic in $\prod_{i=1}^{n} \{w \in \mathbb{C}^n : Re(w_i) < 1\}$
and bounded in its closure.

(v) $\Lambda(iv) \in C_0(\mathbb{R}^n), (v \in \mathbb{R}^n)$. i.e. $\Lambda(iv) \in C(\mathbb{R}^n)$ and $\lim_{|v|\to\infty} \Lambda(iv) = 0$.

(Proof) Without loss of genelarity, we can assume that $n = 1$.

(i) By Theorem 2, $\lambda_m = \dfrac{1}{m!} \displaystyle\int_0^\infty e^{-s} \tilde{F}(s) s^m ds.$

Since $e^{-s}s^m \leq e^{-m}m^m$, we have
$|\lambda_m| \leq \dfrac{1}{m!} e^{-m} m^m \displaystyle\int_0^\infty |\tilde{F}(s)| ds.$

By Stirling's formula $m! \sim \sqrt{2\pi m} e^{-m} m^m$, for sufficiently large m,
$|\lambda_m| \leq C \dfrac{1}{\sqrt{m}}$ valids.

(iii) $\Lambda(w) = \displaystyle\sum_{m=0}^{\infty} \lambda_m w^m = \sum_{m=0}^{\infty} \frac{w^m}{m!} \int_0^\infty e^{-s} s^m \tilde{F}(s) ds =$

$$\int_0^{\infty} e^{-s}\tilde{F}(s)\sum_{m=0}^{\infty}\frac{(ws)^m}{m!}ds = \int_0^{\infty} e^{-s(1-w)}\tilde{F}(s)ds.$$

(iv) For $Re(w) \leq 1$, we have

$$|\Lambda(w)| \leq \int_0^{\infty} |e^{-s(1-w)}||\tilde{F}(s)|ds \leq ||\tilde{F}||_{L^1}.$$

(v) Since $\Lambda(iv)$ is Fourier transform of L^1 function $e^{-s}\tilde{F}(s)$, it is in $C_0(\mathbb{R}^n)$ by Riemann - Lebesgue theorem.

Theorem 6([21])

$$\tilde{F}(s) = (2\pi)^{-1}e^s \int_{-\infty}^{+\infty} e^{-isv}\Lambda(iv)dv,$$

valids in distribution sense.

(Proof) For the simplicity, we put $n = 1$.

By (iii) in Proposition 5, we have

$$\Lambda(iv) = \int_0^{\infty} e^{-s(1-iv)}\tilde{F}(s)ds = \int_0^{\infty} e^{isv}e^{-s}\tilde{F}(s)ds, \quad (v \in \mathbb{R}).$$

This means that $\Lambda(iv)$ is the inverse Fourier transform of integrable function $e^{-s}\tilde{F}(s)$. Since $\Lambda(iv)$ is continuous bounded function, $\Lambda(iv)$ is tempered distribution. Hence as tempered distribution we have $\tilde{F}(s) = e^s F(\Lambda(iv))(s)$.

Example([24]) $F(w, \bar{w}) = e^{\frac{a-1}{2a}(|w|^2)} \quad (0 < a < 1)$.

$$\lambda_m = a^{m+1}, \quad \lambda(z) = a^{z+1}, \quad \Lambda(w) = \frac{a}{1-aw},$$

7.3 Conclusion

1. Daubechies operator in Bargmann - Fock space $B \circ P_F \circ B^{-1}$ is anti - Wick(Toeplitz) operator.

2. Applying the results of the eigenvalue problem of anti - Wick(Toeplitz) operator in Bargmann - Fock space, we can derive Daubechies' results more easily.

3. For anti - Wick operator T_F with polyradial symbols, we can reconstruct polyradial symbol function $F(w, \bar{w})$ from eigenvalues of T_F.

Remark For the details of our study, we refer the reader to [21], [22], [23], [24], [25], [26], [27], [28].

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