数理解析研究所講究録
第2058巻 2017年 120-129

THE L^p-APPROACH TO GLOBAL STRONG WELL-POSEDNESS OF THE PRIMITIVE EQUATIONS OF OCEAN DYNAMICS

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ABSTRACT. In this short note we summarize recent results on the L^p-approach to the primitive equations. By this approach, one obtains global strong well-posedness results for the primitive equations for arbitrarly large data in $D((-A_p)^{1/p})$ for $1<p<\infty$, where A_p denotes the hydrostatic Stokes operator on $L^p_{\overline{\sigma}}(\Omega)$, and $\Omega\subset\mathbb{R}^3$ is a cylindrical domain subject to mixed, periodic Dirichlet and Neumann boundary conditions. The above space $D((-A_p)^{1/p})$ may be identified by a Bessel potential space on Ω, satisfying certain boundary conditions. Furthermore $-A_p$ admits a bounded H^∞-calculus on $L^p_{\overline{\sigma}}(\Omega)$ for all $p\in(1,\infty)$ with H^∞-angle 0 and in particular one obtains thus maximal L^q-L^p- regularity estimates for the linearized primitive equations.

1. INTRODUCTION

The primitive equations for ocean and atmospheric dynamics were introduced by Lions, Teman and Wang in a series of articles [27–29] and they serve since then as a fundamental model for many geophysical flows. This set of equations describing the conservation of momentum and mass of a fluid, assuming hydrostatic balance of the pressure, coupled to the equations for temperature as well as salinity, are given by

$$(1.1)\qquad \begin{cases} \partial_t v + u\cdot\nabla v - \Delta v + \nabla_H \pi &= f, \qquad \text{in } \Omega\times(0,T),\\ \operatorname{div} u &= 0, \qquad \text{in } \Omega\times(0,T),\\ \partial_t\tau + u\cdot\nabla\tau - \Delta\tau &= g_\tau, \qquad \text{in } \Omega\times(0,T),\\ \partial_t\sigma + u\cdot\nabla\sigma - \Delta\sigma &= g_\sigma, \qquad \text{in } \Omega\times(0,T),\\ \partial_z\pi + 1 - \beta_\tau(\tau-1) + \beta_\sigma(\sigma-1) &= 0, \qquad \text{in } \Omega\times(0,T), \end{cases}$$

with initial conditions $v(0)=a$, $\tau(0)=b_\tau$, $\sigma(0)=b_\sigma$ and forcing terms f, g_τ and g_σ. Here $\Omega = G\times(-h,0)\subset\mathbb{R}^3$, with $G=(0,1)\times(0,1)$. The velocity u of the fluid is described by $u=(v,w)$, where $v=(v_1,v_2)$ denotes the horizontal component and w the vertical one. In addition, the temperature and salinity are denoted by τ and σ, respectively, and π denotes the pressure of the fluid. Moreover, we assume $\beta_\tau,\beta_\sigma>0$. Denoting the horizontal coordinates by $x,y\in G$ and the vertical one by $z\in(-h,0)$, we use the notation $\nabla_H=(\partial_x,\partial_y)^T$, whereas Δ denotes the three dimensional Laplacian and ∇ and div the three dimensional gradient and divergence operators.

The above system is complemented by the boundary conditions

$$(1.2)\qquad \begin{cases} \partial_z v = 0, \quad w=0, \quad \partial_z\tau+\alpha\tau=0, \quad \partial_z\sigma=0 & \text{on } \Gamma_u\times(0,\infty),\\ v=0, \quad w=0, \quad \partial_z\tau=0, \quad \partial_z\sigma=0 & \text{on } \Gamma_b\times(0,\infty),\\ v,\pi,\tau,\sigma \quad \text{are periodic} & \text{on } \Gamma_l\times(0,\infty), \end{cases}$$

where

$$\Gamma_u = G\times\{0\}, \quad \Gamma_b = G\times\{-h\} \quad \text{and} \quad \Gamma_l = \partial G\times(-h,0),$$

2010 *Mathematics Subject Classification.* Primary: 35Q35; Secondary: 76D03, 47D06, 86A05.

and $\alpha > 0$.

The rigorous analysis of the primitive equations started with the pioneering work of Lions, Temam and Wang [27–29], who proved the existence of a global weak solution for this set of equations for initial data $a \in L^2$ and $b_\tau \in L^2$, $b_\sigma \in L^2$. For recent results on the uniqueness problem for global weak solutions, we refer to the work of Li and Titi [26] and Kukavica, Pei, Rusin and Ziane [21].

The existence of a local, strong solution for the decoupled velocity equation with data $a \in H^1$ was proved by Guillén-González, Masmoudi and Rodiguez-Bellido in [14].

In 2007, Cao and Titi [4] proved a breakthrough result for this set of equation which says, roughly speaking, that there exists a unique, global strong solution to the primitive equations for *arbitrary* initial data $a \in H^1$ and $b_\tau \in H^1$ neglecting salinity. Their proof is based on *a priori* H^1-bounds for the solution, which in turn are obtained by $L^\infty(L^6)$ energy estimates.

Kukavica and Ziane considered in [23, 24] the primitive equations subject to the boundary conditions on $\Gamma_u \cup \Gamma_b$ as in (1.2) and they proved global strong well-posedness of the primitive equations with respect to arbitrary large H^1-data. For a different approach see also Kobelkov [20].

For recent results dealing with only horizontal viscosity and diffusion or with horizontal or vertical eddy diffusivity, we refer to the work of Cao, Li and Titi in [5–7]. Here, global well-posedness results are established for initial data in H^2.

For local well-posedness results concerning the inviscid primitive equations, we refer to Brenier [3], Masmoudi and Wong [31], Kukavica, Temam, Vicol and Ziane [22] as well as Hamouda, Jung and Temam [15].

Recently, an L^p-approach for the primitive equations was developed in [16], [17] and [13] and it is the aim of this note to describe and summarize the results obtained by this approach.

Roughly speaking, the existence of a *unique, global strong solution* to the primitive equations was proved in [16] [17] for initial data $a \in V_{1/p,p}$ for $p \in (1, \infty)$. Here, $V_{1/p,p}$ denotes the complex interpolation space between the ground space X_p and the domain of the hydrostatic Stokes operator, which was introduced and investigated in [16] and [13].

Choosing in particular $p = 2$, the space of initial data $V_{1/2,2}$ coincides with the space V introduced by Cao and Titi in [4] (up to a compatibility condition due to different boundary conditions), see also [4, 14, 23, 33]. Note that $V_{1/p,p} \hookrightarrow H^{2/p,p}(\Omega)^2$ for all $p \in (1, \infty)$. Hence, choosing p large, one obtains a global well-posedness result for initial data a having less differentiability properties than $H^1(\Omega)$.

At this point, we also would like to draw the attention of the reader to the recent the survey article by Li and Titi [30] on the primitive equations.

2. Global existence in the non-isothermal situation

The primitive equations may be reformulated equivalently as

$$
\left\{
\begin{array}{rll}
\partial_t v + v \cdot \nabla_H v + w \cdot \partial_z v - \Delta v + \nabla_H \pi_s & = f + \Pi(\tau, \sigma), & \text{in } \Omega \times (0,T), \\
\mathrm{div}_H \overline{v} & = 0, & \text{in } \Omega \times (0,T), \\
\partial_t \tau + v \cdot \nabla_H \tau + w \cdot \partial_z \tau - \Delta \tau & = g_\tau, & \text{in } \Omega \times (0,T), \\
\partial_t \sigma + v \cdot \nabla_H \sigma + w \cdot \partial_z \sigma - \Delta \sigma & = g_\sigma, & \text{in } \Omega \times (0,T),
\end{array}
\right. \tag{2.1}
$$

using the notation

$$
\mathrm{div}_H v = \partial_x v_1 + \partial_y v_2 \quad \text{and} \quad \overline{v} := \frac{1}{h} \int_{-h}^{0} v(\cdot, \cdot, \xi) d\xi,
$$

and where we took into account the boundary condition $w = 0$ on Γ_b. Making use of the boundary condition $w = 0$ on Γ_u, the vertical component w of the velocity u is determined by

$$w = -\int_{-h}^{z} \mathrm{div}_H v(\cdot,\cdot,\xi)d\xi.$$

Furthermore, the pressure π is determined by the surface pressure $\pi_s(x,y) = \pi(x,y,-h)$, while the part of the pressure due to temperature and salinity is given by

$$\Pi(\tau,\sigma) = -\nabla_H \int_{-h}^{z} \beta_\tau\tau(\cdot,\xi) - \beta_\sigma\sigma(\cdot,\xi)d\xi, \quad \beta_\tau,\beta_\sigma > 0.$$

Periodic boundary conditions in the horizontal direction are modeled using function spaces as in [16, Section 2].

The linearized problem for the velocity is given by the *hydrostatic Stokes equation*

$$\begin{aligned} \partial_t v - \Delta v + \nabla_H \pi_s &= f, \\ \mathrm{div}_H \overline{v} &= 0, \end{aligned}$$

with initial value $v(0) = a$ and boundary conditions as in (1.2). The study of the hydrostatic Stokes system started with thw work of Ziane [35,36], who considered the L^2 situation. The general L^p setting for $p \in (1,\infty)$ has been studied in detail in [16, Section 3 and 4]. In particular, it has been shown there that the *hydrostatic solenoidal space*

$$L^p_{\overline{\sigma}}(\Omega) = \overline{\{v \in C^\infty_{per}(\Omega)^2 \mid \mathrm{div}_H \overline{v} = 0\}}^{L^p(\Omega)^2}$$

is a closed subspace of $L^p(\Omega)^2$, compare [16, Proposition 4.3]. Furthermore, there exists a continuous projection P_p onto it – called the *hydrostatic Helmholtz projection*, and one has $L^p_{\overline{\sigma}}(\Omega) = \mathrm{Ran}\, P_p$. In particular,

$$L^p_{\overline{\sigma}}(\Omega) = \{v \in L^p(\Omega)^2 \mid \langle \overline{v}, \nabla_H \pi_s\rangle_{L^{p'}(G)} = 0 \text{ for all } \pi_s \in H^{1,p'}_{per}(G)\},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Following [16], we then define the *hydrostatic Stokes operator* A_p by

$$A_p v := P_p \Delta v, \quad D(A_p) := \{v \in H^{2,p}_{per}(\Omega)^2 \mid (\partial_z v)|_{\Gamma_u} = 0, v|_{\Gamma_b} = 0\} \cap L^p_{\overline{\sigma}}(\Omega).$$

Furthermore, we define the operators Δ_τ on $L^q(\Omega)$ for $\alpha > 0$ and Δ_σ by

$$\begin{aligned} \Delta_\tau \tau = \Delta\tau, &\quad D(\Delta_\tau) = \{\tau \in H^{2,q_\tau}_{per}(\Omega) \mid (\partial_z\tau + \alpha\tau)\,|_{\Gamma_u} = 0, \quad \partial_z\tau\,|_{\Gamma_b} = 0\}, \\ \Delta_\sigma \tau = \Delta\sigma, &\quad D(\Delta_\sigma) = \{\sigma \in H^{2,q_\sigma}_{per}(\Omega) \mid \partial_z\sigma\,|_{\Gamma_u} = 0, \quad \partial_z\sigma\,|_{\Gamma_b} = 0\}. \end{aligned}$$

Resolvent estimates for A_p within the L^p-context were obtained in [16, Theorem 3.1] and the operators Δ_τ and Δ_σ were investigated in detail by Nau in [32, Section 8.2.2], also in the L^q-context. For the precise definition of the periodic Sobolev spaces we refer to [16]. We thus obtain the following result.

Proposition 2.1. ([16], [32]). *Let $p \in (1,\infty)$. Then the operator A_p generates an analytic semigroup T_p on $L^p_{\overline{\sigma}}(\Omega)$, which is exponentially stable with decay rate $\beta_v > 0$. Furthermore, the operators Δ_τ and Δ_σ are generators of analytic contraction semigroups T_τ and T_σ on $L^p(\Omega)$ and T_τ is exponentially stable with decay rate $\beta_\tau > 0$.*

After reformulating the original system (1.1) and (1.2) into its equivalent form (2.1), we are now in the position state the following result.

Theorem 2.2 (Existence of unique, global strong solutions, [17]).
Let $p, q_\tau, q_\sigma \in (1,\infty)$ with $q_\tau, q_\sigma \in [\frac{2p}{3}, p] \cap (1,p]$ and suppose that

$$f \in H^{1,2}_{loc}((0,\infty); L^p(\Omega)^2 \cap L^2(\Omega)^2),$$
$$g_\tau \in H^{1,2}_{loc}((0,\infty); L^{q_\tau}(\Omega) \cap L^2(\Omega)), \qquad g_\sigma \in H^{1,2}_{loc}((0,\infty); L^{q_\sigma}(\Omega) \cap L^2(\Omega)).$$

a) Assume that

$$a \in \{u \in H^{2/p,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) \mid v\,|_{\Gamma_b} = 0\}, \quad b_\tau \in H^{2/q_\tau, q_\tau}_{per}(\Omega), \quad b_\sigma \in H^{2/q_\sigma, q_\sigma}_{per}(\Omega).$$

Then there is a unique, global, strong solution to (2.1) *and* (1.2) *satisfying*

$$\begin{aligned} v &\in C^1((0,\infty); L^p_{\overline{\sigma}}(\Omega)) \cap C^0((0,\infty); D(A_p)),\\ \pi_s &\in C^0((0,\infty); H^{1,p}_{per}(G) \cap L^p_0(G)),\\ \tau &\in C^1((0,\infty); L^{q_\tau}(\Omega)) \cap C^0((0,\infty); D(\Delta_\tau)),\\ \sigma &\in C^1((0,\infty); L^{q_\sigma}(\Omega)) \cap C^0((0,\infty); D(\Delta_\sigma)). \end{aligned}$$

b) If in addition

$$a \in D(A_p) \quad \text{and} \quad b_\tau \in D(\Delta_\tau), \quad b_\sigma \in D(\Delta_\sigma)$$

then the above solution extends to $[0,\infty)$.

Considering the primitive equations without salinity we obtain furthermore the following result.

Theorem 2.3 (Decay at infinity, [17]).
In addition to the assumptions of Theorem 2.2, let $b_\sigma = 0$ and $g_\sigma = 0$, and assume that there are $\beta_f \geq \beta_v$, $\beta_{g_\tau} \geq \beta_\tau$, such that

$$\|f\|_{L^p(\Omega)^2} = O(e^{-\beta_f t}) \text{ and } \|g_\tau\|_{L^{q_\tau}(\Omega)} = O(e^{-\beta_g t}), \text{ as } t \to \infty,$$

where β_v, β_τ are given as in Proposition 2.1. Then the strong solution (v, π_s, τ) to (2.1) *and* (1.2) *satisfies*

$$\|\partial_t v\|_{L^p} + \|\Delta v\|_{L^p} = O(e^{-\beta_v t}), \quad \|\partial_t \tau\|_{L^{q_\tau}} + \|\Delta \tau\|_{L^{q_\tau}} = O(e^{-\beta_\tau t}), \quad \|\nabla_H \pi_s\|_{L^p} = O(e^{-\beta t})$$

as $t \to \infty$ and where $\beta := \min\{\beta_v, \beta_\tau\}$.

The strategy to construct a unique, global, strong solution to (2.1) and (1.2) within the L^p-setting is to consider the L^2-situation first and to prove *a priori* estimates. In the second step we consider then the existence of unique, strong, local L^p solution to (2.1) and (1.2), which due to the regularization properties of the underlying linear equation, lies after short time, inside L^2.

Proposition 2.4. *Let $a \in D(A_2)$, $b \in D(\Delta_\zeta)$ for $q = 2$, and*

$$f \in H^{1,2}((0,T); L^2(\Omega)^2), \quad g \in H^{1,2}((0,T); L^2(\Omega)^2).$$

Assume that v, π_s, ζ is a strong solutions to (2.1) *and* (1.2) *on $[0,T]$. Then there are functions $B^v_{H^2}$, $B^{\pi_s}_{H^1}$, $B^\zeta_{H^2}$, continuous on $[0,T]$, such that for all $t \in [0,T]$*

$$\|\zeta(t)\|^2_{H^2(\Omega)^2} \leq B^\tau_{H^2}(t), \quad \|v(t)\|^2_{H^2(\Omega)} \leq B^v_{H^2}(t), \quad \|\pi_s(t)\|^2_{H^1(G)} \leq B^{\pi_s}_{H^1}(t),$$

where the bounds depend on $\|b\|_{H^2}$, $\|a\|_{H^2}$, $\|f\|_{H^{1,2}(L^2)}$, $\|g\|_{H^{1,2}(L^2)}$ and T, only.

The explicit characterization of the initial data in Theorem 2.2 for which we obtain global strong well-posedness of the primitive equations relies on the following characterization of the complex interpolation space

$$V_{\theta,p} := [L^p_{\overline{\sigma}}(\Omega), D(A_p)]_\theta,$$

which arises in the construction of local solutions. Here $0 \leq \theta \leq 1$ and $[\cdot,\cdot]_\theta$ denotes the complex interpolation functor; see [16]. For $p, q \in (1,\infty)$, these spaces are characterized as follows.

Proposition 2.5. *If $p, q \in (1,\infty)$, then*

$$V_{\theta,p} = \begin{cases} \{H^{2\theta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) \mid \partial_z v\,|_{\Gamma_u} = 0, v\,|_{\Gamma_b} = 0\}, & 1/2 + 1/2p < \theta \leq 1, \\ \{H^{2\theta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) \mid v\,|_{\Gamma_b} = 0\}, & 1/2p < \theta < 1/2 + 1/2p, \\ H^{2\theta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega), & \theta < 1/2p, \end{cases}$$

We note that, by the work of Amann [1], results on the interpolation of boundary conditions for Sobolev spaces are known for second order elliptic operators on domains with C^∞-boundaries subject to mixed boundary conditions on disjoint parts of the boundaries. The proof of above assertion relies then on the construction of suitable retractions of interpolation couples transfering results from such a situation to the one considered here.

3. Properties of the isothermal hydrostatic Stokes equation

Starting from the fact that the negative hydrostatic Stokes operator $-A_p$ in $L^p_{\overline{\sigma}}(\Omega)$ for $1 < p < \infty$ is a sectorial operator of spectral angle 0, see Proposition 2.1, it is an interesting question to ask whether $-A_p$ admits a bounded H^∞-calculus on $L^p_{\overline{\sigma}}(\Omega)$. Here we consider the situation where the underlying domain is a cylindrical domain with laterally periodic boundary conditions and with Dirichlet and/or Neumann boundary conditions on the bottom and top part of $\partial\Omega$.

In [13] an affirmative answer to this question was given and it was shown in particular that $-A_p$ admits a bounded H^∞-calculus on $L^p_{\overline{\sigma}}(\Omega)$ with H^∞-angle equal to 0 by means of a perturbation argument. As a consequence, one obtain maximal $L^q - L^p$-regularity estimates for the linearized primitive equations. For a recent survey concerning regularity results for the classsical Stokes equation, we refer to [18].

Combining the explicit description of the complex interpolation spaces $[L^p_{\overline{\sigma}}(\Omega), D(A_p)]_\theta$ given above in Proposition 2.5 with the existence of a bounded H^∞-calculus for $-A_p$ implies further that the domains of the fractional powers $(-A_p)^\theta$ can be characterized explicitly as Bessel potential spaces satisfying appropriate boundary conditions depending on the value of the interpolation parameter $\theta \in [0,1]$. We finally state that the hydrostatic Stokes semigroup satisfies global $L^p - L^q$-smoothing estimates, similarly to the well-known siutation of the classical Stokes semigroup.

We consider again the linearization of equation (2.1), the *hydrostatic Stokes equations*, which are given by

$$\begin{cases} \partial_t v - \Delta v + \nabla_H \pi_s = f, & \text{in } \Omega \times (0,T), \\ \operatorname{div}_H \overline{v} = 0, & \text{in } \Omega \times (0,T), \\ v(0) = v_0 & \text{in } \Omega. \end{cases} \tag{3.1}$$

These equations are supplemented by the mixed boundary conditions on

$$\Gamma_a = G \times \{a\}, \quad \Gamma_b = G \times \{b\} \quad \text{and} \quad \Gamma_l = \partial G \times (a,b),$$

i.e. the bottom, upper and lateral parts of the boundary $\partial\Omega$, respectively, are given by

$$v, \pi_s \text{ are periodic on } \Gamma_l \times (0,\infty),$$
$$v = 0 \text{ on } \Gamma_D \times (0,\infty) \quad \text{and} \quad \partial_z v = 0 \text{ on } \Gamma_N \times (0,\infty),$$

where Dirichlet, Neumann and mixed boundary conditions are comprised by the notation

$$\Gamma_D \in \{\emptyset, \Gamma_a, \Gamma_b, \Gamma_a \cup \Gamma_b\} \quad \text{and} \quad \Gamma_N = (\Gamma_a \cup \Gamma_b) \setminus \Gamma_D.$$

Using this notation, the *hydrostatic Stokes operator* A_p in $L^p_{\overline{\sigma}}(\Omega)$ is then given by

$$A_p v = P_p \Delta v, \quad D(A_p) = \{v \in H^{2,p}_{per}(\Omega)^2 : \partial_z v\big|_{\Gamma_N} = 0,\ v\big|_{\Gamma_D} = 0\} \cap L^p_{\overline{\sigma}}(\Omega).$$

Let us recall that $\Gamma_D \neq \emptyset$ means that Dirichlet conditions are imposed on either Γ_a, Γ_b or $\Gamma_a \cup \Gamma_b$ with Neumann conditions on the remaining part of $\Gamma_a \cup \Gamma_b$.

The following result was proved in [13].

Theorem 3.1. ([13]) *Let $p \in (1,\infty)$ and $\nu \geq 0$. Then the operator $-A_p + \nu$ admits a bounded H^∞-calculus on $L^p_{\overline{\sigma}}(\Omega)$ with $\phi^\infty_A = 0$ provided $\nu > 0$. If $\Gamma_D \neq \emptyset$, then the above assertion holds true even for $\nu = 0$.*

Corollary 3.2. *Let $p \in (1,\infty)$ and $\nu \geq 0$. Then the operator $-A_p + \nu$ admits a bounded $\mathcal{R}H^\infty$-calculus on $L^p_{\overline{\sigma}}(\Omega)$ with $\phi^{\mathcal{R}\infty}_A = 0$ provided $\nu > 0$. If $\Gamma_D \neq \emptyset$, then the above assertion holds true even for $\nu = 0$.*

The existence of the bounded H^∞-calculus for $-A_p$ implies that

$$D((-A_p)^\theta) = [L^p_{\overline{\sigma}}(\Omega), D(A_p)]_\theta, \quad \theta \in [0,1],$$

where $[\cdot,\cdot]_\theta$ denotes the complex interpolation functor. Since $D(A_p) \subset H^{2,p}(\Omega)^2$, we may conclude that $D(-A^\theta_p) \subset H^{2\theta,p}(\Omega)^2$. In [17, Section 4], a suitable retract to compute the interpolation spaces in terms of boundary conditions was constructed and adapting this to the present situation allows us to characterize the domains of $(-A_p)^\theta$ for $\theta \in [0,1]$ as follows.

Corollary 3.3. *Let $1 < p < \infty$ and $\theta \in [0,1]$ with $\theta \notin \{1/2p, 1/2 + 1/2p\}$. Then*

$$D((-A_p)^\theta) = \begin{cases} \{v \in H^{2\theta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : \partial_z v\big|_{\Gamma_N} = 0,\ v\big|_{\Gamma_D} = 0\}, & 1/2 + 1/2p < \theta \leq 1, \\ \{v \in H^{2\theta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : v\big|_{\Gamma_D} = 0\}, & 1/2p < \theta < 1/2 + 1/2p, \\ \{v \in H^{2\theta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega)\}, & \theta < 1/2p. \end{cases}$$

For the corresponding result for the classical Stokes operator, see [12].

As a further consequence of Theorem 3.1, we obtain maximal $L^q - L^p$-regularity estimates for the linearized primitive equations. For $1 < q < \infty$, $0 < T \leq \infty$ and a closed operator A in a Banach space X consider the Cauchy problem

$$u'(t) + Au(t) = f(t), \quad t \in (0,T), \quad u(0) = u_0, \tag{3.2}$$

where $u_0 \in X_\gamma = (X, D(A))_{1/q',q}$, $1/q' + 1/q = 1$ and $(\cdot,\cdot)_{1/q',q}$ denotes the real interpolation functor. We say that (3.2) admits *maximal L^q-regularity* or $A \in M_q(0,T;X)$, if for each $f \in L^q(0,T;X)$ and $u_0 \in X_\gamma$, the equation (3.2) admits a unique solution u satisfying $u \in W^{1,q}((0,T);X)$ and $Au \in L^q((0,T);X)$.

Corollary 3.4. *Let $p,q \in (1,\infty)$ and $T \in (0,\infty)$. Then $-A_p \in M_q((0,T);L^p_{\overline{\sigma}}(\Omega))$. In particular, A_p is the generator of an analytic semigroup on $L^p_{\overline{\sigma}}(\Omega)$. If $\Gamma_D \neq \emptyset$, then the above assertion also holds true for $T = \infty$.*

Amann considered in [2] real interpolation spaces of second order elliptic operators on smooth domains subject to Dirichlet and/or Neumann boundary conditions on disjoint sets of the boundary and was able to characterize them in terms of boundary values. Using the same retract and co-retract as defined in [17, Section 4], this characterization carries over to the present situation.

Corollary 3.5. *Let $p,q \in (1,\infty)$ with $1/p+2/q \notin \{1,2\}$ and $1/q+1/q'=1$. Then*

$$(L^p_{\overline{\sigma}}(\Omega), D(A_p))_{1/q',q} = \begin{cases} \{v \in B^{2-2/q}_{p,q,per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : \partial_z v\,|_{\Gamma_N}=0, v\,|_{\Gamma_D}=0\}, & 1+1/p<2-2/q\le 2,\\ \{v \in B^{2-2/q}_{p,q,per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : v\,|_{\Gamma_D}=0\}, & 1/p<2-2/q<1+1/p,\\ \{v \in B^{2-2/q}_{p,q,per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega)\}, & 0<2-2/q<1/p. \end{cases}$$

Considering $(-A_p)^{1/2}$ we obtain from Corollary 3.3 the L^p-boundedness of the *hydrostatic Riesz transformations* associated with A_p.

Corollary 3.6. *Let $1<p<\infty$. Then the hydrostatic Riesz transform*

$$R_p\colon L^p_{\overline{\sigma}}(\Omega) \to L^p(\Omega)^{2\times 2} \quad \textit{given by} \quad R_p v := \nabla(-A_p)^{-1/2}v$$

is bounded provided $\Gamma_D \neq \emptyset$.

We next state the global $L^p - L^q$-smoothing properties of the hydrostatic Stokes semigroup.

Proposition 3.7. ([13]). *Let $\Gamma_D \neq \emptyset$ and $p,q \in (1,\infty)$ such that $p \le q$. Then there exists a constant $C>0$ such that*

$$\begin{aligned} \|e^{tA_p}P_p f\|_{L^q(\Omega)^2} &\le Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\Omega)^2}, && \textit{for } f \in L^p(\Omega)^2, \quad t>0,\\ \|\nabla e^{tA_p}P_p f\|_{L^q(\Omega)^2} &\le Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|f\|_{L^p(\Omega)^2}, && \textit{for } f \in L^p(\Omega)^2, \quad t>0,\\ \|e^{tA_p}P_p \operatorname{div} f\|_{L^q(\Omega)^2} &\le Ct^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}}\|f\|_{L^p(\Omega)^{2\times 2}}, && \textit{for } f \in L^p(\Omega)^{2\times 2}, \quad t>0. \end{aligned}$$

The proof of Theorem 3.1 is based on perturbation methods. The key observation is that A_p may be represented as

$$A_p v = \Delta v + \nabla_H \Delta_H^{-1}\operatorname{div}_H D_z v\,|_{\Gamma_D}, \quad v \in D(A_p),$$

where

$$D_z v\,|_{\Gamma_D} = \tfrac{1}{b-a}\left(\gamma(b)\partial_z v\,|_{\Gamma_b} - \gamma(a)\partial_z v\,|_{\Gamma_a}\right),$$

and for $c \in \{a,b\}$ we set $\gamma(c)=1$ if $\Gamma_c \subset \Gamma_D$ and $\gamma(c)=0$, otherwise.

In order to obtain the above representation of A_p, we consider for $\lambda \in \Sigma_\pi$ and $f \in L^p(\Omega)$ the resolvent problem for the hydrostatic Stokes equation, which is given by

$$\begin{aligned} \lambda v - \Delta v + \nabla_H \pi &= f, && \text{in } \Omega,\\ \operatorname{div}_H \overline{v} &= 0, && \text{in } \Omega, \end{aligned} \tag{3.3}$$

subject to the boundary conditions

$$v,\pi \text{ are periodic on } \Gamma_l,$$
$$v=0 \text{ on } \Gamma_D \quad \text{and} \quad \partial_z v = 0 \text{ on } \Gamma_N,$$

where Dirichlet, Neumann and mixed boundary conditions are comprised by the notation

$$\Gamma_D \in \{\emptyset, \Gamma_a, \Gamma_b, \Gamma_a \cup \Gamma_b\} \quad \text{and} \quad \Gamma_N = (\Gamma_a \cup \Gamma_b) \setminus \Gamma_D.$$

We consider in the following only the case, where $\Gamma_D \neq \emptyset$. Taking the vertical average of (3.3) yields

$$\begin{aligned} \lambda\overline{v} - \Delta_H \overline{v} + \nabla_H \pi &= \overline{f} + D_z v \,|_{\Gamma_D}, \\ \mathrm{div}_H \overline{v} &= 0, \end{aligned} \tag{3.4}$$

and applying div_H implies

$$\nabla_H \pi = \nabla_H \Delta_H^{-1}\, \mathrm{div}_H \overline{f} + \nabla_H \Delta_H^{-1}\, \mathrm{div}_H D_z v \,|_{\Gamma_D}\,. \tag{3.5}$$

Inserting this expression for $\nabla_H \pi$ into (3.3) yields

$$\lambda v - \Delta v + \nabla_H \Delta_H^{-1} \mathrm{div}_H D_z v \,|_{\Gamma_D} = f - \nabla_H \Delta_H^{-1}\, \mathrm{div}_H \overline{f}.$$

For $f \in L^p(\Omega)$ we interpret this equation now as operator equation in $L^p(\Omega)$ as

$$\lambda v - \Delta_p v - B_p v = P_p f,$$

where P_p denotes the hydrostatic Helmholtz projection as described above and

$$B_p v := -\nabla_H \Delta_H^{-1} \mathrm{div}_H D_z v \,|_{\Gamma_D} \quad \text{with } D(B_p) := H^{1+1/p+\delta,p}(\Omega)^2$$

for some $\delta \in (0, 1 - 1/p)$. Obviously, $D(\Delta_p) \subset D(B_p)$. Moreover, we have

$$D(B_p) \overset{D_z \cdot |_{\Gamma_D}}{\longrightarrow} B^{\delta}_{pp}(G)^2 \cong W^{\delta,p}(G)^2 \hookrightarrow L^p(G)^2 \overset{-\nabla_H \Delta_H^{-1} \mathrm{div}_H}{\longrightarrow} L^p(G)^2 \hookrightarrow L^p(\Omega)^2,$$

where $W^{\delta,p}(G)$ denotes the Sobolev-Slobodeckii space on G of order δ. Boundedness of the trace operator, interpolation and Young's inequality imply

$$\|B_p v\|_{L^p(\Omega)^2} \le \varepsilon \|\Delta_p v\|_{L^p(\Omega)^2} + C_\varepsilon \|v\|_{L^p(\Omega)^2}, \quad v \in D(\Delta_p),$$

for $\varepsilon > 0$ arbitrarily small and some $C_\varepsilon > 0$. Therefore, B_p is a relatively bounded perturbation of Δ_p. Perturbation results for the H^∞-calculus, see e.g. [9], [19], [34], imply then the assertion of Theorem 3.1.

The proof of the global $L^p - L^q$-estimates for the hydrostatic semigroup given in Proposition 3.7 is based on the following lemma.

Lemma 3.8. *There is a continuous extension operator $S\colon L^p(\Omega) \to L^p(\mathbb{R}^3)$, $p \in (1, \infty)$, which is also continuous with respect to the $H^{s,p}$-norm for all $s \in [0, \infty)$. In particular, $[L^p(\Omega), H^{2,p}(\Omega)]_\theta = H^{2\theta,p}(\Omega)$ for $\theta \in [0, 1]$.*

Having Lemma 3.8 in hand, the proof of Proposition is now rather short. Setting $\alpha := 3(\frac{1}{p} - \frac{1}{q})$ and assuming $|\frac{1}{p} - \frac{1}{q}| < \frac{2}{3}$, the first inequality follows from

$$\begin{aligned} \|e^{tA_p} P_p f\|_{L^q(\Omega)^2} &\le C \|e^{tA_p} P_p f\|_{H^{\alpha,p}(\Omega)^2} \le C \|e^{tA_p} P_p f\|^{1-\frac{\alpha}{2}}_{L^p(\Omega)^2} \|e^{tA_p} P_p f\|^{\frac{\alpha}{2}}_{H^{2,p}(\Omega)^2} \\ &\le C \|f\|^{1-\frac{\alpha}{2}}_{L^p(\Omega)^2} \|A_p e^{tA_p} P_p f\|^{\frac{\alpha}{2}}_{L^p(\Omega)^2} \le C \|f\|^{1-\frac{\alpha}{2}}_{L^p(\Omega)^2} \left(t^{-1} \|f\|_{L^p(\Omega)^2}\right)^{\frac{\alpha}{2}} \\ &= C t^{-\frac{3}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} \|f\|_{L^p(\Omega)^2}, \quad t > 0, \end{aligned}$$

where we used the Sobolev embedding $H^{\alpha,p}(\Omega) \hookrightarrow L^q(\Omega)$, Lemma 3.8 and the fact that the semigroup e^{tA_p} is bounded analytic. Iterating, we obtain the first inequality for all $1 < p \le q < \infty$. The other inequalities follow similarly.

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