Prog. Theor. Exp. Phys. **2017**, 063B09 (24 pages)
DOI: 10.1093/ptep/ptx086

Axial U(1) current in Grabowska and Kaplan's formulation

Yu Hamada* and Hikaru Kawai*

Department of Physics, Kyoto University, Kyoto 606-8502, Japan
*E-mail: yu.hamada@gauge.scphys.kyoto-u.ac.jp; hkawai@gauge.scphys.kyoto-u.ac.jp

Received May 2, 2017; Revised May 22, 2017; Accepted May 29, 2017; Published June 30, 2017

Recently, Grabowska and Kaplan [Phys. Rev. Lett. **116**, 211602 (2016); Phys. Rev. D **94**, 114504 (2016)] suggested a nonperturbative formulation of a chiral gauge theory, which consists of the conventional domain-wall fermion and a gauge field that evolves by gradient flow from one domain wall to the other. In this paper, we discuss the U(1) axial-vector current in 4 dimensions using this formulation. We introduce two sets of domain-wall fermions belonging to complex conjugate representations so that the effective theory is a 4D vector-like gauge theory. Then, as a natural definition of the axial-vector current, we consider a current that generates simultaneous phase transformations for the massless modes in 4 dimensions. However, this current is exactly conserved and does not reproduce the correct anomaly. In order to investigate this point precisely, we consider the mechanism of the conservation. We find that this current includes not only the axial current on the domain wall but also a contribution from the bulk, which is nonlocal in the sense of 4D fields. Therefore, the local current is obtained by subtracting the bulk contribution from it.

Subject Index B01, B33, B38

1. Introduction

Formulating a chiral gauge theory nonperturbatively is a long-standing problem [1–15]. Recently [16,17], Grabowska and Kaplan suggested a formulation that consists of the domain-wall fermion in $2n+1$ dimensions and a gauge field that evolves by gradient flow from one domain wall to the other. A long-distance flow makes the gauge field pure gauge, and thus one of the massless modes ("fluffy mirror fermion" or "fluff") does not couple with the gauge field. Therefore, we obtain a chiral gauge theory including only the other massless mode that couples with the gauge field. However, the heavy modes in the bulk induce some terms that cannot be renormalized to the 4D Lagrangian. To cancel the bulk terms, Grabowska and Kaplan introduced a subtracting field, which has a loop factor +1 and a constant mass. It is known that the cancellation is not complete, but there remains a Chern–Simons-like term [18,19]. However, if the anomaly-free condition $d^{abc} = 0$ is satisfied, the Chern–Simons-like term vanishes and then we obtain the 4D local theory.

In order to investigate the consistency of this formulation, we consider a vector-like theory by introducing two sets of domain-wall fermions belonging to complex conjugate representations [17,20–22] (H. Suzuki and O. Morikawa, personal communication). Each of the fermions induces one left-handed physical fermion and one right-handed fluff fermion. Therefore, if the fluffs are decoupled correctly, we have a 4D vector-like gauge theory with one right-handed and one left-handed chiral fermion after we apply the charge conjugation to one of the physical fermions. In this paper, we consider the U(1) axial-vector current and discuss how the anomaly arises. We first define a current that generates simultaneous phase transformations for the left-handed physical fermions in 4 dimensions. From the

Funded by SCOAP^3

viewpoint of the effective theory, this current looks like the U(1) axial-vector current. However, it is pointed out in Ref. [20] that this current is exactly conserved and does not reproduce the correct anomaly (H. Suzuki and O. Morikawa, personal communication). In order to solve this paradox, we investigate the mechanism of the conservation. We find that this current contains a bulk contribution in addition to the axial-vector current on the domain wall. Therefore, the proper local current is obtained by subtracting the bulk part.

This paper is organized as follows. In Sect. 2, we review the formulation of Grabowska and Kaplan in the lattice space. In Sect. 3, we consider a regularization of this formulation in the continuum space in order to simplify the calculations in the subsequent sections. We find that one needs to introduce Pauli–Villars fields for both the domain-wall fermion and the subtracting field. In Sect. 4, we calculate the 1-loop effective action to the quadratic order in the gauge fields, and confirm that the effective action consists of three parts. One is equal to the effective action of a chiral fermion with Pauli–Villars-like regularization. The second is the Chern–Simons term in the bulk. The third are various divergent terms, which will be canceled by our regularization. In Sect. 5, we discuss the axial-vector current in 4 dimensions. In Sect. 6, we give a summary and conclusions.

2. Review of Grabowska and Kaplan's method

We review the formulation of Grabowska and Kaplan. There are recent studies [21–23] based on this formulation. In this section, we consider the lattice space although we use the symbols in the continuum space. We will discuss the continuum regularization in the next section.

We start with a domain-wall fermion in $2n$+1 dimensions:

$$\mathcal{L} = \bar{\psi}\left[\partial\!\!\!/_{2n+1} - M\epsilon(s)\right]\psi. \tag{2.1}$$

Here ψ is the Dirac field with 2^n components. s is the $2n$+1th coordinate, $s \in [-L, L]$ with periodic boundary condition, and $\epsilon(s) = \mathrm{sgn}(s)$. In the limit of $L \to \infty$, two massless modes are localized on the $2n$-dimensional wall $s = 0$ and $s = L$, which have the chirality -1 and $+1$ respectively. The heavy modes that live in the bulk will be decoupled classically in the limit of $M \to \infty$. In order to obtain a chiral gauge theory, in which only the left-handed mode couples with the gauge field, the $2n$+1-dimensional gauge field $\bar{A}_\mu$ is constructed by the gradient flow [24–26] from $s = 0$ to $s = \pm L$:

$$\partial_s \bar{A}_\nu(x, s) = \frac{\epsilon(s)}{M'} D_\mu \bar{F}_{\mu\nu}, \tag{2.2}$$

with $\bar{A}_\mu(x, 0) = A_\mu(x)$, $\mu, \nu = 1, \ldots, 2n$, and $\bar{A}_{2n+1} = 0$. $\bar{F}_{\mu\nu}$ is the field strength of the gauge field $\bar{A}_\mu$. We assume that $M' \gg M$ so that $\bar{A}_\mu(x, s)$ is close to $A_\mu(x)$ near the domain wall $|s| \lesssim 1/M$. Since the gradient flow damps the physical degrees of freedom, the gauge field $\bar{A}_\mu$ becomes pure gauge[1] at $s = L$ in the limit of $L \to \infty$. Thus the right-handed mode on $s = L$ is decoupled and we obtain the $2n$-dimensional chiral gauge theory if the bulk degrees of freedom are decoupled.

[1] More precisely, it is also possible for the gauge field to attain an instanton configuration. We do not consider this case in this paper.

In order to cancel the bulk degrees of freedom[2], we introduce a "subtracting field"[3], which has a loop factor +1 and a constant mass $-M$. This setting is equivalent to defining the fermion determinant as follows:

$$\Delta(A) \equiv \frac{\det\left(\not{D}^{(R)}_{2n+1} - M\epsilon(s)\right)}{\det\left(\not{D}^{(R)}_{2n+1} + M\right)}, \tag{2.3}$$

where $\not{D}^{(R)}_{2n+1}$ is the $2n$+1-dimensional Dirac operator belonging to the representation R. Indeed the terms that are even functions of M in the bulk are canceled. On the other hand, for the odd terms, the parity anomaly survives and the effective action contains a bulk term [18,19]:

$$S^{(\mathrm{CS})}_{2n+1} = c_{2n+1}\frac{M}{|M|}\int [\epsilon(s)+1]\,\omega_{2n+1}. \tag{2.4}$$

Here

$$c_{2n+1} = \frac{i^n}{2^{n+1}\pi^n(n+1)!}, \tag{2.5}$$

and ω_{2n+1} is the $2n$+1-dimensional Chern–Simons form. $S^{(\mathrm{CS})}_{2n+1}$ vanishes if the representation satisfies the condition for the anomaly cancellation in $2n$ dimensions.

In order to perform the calculation easily, we consider a continuum version of this formulation in the following sections.

3. Regularization in the continuum formulation

In this section, we regularize the formulation given in Sect. 2 in the continuum space. The bare effective action corresponding to Eq. (2.3) is given by[4]

$$\log\Delta(A) = \mathrm{Tr}\log\left(\not{D}_{2n+1} - M\epsilon(s)\right) - \mathrm{Tr}\log\left(\not{D}_{2n+1} + M\right). \tag{3.1}$$

Here, we adopt the Pauli–Villars regularization[5].

We regularize the domain-wall fermion and the subtracting field respectively as follows:

$$\mathrm{Tr}\log\left(\not{D}_{2n+1} - M\epsilon(s)\right) \to \mathrm{Tr}\log\left(\not{D}_{2n+1} - M\epsilon(s)\right) + \sum_i C_i \mathrm{Tr}\log\left(\not{D}_{2n+1} - M_i\epsilon(s)\right), \tag{3.2}$$

$$\mathrm{Tr}\log\left(\not{D}_{2n+1} + M\right) \to \mathrm{Tr}\log\left(\not{D}_{2n+1} + M\right) + \sum_i C'_i \mathrm{Tr}\log\left(\not{D}_{2n+1} + M'_i\right). \tag{3.3}$$

Note that, while the subtracting field is regularized as usual, the domain-wall fermion is regularized by additional domain-wall fermions with mass $M_i\epsilon(s)$. The parameters C_i, M_i, C'_i, M'_i will be determined

[2] For the case that the gauge field is constant in the s direction, $\bar{A}_\mu(x,s) = A_\mu(x)$, we do not have to cancel the bulk terms because they can be absorbed in the 4D Lagrangian. However, this is not possible when we consider the gradient flow.

[3] In Refs. [16,17], this field is called a "Pauli–Villars field". But we distinguish this field from a conventional Pauli–Villars field, whose role is regularization.

[4] We drop the superscript "(R)" in $\not{D}_{2n+1}$.

[5] The dimensional regularization cannot be used for the 2^n-component Dirac field in $2n+1$ dimensions.

later so that the regularized effective action converges as usual. Here, we choose $C'_i = C_i$ and $M'_i = M_i$ so that the Pauli–Villars fields do not generate extra bulk effective action. In other words, we introduce pairs of Pauli–Villars fields consisting of a domain-wall fermion and a subtracting field, which we call Pauli–Villars pairs. Thus the regularized effective action is

$$\begin{aligned}\log\Delta(A)_{\text{reg.}} &= \mathrm{Tr}\log\left(\not{D}_{2n+1} - M\epsilon(s)\right) - \mathrm{Tr}\log\left(\not{D}_{2n+1} + M\right)\\ &\quad + \sum_i C_i\left[\mathrm{Tr}\log\left(\not{D}_{2n+1} - M_i\epsilon(s)\right) - \mathrm{Tr}\log\left(\not{D}_{2n+1} + M_i\right)\right].\end{aligned} \tag{3.4}$$

Let us write down the condition for the effective action to converge. For a necessary condition, divergences arising on the walls should be canceled. As we will see in Eq. (4.76), a pair of a domain-wall fermion and a subtracting field behaves like a chiral fermion with a Pauli–Villars-like field[6] around $s = 0$. Therefore, all pairs including the Pauli–Villars pairs give the following contribution to the effective action from the near-wall region:

$$\begin{aligned}&\mathrm{Tr}\log\left(\not{D}_{2n+1} - M\epsilon(s)\right) - \mathrm{Tr}\log\left(\not{D}_{2n+1} + M\right)\\ &\quad + \sum_i C_i\left[\mathrm{Tr}\log\left(\not{D}_{2n+1} - M_i\epsilon(s)\right) - \mathrm{Tr}\log\left(\not{D}_{2n+1} + M_i\right)\right]\\ &\xrightarrow{\text{around } s=0} \left[\mathrm{Tr}\log(\not{D}_{2n}P_- + \not{\partial}_{2n}P_+) - \mathrm{Tr}\log(\not{D}_{2n}P_- + \not{\partial}_{2n}P_+ - M)\right]\\ &\quad + \sum_i C_i\left[\mathrm{Tr}\log(\not{D}_{2n}P_- + \not{\partial}_{2n}P_+) - \mathrm{Tr}\log(\not{D}_{2n}P_- + \not{\partial}_{2n}P_+ - M_i)\right],\end{aligned} \tag{3.5}$$

where P_- and P_+ are the chirality projection operators. $\mathrm{Tr}\log(\not{D}_{2n}P_- + \not{\partial}_{2n}P_+)$ and $\mathrm{Tr}\log(\not{D}_{2n}P_- + \not{\partial}_{2n}P_+ - M)$ are the effective action of the left-handed chiral fermion and the Pauli–Villars-like field, respectively. We will derive Eq. (3.5) in Sect. 4. The conditions to cancel the divergences in Eq. (3.5) are[7]

$$\begin{aligned} M + \sum_i C_i M_i &= 0,\\ M^2 + \sum_i C_i (M_i)^2 &= 0,\\ M^3 + \sum_i C_i (M_i)^3 &= 0,\\ &\vdots \quad . \end{aligned} \tag{3.6}$$

Note that the leading divergences in Eq. (3.5), which are independent of M and M_i, are canceled in each pair.

Equations (3.6) are also sufficient to cancel the divergences from the bulk. In the bulk region $-L < s < 0$, the cancellation is trivial because the domain-wall fermions and the subtracting fields

[6] This $2n$-dimensional Pauli–Villars-like field is not the Pauli–Villars field that we have introduced in Eq. (3.2) and Eq. (3.3).

[7] Generally, we need d conditions in d dimensions.

have the same mass in each pair. In the bulk region $0 < s < L$, Eq. (3.4) reduces to

$$\begin{aligned}&\mathrm{Tr}\log\left(\not{D}_{2n+1} - M\right) - \mathrm{Tr}\log\left(\not{D}_{2n+1} + M\right)\\&\quad + \sum_i C_i\left[\mathrm{Tr}\log\left(\not{D}_{2n+1} - M_i\right) - \mathrm{Tr}\log\left(\not{D}_{2n+1} + M_i\right)\right].\end{aligned} \tag{3.7}$$

In Eq. (3.7), terms that are even functions of M and M_i are trivially canceled. On the other hand, the odd terms are canceled if the following conditions are satisfied:

$$\begin{aligned} M + \sum_i C_i M_i &= 0,\\ M^3 + \sum_i C_i (M_i)^3 &= 0,\\ &\vdots\quad , \end{aligned} \tag{3.8}$$

which are part of Eq. (3.6). Therefore, Eq. (3.6) is the necessary and sufficient condition for the effective action to converge.

However, we need to prevent the Pauli–Villars fields from changing the physical degrees of freedom. In fact, each of the Pauli–Villars pairs induces a massless mode on the wall and a Chern–Simons term in the bulk, which will not be decoupled even if we take the limit $M_i \to \infty$. Thus one observes $\sum_i C_i$ additional massless modes and Chern–Simons terms. These extra contributions vanish by imposing an additional condition:

$$\sum_i C_i = 0, \tag{3.9}$$

which we will confirm in Eq. (4.79).

Thus we conclude that a continuum version of the regularized effective action is given by Eq. (3.4) with Eqs. (3.6) and (3.9).

4. Calculation of the effective action

In this section, we calculate the regularized effective action, Eq. (3.4), by expanding with respect to the gauge field $\bar{A}_\mu$. In order to do this, it is sufficient to calculate one pair of a domain-wall fermion and a subtracting field:

$$\mathrm{Tr}\log(\not{D}_{2n+1} - M\epsilon(s)) - \mathrm{Tr}\log(\not{D}_{2n+1} + M). \tag{4.1}$$

The other pairs are obtained by replacing the mass and loop factor. As we will see later, Eq. (4.1) consists of three parts. One is the effective action of the $2n$-dimensional chiral fermions with a Pauli–Villars-like regularization. This confirms that the massless modes localized on the walls behave as chiral fermions even at the quantum level. The second is the Chern–Simons term in $2n$+1 dimensions. The third are various divergent terms, which will be canceled after summing up with the Pauli–Villars pairs.

4.1. Propagator of the domain-wall fermion

We begin with deriving the propagator of the domain-wall fermion in the continuum[8]. As we will see below, this propagator can be regarded as the sum of two processes. One is the bulk propagation with a constant mass $\pm M$. The other is the massless propagation along the domain walls. Thus this propagator includes both the heavy bulk modes and the massless domain-wall modes.

The propagator is a solution of the following equation:

$$\left[i\not{p} + \gamma^5 \partial_s - \epsilon(s) M\right] G(p, s; s') = \delta(s - s'), \tag{4.2}$$

where $G(p, s; s')$ is the Fourier transform of the propagator in $2n$ directions:

$$G(x, s; x', s') = \int \frac{d^{2n}p}{(2\pi)^{2n}}\, e^{-ip\cdot(x-x')}\, G(p, s; s'). \tag{4.3}$$

We use the symbol γ^5 as the chirality matrix even in $2n$+1 dimensions, i.e., $\gamma^5 \equiv \gamma^1 \cdots \gamma^{2n}$.

In order to concentrate on the modes around $s = 0$, we take the limit $L \to \infty$, and obtain the following expression (see Appendix A):

$$G(p, s; s') = \begin{cases} S^{(+)}(p, s - s') \ + \ \mathcal{D}^{(+)}(p)\, e^{-(s'+s)\sqrt{p^2+M^2}} & (0 < s, s') \\ S^{(-)}(p, s - s') \ + \ \mathcal{D}^{(-)}(p)\, e^{(s'+s)\sqrt{p^2+M^2}} & (s, s' < 0) \\ \mathcal{D}^{(-+)}(p)\, e^{(s-s')\sqrt{p^2+M^2}} & (s < 0 < s') \\ \mathcal{D}^{(+-)}(p)\, e^{(s'-s)\sqrt{p^2+M^2}} & (s' < 0 < s), \end{cases} \tag{4.4}$$

where

$$\begin{aligned} S^{(+)}(p, s - s') = &-\theta(s - s')\, \frac{i\not{p} + M - \sqrt{p^2 + M^2}\gamma^5}{2\sqrt{p^2 + M^2}}\, e^{(s'-s)\sqrt{p^2+M^2}} \\ &- \theta(s' - s)\, \frac{i\not{p} + M + \sqrt{p^2 + M^2}\gamma^5}{2\sqrt{p^2 + M^2}}\, e^{(s-s')\sqrt{p^2+M^2}}, \end{aligned} \tag{4.5}$$

$$\mathcal{D}^{(+)}(p) = -\frac{i\not{p} M (M + \sqrt{p^2 + M^2}\gamma^5)}{2p^2\sqrt{p^2 + M^2}} + \frac{M}{2\sqrt{p^2 + M^2}}, \tag{4.6}$$

$$\mathcal{D}^{(-+)}(p) = -\frac{i\not{p}(\sqrt{p^2 + M^2} + M\gamma^5)}{2p^2} - \frac{1}{2}\gamma^5, \tag{4.7}$$

and $\theta(s - s')$ is the step function. $S^{(-)}$ is obtained by replacing $M \to -M$ in $S^{(+)}$. $\mathcal{D}^{(-)}, \mathcal{D}^{(+-)}$ are obtained from $\mathcal{D}^{(+)}, \mathcal{D}^{(-+)}$, respectively, by replacing $M \to -M$ and $\gamma^5 \to -\gamma^5$. Note that $S^{(+)}$ and $S^{(-)}$ are the conventional propagators in $2n$+1 dimensions with constant mass M and $-M$, respectively, and represent the heavy modes. The other terms in Eq. (4.4) represent the massless modes localized on the wall $s = 0$.

These results are consistent with the physical intuition that the propagator $G(p, s; s')$ reduces to the conventional one with constant mass $\pm M$ in the region far from the domain wall, $s, s' \ll -1/M$ or $1/M \ll s, s'$.

[8] The propagator in the lattice theory is derived in Refs. [11,12].

4.2. Calculation of effective action: vacuum polarization

Let us expand Eq. (4.1) as follows:

$$\mathrm{Tr}\log(\not{D}_{2n+1} - \epsilon(s)M) - \mathrm{Tr}\log(\not{D}_{2n+1} + M) \tag{4.8}$$

$$= \sum_m \frac{1}{m} \left(\prod_i^m \int \frac{d^d k_i}{(2\pi)^d}\right) \left(\prod_i^m \int_{-L}^{L} ds_i\right) (2\pi)^d\, \delta^{(d)}(\sum_i k_i)$$

$$\mathrm{tr}\left[\bar{A}_{\mu_1}(k_1, s_1)\cdots \bar{A}_{\mu_m}(k_m, s_m)\right]\, \Gamma^{\mu_1\cdots\mu_m}(\{k_i\},\{s_i\}) \tag{4.9}$$

$$\equiv \sum_m \frac{1}{m} I_m^{(L)}, \tag{4.10}$$

where $d = 2n$ and $\Gamma^{\mu_1\cdots\mu_m}$ is the sum of the fermion loops with m vertices for the domain-wall fermion and the subtracting field. Note that $k_i (i = 1, \ldots, m)$ are the $2n$-dimensional momenta, and s_i are the $2n$+1th coordinates. As in the previous section, we take the limit $L \to \infty$ and consider

$$\lim_{L\to\infty} I_m^{(L)} \equiv I_m. \tag{4.11}$$

In the following, we give an explicit calculation for I_2, which is nothing but the vacuum polarization loop:

$$I_2 = \int \frac{d^d k}{(2\pi)^d} \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} ds'\ \mathrm{tr}\left[\bar{A}_\mu(-k, s')\bar{A}_\nu(k, s)\right] \Gamma^{\mu\nu}(k, s, s'), \tag{4.12}$$

where

$$\Gamma^{\mu\nu}(k, s, s') = \int \frac{d^d p}{(2\pi)^d} \left[\mathrm{tr}\left[\gamma^\mu G(p, s'; s)\gamma^\nu G(p', s; s')\right] - \mathrm{tr}\left[\gamma^\mu S^{(-)}(p, s'; s)\gamma^\nu S^{(-)}(p', s; s')\right]\right]. \tag{4.13}$$

Here p' stands for $p + k$ so that we must substitute $p' = p + k$ in Eq. (4.13) before integrating with respect to p. The second term in Eq. (4.13) comes from the subtracting field.

It is convenient to divide the range of s and s' into six regions:

$$\mathrm{I} : \{s' < s < 0\} \oplus \mathrm{II} : \{s < 0 < s'\} \oplus \mathrm{III} : \{0 < s' < s\} \oplus \mathrm{I}' \oplus \mathrm{II}' \oplus \mathrm{III}', \tag{4.14}$$

where the regions I, II, III correspond to diagrams in Fig. 1. I′, II′, III′ are obtained by interchanging $s \leftrightarrow s'$ in I, II, III, respectively.

We denote the contribution from region I by I:

$$\mathrm{I} \equiv \iint_{(\mathrm{I})} \mathrm{tr}\left[\bar{A}_\mu(-k, s')\bar{A}_\nu(k, s)\right]\ \Gamma^{\mu\nu}(k, s, s'). \tag{4.15}$$

In this region, the propagator G can be written as (see Eq. (4.4)):

$$G(p', s; s') = S^{(-)}(p', s - s') + D^{(-)}(p', s + s'), \tag{4.16}$$

and

$$G(p, s'; s) = S^{(-)}(p, s' - s) + D^{(-)}(p, s + s'), \tag{4.17}$$

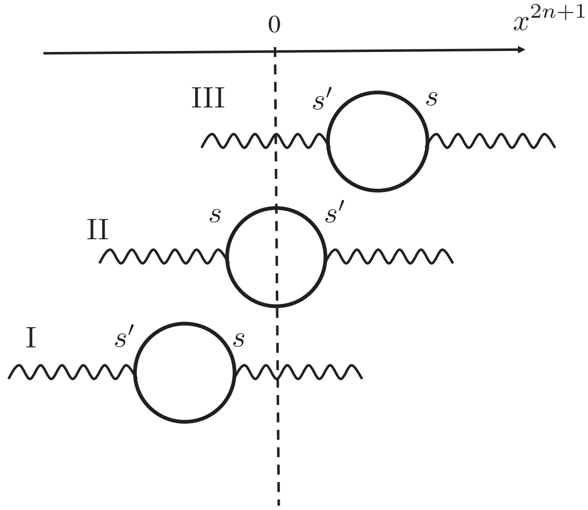

Fig. 1. Feynman diagrams for three regions I, II, III.

where

$$D^{(-)}(p, s+s') \equiv \mathcal{D}^{(-)}(p)\, e^{(s'+s)\sqrt{p^2+M^2}}. \tag{4.18}$$

Thus $\Gamma^{\mu\nu}(k, s, s')$ is[9]

$$\Gamma^{\mu\nu}(k, s, s') \tag{4.19}$$

$$= \int \frac{d^d p}{(2\pi)^d} \left[\mathrm{tr}\left[\gamma^\mu G\, \gamma^\nu G'\,\right] - \mathrm{tr}\left[\gamma^\mu S^{(-)} \gamma^\nu S'^{(-)}\right]\right] \tag{4.20}$$

$$= \int \frac{d^d p}{(2\pi)^d} \left[\mathrm{tr}\left[\gamma^\mu \left(S^{(-)} + D^{(-)}\right) \gamma^\nu \left(S'^{(-)} + D'^{(-)}\right)\right] - \mathrm{tr}\left[\gamma^\mu S^{(-)} \gamma^\nu S'^{(-)}\right]\right] \tag{4.21}$$

$$= \int \frac{d^d p}{(2\pi)^d} \left[\mathrm{tr}\left[\gamma^\mu D^{(-)} \gamma^\nu D'^{(-)}\right] + \mathrm{tr}\left[\gamma^\mu D^{(-)} \gamma^\nu S'^{(-)}\right] + \mathrm{tr}\left[\gamma^\mu S^{(-)} \gamma^\nu D'^{(-)}\right]\right] \tag{4.22}$$

$$\equiv \int \frac{d^d p}{(2\pi)^d}\, T^{(-)}_{\mathrm{local}}(p, p', s, s'), \tag{4.23}$$

where $T^{(-)}_{\mathrm{local}}(p, p', s, s')$ depends on s, s' as follows:

$$\begin{aligned} T^{(-)}_{\mathrm{local}}(p, p', s, s') = {} & \alpha(p, p')\, e^{(s+s')(\sqrt{p^2+M^2} + \sqrt{p'^2+M^2})} \\ & + \beta(p, p')\, e^{(s'+s)\sqrt{p^2+M^2}}\, e^{(s'-s)\sqrt{p'^2+M^2}} \\ & + \gamma(p, p')\, e^{(s'+s)\sqrt{p'^2+M^2}}\, e^{(s'-s)\sqrt{p^2+M^2}}. \end{aligned} \tag{4.24}$$

Here, α, β, γ are functions of p, p' and obtained from $\mathrm{tr}\left[\gamma^\mu D^{(-)} \gamma^\nu D'^{(-)}\right]$, $\mathrm{tr}\left[\gamma^\mu D^{(-)} \gamma^\nu S'^{(-)}\right]$, $\mathrm{tr}\left[\gamma^\mu S^{(-)} \gamma^\nu D'^{(-)}\right]$, respectively (see Appendix B). Note that the bulk term from the domain-wall fermion, $\mathrm{tr}\left[\gamma^\mu S^{(-)} \gamma^\nu S^{(-)}\right]$, has been canceled by the subtracting field, and there remains only

[9] We drop the arguments p, p', s, s' for simplicity. The symbols without primes such as $G, S^{(-)}, D^{(-)}$ mean that their arguments are $(p, s'; s)$. On the other hand, the symbols with primes stand for the arguments $(p', s; s')$.

$T^{(-)}_{\text{local}}$, which damps exponentially in s, s'. Therefore, $\Gamma^{\mu\nu}(k,s,s')$ in region I has values only in $-1/M \lesssim s', s < 0$.

Using this fact, we evaluate the integral with respect to s, s' in Eq. (4.15) as follows. First, we approximate that $\bar{A}(x,s)$, which evolves by the gradient flow, is constant in the region $-1/M \lesssim s', s < 0$. Thus we can write

$$\mathrm{I} = \iint_{(\mathrm{I})} \mathrm{tr}\left[\bar{A}_\mu(-k,s')\bar{A}_\nu(k,s)\right] \int \frac{d^dp}{(2\pi)^d}\, T^{(-)}_{\text{local}}(p,p',s,s') \tag{4.25}$$

$$\sim \mathrm{tr}\left[\bar{A}_\mu(-k,0)\bar{A}_\nu(k,0)\right] \iint_{(\mathrm{I})} \int \frac{d^dp}{(2\pi)^d}\, T^{(-)}_{\text{local}}(p,p',s,s') \tag{4.26}$$

$$= \mathrm{tr}\left[A_\mu(-k)A_\nu(k)\right] \int \frac{d^dp}{(2\pi)^d} \iint_{(\mathrm{I})} T^{(-)}_{\text{local}}(p,p',s,s'). \tag{4.27}$$

The approximation "$\sim$" will be exact if we take the limit $M' \to \infty$. Then, e.g., for the first term in Eq. (4.24), we have

$$\iint_{(\mathrm{I})} \alpha(p,p')\, e^{(s+s')(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})} \tag{4.28}$$

$$= \int_{-\infty}^{0} ds \int_{-\infty}^{s} ds'\, \alpha(p,p')\, e^{(s+s')(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})} \tag{4.29}$$

$$= \alpha(p,p') \int_{-\infty}^{0} ds \left(\left. \frac{e^{(s'+s)(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})}}{\sqrt{p^2+M^2}+\sqrt{p'^2+M^2}} \right|_{s'=s} \right) \tag{4.30}$$

$$= \alpha(p,p') \left. \frac{e^{2s(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})}}{2(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})^2} \right|_{s=0} \tag{4.31}$$

$$= \alpha(p,p') \frac{1}{2(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})^2}. \tag{4.32}$$

The other exponentials in Eq. (4.24) can be integrated in the same way and we obtain the expression Eq. (B.11).

We denote the contribution from region II by II:

$$\mathrm{II} \equiv \iint_{(\mathrm{II})} \mathrm{tr}\left[\bar{A}_\mu(-k,s')\bar{A}_\nu(k,s)\right] \Gamma^{\mu\nu}(k,s,s'). \tag{4.33}$$

The propagator G in this region is given by (see Eq. (4.4))

$$G(p',s;s') = \mathcal{D}^{(-+)}(p')\, e^{(s-s')\sqrt{p'^2+M^2}} \tag{4.34}$$

$$\equiv D^{(-+)}(p', s-s'), \tag{4.35}$$

$$G(p,s';s) = \mathcal{D}^{(+-)}(p)\, e^{(s-s')\sqrt{p^2+M^2}} \tag{4.36}$$

$$\equiv D^{(+-)}(p, s-s'). \tag{4.37}$$

Thus $\Gamma^{\mu\nu}(k,s,s')$ is

$$\Gamma^{\mu\nu}(k,s,s') \tag{4.38}$$
$$= \int \frac{d^dp}{(2\pi)^d}\left[\mathrm{tr}\left[\gamma^\mu G(p,s';s)\,\gamma^\nu G'(p',s;s')\,\right] - \mathrm{tr}\left[\gamma^\mu S^{(-)}\gamma^\nu S'^{(-)}\right]\right] \tag{4.39}$$
$$= \int \frac{d^dp}{(2\pi)^d}\left[\mathrm{tr}\left[\gamma^\mu D^{(+-)}\gamma^\nu D'^{(-+)}\right] - \mathrm{tr}\left[\gamma^\mu S^{(-)}\gamma^\nu S'^{(-)}\right]\right]. \tag{4.40}$$

Note that $\Gamma^{\mu\nu}$ has values only when $-1/M \lesssim s < 0 < s' \lesssim 1/M$ because the integrand in Eq. (4.40) is proportional to $\exp[(s-s')(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})]$. Therefore, the calculation can be performed similarly to region I, and we obtain the resulting expression Eq. (B.13).

We denote the contribution from region III by III:

$$\mathrm{III} \equiv \iint_{(\mathrm{III})} \mathrm{tr}\left[\bar{A}_\mu(-k,s')\bar{A}_\nu(k,s)\right]\,\Gamma^{\mu\nu}(k,s,s'). \tag{4.41}$$

In this region, the propagator G can be written as

$$G(p',s;s') = S^{(+)}(p',s-s') + D^{(+)}(p',s+s'), \tag{4.42}$$

and

$$G(p,s';s) = S^{(+)}(p,s'-s) + D^{(+)}(p,s+s'), \tag{4.43}$$

where

$$D^{(+)}(p,s+s') \equiv \mathcal{D}^{(+)}(p)\; e^{-(s'+s)\sqrt{p^2+M^2}}. \tag{4.44}$$

Thus $\Gamma^{\mu\nu}(k,s,s')$ is calculated as

$$\Gamma^{\mu\nu}(k,s,s') \tag{4.45}$$
$$= \int \frac{d^dp}{(2\pi)^d}\left[\mathrm{tr}\left[\gamma^\mu G\,\gamma^\nu G'\,\right] - \mathrm{tr}\left[\gamma^\mu S^{(-)}\gamma^\nu S'^{(-)}\right]\right] \tag{4.46}$$
$$= \int \frac{d^dp}{(2\pi)^d}\left[\mathrm{tr}\left[\gamma^\mu\left(S^{(+)}+D^{(+)}\right)\gamma^\nu\left(S'^{(+)}+D'^{(+)}\right)\right] - \mathrm{tr}\left[\gamma^\mu S^{(-)}\gamma^\nu S'^{(-)}\right]\right] \tag{4.47}$$
$$= \int \frac{d^dp}{(2\pi)^d}\left[\mathrm{tr}\left[\gamma^\mu D^{(+)}\gamma^\nu D'^{(+)}\right] + \mathrm{tr}\left[\gamma^\mu D^{(+)}\gamma^\nu S'^{(+)}\right] + \mathrm{tr}\left[\gamma^\mu S^{(+)}\gamma^\nu D'^{(+)}\right]\right]$$
$$+ \int \frac{d^dp}{(2\pi)^d}\left[\mathrm{tr}\left[\gamma^\mu S^{(+)}\gamma^\nu S'^{(+)}\right] - \mathrm{tr}\left[\gamma^\mu S^{(-)}\gamma^\nu S'^{(-)}\right]\right] \tag{4.48}$$
$$\equiv \int \frac{d^dp}{(2\pi)^d}\left[T^{(+)}_{\mathrm{local}}(p,p',s,s') + T_{\mathrm{bulk}}(p,p',s,s')\right], \tag{4.49}$$

where

$$T_{\text{bulk}} \equiv \text{tr}\left[\gamma^\mu S^{(+)} \gamma^\nu S'^{(+)}\right] - \text{tr}\left[\gamma^\mu S^{(-)} \gamma^\nu S'^{(-)}\right], \tag{4.50}$$

$$T^{(+)}_{\text{local}} \equiv \text{tr}\left[\gamma^\mu D^{(+)} \gamma^\nu D'^{(+)}\right] + \text{tr}\left[\gamma^\mu D^{(+)} \gamma^\nu S'^{(+)}\right] + \text{tr}\left[\gamma^\mu S^{(+)} \gamma^\nu D'^{(+)}\right] \tag{4.51}$$

$$\begin{aligned} &= \alpha(p,p')\, e^{-(s+s')(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})} \\ &\quad + \beta(p,p')\, e^{-(s'+s)\sqrt{p^2+M^2}}\, e^{(s'-s)\sqrt{p'^2+M^2}} \\ &\quad + \gamma(p,p')\, e^{-(s'+s)\sqrt{p'^2+M^2}}\, e^{(s'-s)\sqrt{p^2+M^2}}. \end{aligned} \tag{4.52}$$

Here, α, β, γ in Eq. (4.52) are the same as those in Eq. (4.24) (see Appendix B). Note that $T^{(+)}_{\text{local}}$, which is localized on $s = 0$, gives the same contribution as $T^{(-)}_{\text{local}}$ after integrating over s and s'. On the other hand, T_{bulk} is the bulk term, which does not vanish, unlike region I, because of the opposite signs of the masses. We will discuss this point in the next subsection.

Next, we consider the other regions I$'$, II$'$, III$'$. The net effect of interchanging $s \leftrightarrow s'$ is to change the signs of γ^5 in $S^{(+)}$ and $S^{(-)}$ (see Eq. (4.5)). Therefore the contributions from I$'$, II$'$, III$'$ are obtained from those of I, II, III, respectively, by changing the signs of γ^5 in $S^{(+)}$ and $S^{(-)}$ (see the discussions below Eqs. (B.11) and (B.13)).

All contributions (Eqs. (B.11), (B.13), (4.49)) sum up to

$$I_2 = \int \frac{d^dk}{(2\pi)^d} \left(\text{I} + \text{II} + \text{III} + \text{I}' + \text{II}' + \text{III}'\right) \tag{4.53}$$

$$= \int \frac{d^dk}{(2\pi)^d}\, \text{tr}[A_\mu(-k)A_\nu(k)] \left(\Pi^{\mu\nu}_{(\text{nonanomalous})} + \Pi^{\mu\nu}_{(\text{anomalous})}\right) + I_2^{\text{bulk}}(\bar{A}), \tag{4.54}$$

where

$$\Pi^{\mu\nu}_{(\text{nonanomalous})} \equiv \int \frac{d^dp}{(2\pi)^d} \left[\frac{2^n M^2 (p^2 + p'^2 + M^2) N^{\mu\nu}}{2p^2 p'^2 (p^2+M^2)(p'^2+M^2)} - \frac{2^n M^2 \delta^{\mu\nu}}{2(p^2+M^2)(p'^2+M^2)}\right], \tag{4.55}$$

$$\Pi^{\mu\nu}_{(\text{anomalous})} \equiv \int \frac{d^dp}{(2\pi)^d} \frac{-M\, \text{tr}\left[\gamma^\mu \not{p} \gamma^\nu \not{p}' \gamma^5\right] \left(p^2 + p'^2 + M^2 + \sqrt{p^2+M^2}\sqrt{p'^2+M^2}\right)}{2p^2p'^2\sqrt{p^2+M^2}\sqrt{p'^2+M^2}(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})}, \tag{4.56}$$

$$I_2^{\text{bulk}}(\bar{A}) \equiv \int \frac{d^dk}{(2\pi)^d} \int_0^\infty ds \int_0^\infty ds'\, \text{tr}[\bar{A}_\mu(-k,s')\bar{A}_\nu(k,s)] \int \frac{d^dp}{(2\pi)^d}\, T_{\text{bulk}}(p,p',s,s'). \tag{4.57}$$

Here, $N^{\mu\nu} \equiv p \cdot p' \delta^{\mu\nu} - p^\mu p'^\nu - p^\nu p'^\mu$. The first term in Eq. (4.54) represents the contribution from the localized terms. $\Pi^{\mu\nu}_{(\text{anomalous})}$ and $\Pi^{\mu\nu}_{(\text{nonanomalous})}$ are the parts with and without γ^5, respectively. As we will see in the next subsection,

$$\Pi^{\mu\nu}_{(\text{nonanomalous})} + \Pi^{\mu\nu}_{(\text{anomalous})} \tag{4.58}$$

is equal to the vacuum polarization of a left-handed chiral fermion with a Pauli–Villars-like regulator of mass M. I_2^{bulk} represents the contribution from the bulk region $0 < s, s' < \infty$.

Note that there are no leading ultraviolet (UV) divergences, terms that have degree of divergence $d-2$, in Eqs. (4.55)–(4.57). Therefore, all UV divergences are canceled by the Pauli–Villars pairs under the conditions of Eq. (3.6).

4.3. Comparison with chiral fermion in 2n dimensions

We first consider the vacuum polarization of a left-handed chiral fermion:

$$\int \frac{d^d p}{(2\pi)^d} \mathrm{tr}\left[\gamma^\mu P_- \frac{i\not p}{p^2}\gamma^\nu P_- \frac{i\not p'}{p'^2}\right], \tag{4.59}$$

where $P_- = \frac{1-\gamma^5}{2}$. By introducing one Pauli–Villars field, we have

$$\int \frac{d^d p}{(2\pi)^d}\left(\mathrm{tr}\left[\gamma^\mu P_- \frac{i\not p}{p^2}\gamma^\nu P_- \frac{i\not p'}{p'^2}\right] - \mathrm{tr}\left[\gamma^\mu P_- \frac{i\not p + M}{p^2+M^2}\gamma^\nu P_- \frac{i\not p' + M}{p'^2+M^2}\right]\right) \tag{4.60}$$

$$= V^{\mu\nu}_{(\text{nonanomalous})} + V^{\mu\nu}_{(\text{anomalous})}, \tag{4.61}$$

where

$$V^{\mu\nu}_{(\text{nonanomalous})} \equiv \int \frac{d^d p}{(2\pi)^d}\left(\frac{-M^2(p^2+p'^2+M^2)\,\mathrm{tr}\left[\gamma^\mu \not p \gamma^\nu \not p'\right]}{2p^2p'^2(p^2+M^2)(p'^2+M^2)} - \frac{M^2\mathrm{tr}\left[\gamma^\mu\gamma^\nu\right]}{2(p^2+M^2)(p'^2+M^2)}\right), \tag{4.62}$$

$$V^{\mu\nu}_{(\text{anomalous})} \equiv \int \frac{d^d p}{(2\pi)^d}\left(-\frac{\mathrm{tr}\left[\gamma^\mu \not p \gamma^\nu \not p' \gamma^5\right]}{2p^2p'^2} + \frac{\mathrm{tr}\left[\gamma^\mu \not p \gamma^\nu \not p' \gamma^5\right]}{2(p^2+M^2)(p'^2+M^2)}\right). \tag{4.63}$$

The nonanomalous part $V^{\mu\nu}_{(\text{nonanomalous})}$ is precisely equal to Eq. (4.55).

We evaluate the difference between Eq. (4.56) and Eq. (4.63), and show that it is zero in the limit of $M \to \infty$. This is trivial for $d > 2$ since both of them vanish. Thus we consider the case $d = 2$. The difference is calculated as:

$$\begin{aligned} F_M(k) &\equiv V^{\mu\nu}_{(\text{anomalous})} - \Pi^{\mu\nu}_{(\text{anomalous})} \\ &= \int \frac{d^2 p}{(2\pi)^2}\left[-\frac{M^2(p^2+p'^2+M^2)\,\mathrm{tr}\left[\gamma^\mu \not p \gamma^\nu \not p' \gamma^5\right]}{2p^2p'^2(p^2+M^2)(p'^2+M^2)}\right. \\ &\quad \left. + \frac{M\,\mathrm{tr}\left[\gamma^\mu \not p \gamma^\nu \not p' \gamma^5\right]\left(p^2+p'^2+M^2+\sqrt{p^2+M^2}\sqrt{p'^2+M^2}\right)}{2p^2p'^2\sqrt{p^2+M^2}\sqrt{p'^2+M^2}(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})}\right]. \end{aligned} \tag{4.64}$$

Because this integral is finite at $k = 0$, we can expand it around $k = 0$:

$$F_M(k) = F_M(0) + \mathcal{O}\left(\frac{k}{M}\right), \tag{4.65}$$

where

$$F_M(0)$$

$$= \int \frac{d^2 p}{(2\pi)^2}\frac{M\,\mathrm{tr}\left[\gamma^\mu \not p \gamma^\nu \not p \gamma^5\right]}{(p^2)^2(p^2+M^2)^2}\left(-\frac{1}{2}M(2p^2+M^2) + \frac{1}{4}\sqrt{p^2+M^2}(3p^2+2M^2)\right) \tag{4.66}$$

$$\propto \mathrm{tr}\left[\gamma^\mu\gamma^\lambda\gamma^\nu\gamma_\lambda\gamma^5\right] \tag{4.67}$$

$$= 0. \tag{4.68}$$

Thus we obtain

$$\lim_{M\to\infty} F_M(k) = 0. \tag{4.69}$$

Therefore, Eq. (4.58) is equal to Eq. (4.61) in the limit of $M \to \infty$.

We extend the above result to general cases $m > 2$. It is expected that I_m given by Eq. (4.11) is also written as

$$I_m = I_m^{s=0} + I_m^{\rm bulk}, \tag{4.70}$$

where $I_m^{s=0}$ is the m-vertex loop of the left-handed chiral fermion with the Pauli–Villars field on the domain wall $s = 0$. $I_m^{\rm bulk}$ is the m-vertex loop of the heavy mode and the subtracting field in the bulk region $0 < s < \infty$. Note that the divergent terms that are included in $I_m^{s=0}$ and $I_m^{\rm bulk}$ will be canceled by adding the Pauli–Villars pairs. Then, from Eq. (4.70), we have

$$\lim_{L\to\infty} \left[\mathrm{Tr}\log(\not{D}_{2n+1} - \epsilon(s)M) - \mathrm{Tr}\log(\not{D}_{2n+1} + M)\right] \tag{4.71}$$

$$= \sum_m \frac{1}{m} I_m^{s=0} + \sum_m \frac{1}{m} I_m^{\rm bulk}. \tag{4.72}$$

The first term in Eq. (4.72) can be regarded as

$$\sum_m \frac{1}{m} I_m^{s=0} = \mathrm{Tr}\log(\not{D}_{2n}P_- + \not{\partial}_{2n}P_+) - \mathrm{Tr}\log(\not{D}_{2n}P_- + \not{\partial}_{2n}P_+ - M). \tag{4.73}$$

Here, the first and second terms in Eq. (4.73) are the effective actions of the left-handed chiral fermion and the Pauli–Villars field, respectively. On the other hand, the second term in Eq. (4.72) can be written as [18,19]:

$$\sum_m \frac{1}{m} I_m^{\rm bulk} = S_{2n+1}^{\rm (CS)} + \delta S_{2n+1}(M), \tag{4.74}$$

where $S_{2n+1}^{\rm (CS)}$ is the Chern–Simons term given by Eq. (2.4). The UV divergence in $\delta S_{2n+1}(M)$ is canceled after combining with the Pauli–Villars pairs.

So far, we have neglected the domain wall $s = L$ by taking the limit $L \to \infty$. There, a similar result for the right-handed chiral fermion to Eq. (4.73) should be obtained. Therefore the effective action Eq. (4.8) is

$$\mathrm{Tr}\log(\not{D}_{2n+1} - \epsilon(s)M) - \mathrm{Tr}\log(\not{D}_{2n+1} + M) \tag{4.75}$$

$$\begin{aligned} &= \mathrm{Tr}\log(\not{D}_{2n}P_- + \not{\partial}_{2n}P_+)_{s=0} - \mathrm{Tr}\log(\not{D}_{2n}P_- + \not{\partial}_{2n}P_+ - M)_{s=0} \\ &\quad + \mathrm{Tr}\log(\not{D}_{2n}P_+ + \not{\partial}_{2n}P_-)_{s=L} - \mathrm{Tr}\log(\not{D}_{2n}P_+ + \not{\partial}_{2n}P_- - M)_{s=L} \\ &\quad + S_{2n+1}^{\rm (CS)} + \delta S_{2n+1}(M). \end{aligned} \tag{4.76}$$

Here, $()_{s=0}$ and $()_{s=L}$ stand for substituting the gauge fields $\bar{A}(x, s = 0)$ and $\bar{A}(x, s = L)$ into the covariant derivative, respectively.

By adding the Pauli–Villars pairs, the regularized effective action is obtained as follows. For the sake of simplicity, we write Eq. (3.4) as

$$\log \Delta(A)_{\text{reg.}} = \sum_{i=0} C_i \left[\operatorname{Tr} \log \left(\not{D}_{2n+1} - M_i \epsilon(s) \right) - \operatorname{Tr} \log \left(\not{D}_{2n+1} + M_i \right) \right], \tag{4.77}$$

where $C_0 = 1$ and $M_0 = M$. By applying Eq. (4.76) to each pair, we have

$$\begin{aligned}
\log \Delta(A)_{\text{reg.}} = & \sum_{i=0} C_i \left[\operatorname{Tr} \log(\not{D}_{2n} P_- + \not{\partial}_{2n} P_+)_{s=0} - \operatorname{Tr} \log(\not{D}_{2n} P_- + \not{\partial}_{2n} P_+ - M_i)_{s=0} \right] \\
& + \sum_{i=0} C_i \left[\operatorname{Tr} \log(\not{D}_{2n} P_+ + \not{\partial}_{2n} P_-)_{s=L} - \operatorname{Tr} \log(\not{D}_{2n} P_+ + \not{\partial}_{2n} P_- - M_i)_{s=L} \right] \\
& + \sum_{i=0} C_i \, S_{2n+1}^{(\text{CS})} + \sum_{i=0} C_i \, \delta S_{2n+1}(M_i) \qquad (4.78) \\
= & \operatorname{Tr} \log(\not{D}_{2n} P_- + \not{\partial}_{2n} P_+)_{s=0} - \sum_{i=0} C_i \operatorname{Tr} \log(\not{D}_{2n} P_- + \not{\partial}_{2n} P_+ - M_i)_{s=0} \\
& + \operatorname{Tr} \log(\not{D}_{2n} P_+ + \not{\partial}_{2n} P_-)_{s=L} - \sum_{i=0} C_i \operatorname{Tr} \log(\not{D}_{2n} P_+ + \not{\partial}_{2n} P_- - M_i)_{s=L} \\
& + S_{2n+1}^{(\text{CS})}. \qquad (4.79)
\end{aligned}$$

The last term in Eq. (4.78) is UV finite and vanishes in the limit of $M, M_i \to \infty$, which we drop in the following expressions. As argued in Sect. 3, the extra massless modes and Chern–Simons terms have vanished by the condition $\sum_{i=1} C_i = 0$. Thus there are no artificial degrees of freedom. In addition, the regularized effective action Eq. (4.79) converges under the condition of Eq. (3.6).

Note that Eq. (4.79) is gauge invariant because gauge anomalies from the three lines are canceled. For example, for $n = 2$, the gauge variation of the Chern–Simons term is

$$\delta_\chi S_5^{(\text{CS})} = \frac{-1}{48\pi^2} \int d^4x \, \epsilon_{\mu\nu\lambda\rho} \ \operatorname{tr} \left[\chi \ \partial_\mu \left(\bar{A}_\nu \bar{A}_\lambda \bar{A}_\rho + 2\bar{A}_\nu \partial_\lambda \bar{A}_\rho \right) \right] \Big|_{s=0}^{s=L}, \tag{4.80}$$

where χ is the gauge function. On the other hand, the first and second lines in Eq. (4.79) give the anomaly of the left- and right-handed chiral fermions in 4 dimensions, respectively, which cancel with Eq. (4.80). This cancellation agrees with the manifestly gauge-invariant construction, Eq. (3.4).

5. Axial-vector current in vector-like gauge theory

We investigate the consistency of this formulation by introducing two sets of domain-wall fermions belonging to complex conjugate representations. As a simple example, we consider a 5D U(1) gauge theory. We assume that each set of fermions consists of a domain-wall fermion, a subtracting field, and Pauli–Villars pairs. The two domain-wall fermions ψ and ψ' have U(1) charge ± 1, respectively. While left-handed physical fermions are localized on $s = 0$, right-handed fluff fermions are localized on $s = L$. We denote the former coming from ψ and ψ' by q_L and q'_L, respectively. Because the gradient flow makes the fluff fermions decouple, we obtain a 4D effective theory consisting of the left-handed physical fermions q_L, q'_L. Here, the Chern–Simons term vanishes due to the representations, and the effective theory is equivalent to the vector-like theory after applying the charge conjugation: $q_R \equiv q_L'^C$. In the following, we will show that the axial-vector current that is defined naturally does not reproduce the correct anomaly (H. Suzuki and O. Morikawa, personal communication, and Ref. [20]).

One natural way to define such current is to introduce a fictitious U(1) gauge field B_μ that couples to q_L and q'_L with charge +1. Then the current is defined by the variation with respect to the gauge field $B_\mu(x)$. In order to realize it, we consider the bulk U(1) gauge field $\bar{B}_\mu(x,s)$ that couples to ψ and ψ' with charge +1. We assume that $\bar{B}_\mu$ also evolves by the gradient flow from $s=0$ to $s=\pm L$:

$$\partial_s \bar{B}_\nu = \frac{\epsilon(s)}{M''}\partial_\mu \bar{F}^{(B)}_{\mu\nu}\,, \tag{5.1}$$

with $\mu,\nu = 1,\ldots,4$, $\bar{B}_\mu(x,0) = B_\mu(x)$, and $\bar{B}_5 = 0$. $\bar{F}^{(B)}_{\mu\nu}$ denotes the field strength of $\bar{B}_\mu$, and $M'' \gg M$ as M' in Eq. (2.2). Then, we define $J^B_\mu(x)$ by

$$\langle J^B_\mu(x)\rangle_A \equiv \left.\frac{\delta S_{\rm eff}[\bar{A},\bar{B}]}{\delta B_\mu(x)}\right|_{B_\mu=0}, \tag{5.2}$$

where $\mu = 1,\ldots,4$. $S_{\rm eff}[\bar{A},\bar{B}]$ is the effective action obtained by integrating out ψ and ψ'. The symbol $\langle\rangle_A$ stands for the expectation value in the presence of the background gauge field A_μ, which we drop in the expressions below. J^B_μ seems to be the U(1) axial-vector current:

$$J^B_\mu(x) \sim \bar{q}_L\gamma^\mu q_L + \bar{q}'_L\gamma^\mu q'_L \tag{5.3}$$

$$= \bar{q}_L\gamma^\mu q_L - \bar{q}_R\gamma^\mu q_R. \tag{5.4}$$

However, it does not reproduce the correct axial anomaly. Indeed, as we will see below, J^B_μ is exactly conserved (H. Suzuki and O. Morikawa, personal communication, and Ref. [20]):

$$\partial_\mu J^B_\mu(x) = 0. \tag{5.5}$$

On the other hand, from the viewpoint of the 5D theory, this conservation is natural because this current is a Noether current of this system. In order to solve this paradox, we investigate the mechanism of this conservation.

First we discuss how the effective action changes under the gauge transformation of $B_\mu(x)$:

$$B_\mu(x) \mapsto B_\mu(x) + \partial_\mu\chi(x). \tag{5.6}$$

Because $\bar{B}_\mu(x,s)$ is changed as follows:

$$\bar{B}_\mu(x,s) \mapsto \bar{B}_\mu(x,s) + \partial_\mu\chi(x), \tag{5.7}$$

the variation of the effective action $S_{\rm eff}[\bar{A},\bar{B}]$ can be written in the following two ways:

$$\delta S_{\rm eff} = \int d^4x\, \partial_\mu\chi(x)\, \frac{\delta S_{\rm eff}[\bar{A},\bar{B}]}{\delta B_\mu(x)} \tag{5.8}$$

$$= \int d^4x \int ds\, \partial_\mu\chi(x)\, \frac{\delta S_{\rm eff}[\bar{A},\bar{B}]}{\delta \bar{B}_\mu(x,s)}. \tag{5.9}$$

Thus we obtain

$$\partial_\mu J^B_\mu(x) = \int_0^L ds\, \partial_\mu j^B_\mu(x,s), \tag{5.10}$$

where

$$j_\mu^B(x,s) \equiv \left.\frac{\delta S_{\rm eff}[\bar{A},\bar{B}]}{\delta \bar{B}_\mu(x,s)}\right|_{\bar{B}_\mu=0}. \tag{5.11}$$

Note that the region $-L < s < 0$ has no contribution to Eq. (5.10) because no terms are induced there, as we have seen in Sect. 4.2. The above expression indicates that there is a contribution from the bulk to the divergence of the current as well as that from the domain wall, Eq. (5.3).

As we have seen in the previous section, $S_{\rm eff}[\bar{A},\bar{B}]$ consists of the effective action of the chiral fermions on $s = 0, L$ and the Chern–Simons term in the bulk, in the limit of $M \to \infty$. Thus we can write

$$\int_0^L ds\, j_\mu^B(x,s) = J_\mu^{(q_L,q_R)}(x) + J_\mu^{(\rm fluff)}(x) + J_\mu^{(\rm CS)}(x), \tag{5.12}$$

where $J_\mu^{(q_L,q_R)}$, $J_\mu^{(\rm fluff)}$ are currents of the chiral fermions on each boundary, and

$$J_\mu^{(\rm CS)}(x) \equiv \int_0^L ds\, j_\mu^{(\rm CS)}(x,s). \tag{5.13}$$

$j_\mu^{(\rm CS)}(x,s)$ is the Chern–Simons current:

$$j_\mu^{(\rm CS)}(x,s) \equiv \left.\frac{\delta}{\delta \bar{B}_\mu} S_5^{(\rm CS)}(\bar{A},\bar{B})\right|_{\bar{B}=0} \tag{5.14}$$

$$= \frac{-1}{24\pi^2}\, \frac{\delta}{\delta \bar{B}_\mu} \int \omega_5(\bar{A},\bar{B})\Big|_{\bar{B}=0}. \tag{5.15}$$

In the presence of the gauge fields $\bar{A}$ and $\bar{B}$, the Chern–Simons form ω_5 is

$$\int \omega_5(\bar{A},\bar{B}) \tag{5.16}$$

$$= \int [\ \{d(\bar{A}+\bar{B})\}^2\ (\bar{A}+\bar{B}) + \{d(-\bar{A}+\bar{B})\}^2\ (-\bar{A}+\bar{B})\ ] \tag{5.17}$$

$$= \int [\ 2(d\bar{A})^2\bar{B} + 4\, d\bar{A}\ d\bar{B}\ \bar{A} + \mathcal{O}(\bar{B}^2)\ ]. \tag{5.18}$$

Note that the $\bar{B}$-dependent part does not vanish although the anomaly-free condition for $\bar{A}$ is satisfied. By substituting Eq. (5.18) into Eq. (5.15), we obtain

$$j_\mu^{(\rm CS)}(x,s) = \frac{-1}{4\pi^2}\, \epsilon_{\mu abcd}\ \partial_a\bar{A}_b\partial_c\bar{A}_d(x,s), \tag{5.19}$$

where $\mu = 1,\ldots,4$ because $\bar{B}_5 = 0$, and $a,b,c,d = 1,\ldots,5$. Then, the divergence of $J_\mu^{(\rm CS)}$ is calculated as follows:

$$\partial_\mu J_\mu^{(\rm CS)}(x) = \partial_\mu \int_0^L ds\, j_\mu^{(\rm CS)}(x,s) \tag{5.20}$$

$$= \frac{-1}{4\pi^2}\int_0^L ds\ \partial_\mu \left(\epsilon_{\mu abcd}\ \partial_a\bar{A}_b\partial_c\bar{A}_d\right) \tag{5.21}$$

$$= \frac{-1}{2\pi^2}\int_0^L ds\, \epsilon_{\mu 5bcd}\, \left(\partial_\mu \partial_5 \bar{A}_b\right) \partial_c \bar{A}_d \tag{5.22}$$

$$= \frac{1}{4\pi^2}\int_0^L ds\, \partial_5 \left(\epsilon_{5\mu bcd}\, \partial_\mu \bar{A}_b \partial_c \bar{A}_d\right) \tag{5.23}$$

$$= \frac{1}{4\pi^2}\, \epsilon_{\mu\nu\lambda\rho}\, \partial_\mu \bar{A}_\nu \partial_\lambda \bar{A}_\rho \Big|_{s=0}^{s=L} \tag{5.24}$$

$$= \frac{-1}{16\pi^2}\epsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}(x,0), \tag{5.25}$$

with $\mu, \nu, \lambda, \rho = 1, \ldots, 4$. We have used the notion that $\bar{A}_\mu(x, s = L)$ is pure gauge in the last line.

On the other hand, the anomaly of $J_\mu^{(q_L,q_R)}$ is the same as the conventional axial anomaly of the vector-like fermion [27,28]:

$$\partial_\mu J_\mu^{(q_L,q_R)}(x) = \frac{1}{16\pi^2}\epsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}(x, s=0). \tag{5.26}$$

$\partial_\mu J_\mu^{(\text{fluff})}$ is similar, but vanishes because $\bar{A}(x, s = L)$ is pure gauge[10].

Thus the 4D current J_μ^B is conserved as mentioned above:

$$\partial_\mu J_\mu^B(x) = \partial_\mu J_\mu^{(q_L,q_R)}(x) + \partial_\mu J_\mu^{(\text{fluff})} + \partial_\mu J_\mu^{(\text{CS})}(x) \tag{5.27}$$

$$= 0. \tag{5.28}$$

In addition, the current is nonlocal in the sense of the 4D field theory because it includes the bulk contribution. Therefore we cannot regard J_μ^B as the local U(1) axial current in the effective theory.

In order to obtain the local and correctly anomalous current, we subtract the bulk contribution from J_μ^B:

$$J_\mu^{\text{axial}}(x) \equiv J_\mu^B(x) - \int d^4y \int_0^L ds\, j_\nu^{(\text{CS})}(y,s)\, \frac{\delta \bar{B}_\nu(y,s)}{\delta B_\mu(x)}. \tag{5.29}$$

Indeed, Eq. (5.29) can be written as

$$J_\mu^{\text{axial}}(x) = \int d^4y \int_0^L ds\, \left(j_\nu^B(y,s) - j_\nu^{(\text{CS})}(y,s)\right) \frac{\delta \bar{B}_\nu(y,s)}{\delta B_\mu(x)}, \tag{5.30}$$

which is manifestly local and reproduces the correct anomaly:

$$\partial_\mu J_\mu^{\text{axial}} = \frac{1}{16\pi^2}\epsilon_{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho}(x). \tag{5.31}$$

Note that the Chern–Simons current $j_\mu^{(\text{CS})}(x,s)$ and $j_\mu^B(x,s)$ are gauge invariant (see Eqs. (5.19) and (5.11)). Therefore $J_\mu^{\text{axial}}(x)$ is also gauge invariant. This is also true when the gauge group of the gauge field $\bar{A}_\mu$ is non-Abelian. In such case, indeed, the Chern–Simons form is

$$\int \omega_5(\bar{A}, \bar{B}) \tag{5.32}$$

$$= \sum_{R=r,\bar{r}} \int \text{tr}_R \left[(d(\bar{A}+\bar{B}))^2(\bar{A}+\bar{B}) + \frac{3}{2}(\bar{A}+\bar{B})^3 d(\bar{A}+\bar{B}) + \frac{3}{5}(\bar{A}+\bar{B})^5 \right], \tag{5.33}$$

[10] The fluff fermions are indeed decoupled even for the anomaly.

where r and $\bar{r}$ are the representations of the two fermions ψ and ψ', respectively. Thus the Chern–Simons current is written as

$$j_\mu^{(\mathrm{CS})}(x,s) = \frac{-1}{32\pi^2} \sum_{R=r,\bar{r}} \epsilon_{\mu abcd} \, \mathrm{tr}_R \, \bar{F}_{ab}\bar{F}_{cd}, \tag{5.34}$$

which is manifestly gauge invariant.

6. Summary and conclusions

In this paper, we have studied the formulation in Refs. [16,17] in the continuum. In Sect. 3, we have given the regularization by Eq. (3.4) with Eqs. (3.6) and (3.9). The Pauli–Villars pairs could generate extra massless modes on the walls and Chern–Simons terms in the bulk. However, the condition of Eq. (3.9) eliminates these extra contributions.

In Sect. 4, we have calculated the effective action to the quadratic order in the gauge field, and we have found that the effective action consists of three parts. One is the effective action of the chiral fermions on the domain walls with Pauli–Villars-like regularization. The second is the Chern–Simons term in the bulk. The third are divergent terms, which are canceled by the Pauli–Villars pairs.

In Sect. 5, we have argued the axial-vector current in 4 dimensions. We have introduced two sets of domain-wall fermions belonging to complex conjugate representations so that the effective theory is the vector-like gauge theory. Then we have considered the axial-vector current that generates the simultaneous phase transformations for the fermions. This current is exactly conserved, but it contains the contribution from the bulk, which is nonlocal from the viewpoint of the 4D theory. Therefore the local gauge-invariant axial-vector current is obtained by subtracting the bulk part.

Acknowledgements

We would like to thank Hiroshi Suzuki, Okuto Morikawa, and Ryuichiro Kitano for useful discussion at the KEK Theory Workshop 2016.

Funding

Open Access funding: SCOAP^3.

Appendix A. Propagator of the domain-wall fermion

The propagator of the domain-wall fermion is a solution of the following equation:

$$[i\not{p} + \gamma^5 \partial_s - \epsilon(s)M] G(p,s;s') = \delta(s-s'), \tag{A.1}$$

where $G(p,s;s')$ is the Fourier transform of the propagator in $2n$ directions:

$$G(x,s;x',s') = \int \frac{d^{2n}p}{(2\pi)^{2n}} \, e^{-ip\cdot(x-x')} \, G(p,s;s'). \tag{A.2}$$

We first consider the region $s' > 0$. Then we have three cases for s:

$$\begin{cases} \text{(i)} & 0 < s' < s \\ \text{(ii)} & 0 < s < s' \\ \text{(iii)} & s < 0 < s' \end{cases} \tag{A.3}$$

We denote the propagators for (i), (ii), (iii) by $G^{(1)}$, $G^{(2)}$, $G^{(3)}$, respectively. From Eq. (A.1), we have

$$G^{(1)}(p,s;s') = e^{(i\not{p}+M)\gamma^5 s}\, C_1(s'), \tag{A.4}$$

$$G^{(2)}(p,s;s') = e^{(i\not{p}+M)\gamma^5 s}\, C_2(s'). \tag{A.5}$$

$$G^{(3)}(p,s;s') = e^{(i\not{p}-M)\gamma^5 s}\, C_3(s'), \tag{A.6}$$

where C_1, C_2, C_3 are s-independent matrices. Note that the sign of the mass in $G^{(3)}$ is different from the others. We impose the following boundary conditions:

$$\begin{cases} G^{(1)}(s=s') - G^{(2)}(s=s') = \gamma^5 \\ G^{(2)}(s=0) = G^{(3)}(s=0) \\ G^{(1)}(s=L) = G^{(3)}(s=-L). \end{cases} \tag{A.7}$$

The first equation is obtained from Eq. (A.1) by integrating for s around s'. The second is to connect G continuously at $s=0$. The third is the periodic boundary condition. Thus matrices C_1, C_2, C_3 are all determined. Then, by using the identity

$$e^{(i\not{p}+M)\gamma^5 s} = \cosh\left(s\sqrt{p^2+M^2}\right) + \frac{(i\not{p}+M)\gamma^5}{\sqrt{p^2+M^2}} \sinh\left(s\sqrt{p^2+M^2}\right),$$

we obtain

$$G^{(1)}(p,s;s') = \frac{-\sqrt{p^2+M^2}}{2\sinh(L\sqrt{p^2+M^2})}\,\frac{i\not{p}}{p^2}\, e^{-(i\not{p}+M)\gamma^5(s-L)}\, e^{(i\not{p}-M)\gamma^5 s'}, \tag{A.8}$$

$$G^{(2)}(p,s;s') = \frac{-\sqrt{p^2+M^2}}{2\sinh(L\sqrt{p^2+M^2})}\,\frac{i\not{p}}{p^2}\, e^{-(i\not{p}+M)\gamma^5 s}\, e^{(i\not{p}-M)\gamma^5(s'-L)}, \tag{A.9}$$

$$G^{(3)}(p,s;s') = \frac{-\sqrt{p^2+M^2}}{2\sinh(L\sqrt{p^2+M^2})}\,\frac{i\not{p}}{p^2}\, e^{-(i\not{p}+M)\gamma^5(s-s'+L)}. \tag{A.10}$$

In the following, we consider the limit of $L\to\infty$. Then, $G^{(1)}$ becomes

$$\begin{aligned} &G^{(1)}(p,s;s') \\ &= \frac{i\not{p}}{p^2}\,\frac{-\sqrt{p^2+M^2}}{e^{L\sqrt{p^2+M^2}} - e^{-L\sqrt{p^2+M^2}}} \\ &\quad\times\left[\cosh\{(s-L)\sqrt{p^2+M^2}\} - \frac{(i\not{p}+M)\gamma^5}{\sqrt{p^2+M^2}}\sinh\{(s-L)\sqrt{p^2+M^2}\}\right] e^{(i\not{p}-M)\gamma^5 s'} \\ &\to -\frac{i\not{p}}{2p^2}\, e^{-s\sqrt{p^2+M^2}}\left[\sqrt{p^2+M^2} + (i\not{p}+M)\gamma^5\right] \\ &\quad\times\left[\cosh\left(s'\sqrt{p^2+M^2}\right) + \frac{(i\not{p}-M)\gamma^5}{\sqrt{p^2+M^2}}\sinh\left(s'\sqrt{p^2+M^2}\right)\right] \\ &= -\frac{i\not{p} + M - \sqrt{p^2+M^2}\,\gamma^5}{2\sqrt{p^2+M^2}}\, e^{(s'-s)\sqrt{p^2+M^2}} \\ &\quad - \frac{i\not{p}M(i\not{p} + \sqrt{p^2+M^2}\,\gamma^5 + M)}{2p^2\sqrt{p^2+M^2}} e^{-(s+s')\sqrt{p^2+M^2}}. \end{aligned} \tag{A.11}$$

$G^{(2)}$ and $G^{(3)}$ can be calculated similarly. The result can be summarized as

$$\begin{cases} G^{(1)}(p,s;s') = S^{(+)}(p,s-s') + \mathcal{D}^{(+)}(p)\, e^{-(s'+s)\sqrt{p^2+M^2}} & (0<s'<s) \\ G^{(2)}(p,s;s') = S^{(+)}(p,s-s') + \mathcal{D}^{(+)}(p)\, e^{-(s'+s)\sqrt{p^2+M^2}} & (0<s<s') \\ G^{(3)}(p,s;s') = \mathcal{D}^{(-+)}(p)\, e^{(s-s')\sqrt{p^2+M^2}} & (s<0<s'), \end{cases} \tag{A.12}$$

where

$$S^{(+)}(p,s-s') = -\theta(s-s')\,\frac{i\not{p}+M-\sqrt{p^2+M^2}\gamma^5}{2\sqrt{p^2+M^2}}\,e^{(s'-s)\sqrt{p^2+M^2}} \\ -\theta(s'-s)\,\frac{i\not{p}+M+\sqrt{p^2+M^2}\gamma^5}{2\sqrt{p^2+M^2}}\,e^{(s-s')\sqrt{p^2+M^2}}, \tag{A.13}$$

$$\mathcal{D}^{(+)}(p) = -\frac{i\not{p}M(i\not{p}+\sqrt{p^2+M^2}\gamma^5+M)}{2p^2\sqrt{p^2+M^2}}, \tag{A.14}$$

$$\mathcal{D}^{(-+)}(p) = -\frac{i\not{p}(\sqrt{p^2+M^2}-(i\not{p}-M)\gamma^5)}{2p^2}. \tag{A.15}$$

The propagator for $s'<0$ is obtained by replacing $M\to -M$ and $\gamma^5\to-\gamma^5$ in the above expressions (A.12)–(A.15):

$$\begin{cases} G^{(4)}(p,s;s') = S^{(-)}\,(p,s-s') + \mathcal{D}^{(-)}(p)\, e^{(s'+s)\sqrt{p^2+M^2}} & (s<s'<0) \\ G^{(5)}(p,s;s') = S^{(-)}\,(p,s-s') + \mathcal{D}^{(-)}(p)\, e^{(s'-s)\sqrt{p^2+M^2}} & (s'<s<0') \\ G^{(6)}(p,s;s') = \mathcal{D}^{(+-)}(p) & (s'<0<s), \end{cases} \tag{A.16}$$

where

$$S^{(-)}(p,s-s') = -\theta(s-s')\,\frac{i\not{p}-M-\sqrt{p^2+M^2}\gamma^5}{2\sqrt{p^2+M^2}}\,e^{(s'-s)\sqrt{p^2+M^2}} \\ -\theta(s'-s)\,\frac{i\not{p}-M+\sqrt{p^2+M^2}\gamma^5}{2\sqrt{p^2+M^2}}\,e^{(s-s')\sqrt{p^2+M^2}}, \tag{A.17}$$

$$\mathcal{D}^{(-)}(p) = +\frac{i\not{p}M(i\not{p}-\sqrt{p^2+M^2}\gamma^5-M)}{2p^2\sqrt{p^2+M^2}}, \tag{A.18}$$

$$\mathcal{D}^{(+-)}(p) = -\frac{i\not{p}(\sqrt{p^2+M^2}+(i\not{p}+M)\gamma^5)}{2p^2}. \tag{A.19}$$

Appendix B. Vacuum polarization

We give concrete expressions for I, II, III, defined in Eqs. (4.15), (4.33), (4.41), respectively. I is given by

$$\mathrm{I} = \iint_{(\mathrm{I})} \mathrm{tr}\left[\bar{A}_\mu(-k,s')\bar{A}_\nu(k,s)\right] \int \frac{d^dp}{(2\pi)^d}\, T^{(-)}_{\mathrm{local}}(p,p',s,s'), \tag{B.1}$$

where

$$T^{(-)}_{\rm local}(p,p',s,s') \\ = {\rm tr}\left[\gamma^\mu D^{(-)}\gamma^\nu D'^{(-)}\right] + {\rm tr}\left[\gamma^\mu D^{(-)}\gamma^\nu S'^{(-)}\right] + {\rm tr}\left[\gamma^\mu S^{(-)}\gamma^\nu D'^{(-)}\right]. \tag{B.2}$$

Here, ${\rm tr}\left[\gamma^\mu D^{(-)}\gamma^\nu D'^{(-)}\right]$ is calculated as follows:

$${\rm tr}\left[\gamma^\mu D^{(-)}(p,s';s)\gamma^\nu D^{(-)}(p',s;s')\right] \tag{B.3}$$

$$= \frac{-M^2 e^{-(s+s')(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})}}{4p^2p'^2\sqrt{p^2+M^2}\sqrt{p'^2+M^2}} \\ \times {\rm tr}\left[\gamma^\mu \not{p}(i\not{p}-\sqrt{p^2+M^2}\gamma^5 - M)\gamma^\nu \not{p}'(i\not{p}'-\sqrt{p'^2+M^2}\gamma^5 - M)\right] \tag{B.4}$$

$$\equiv \alpha(p,p')\ e^{(s+s')(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})}, \tag{B.5}$$

where

$$\alpha(p,p') = \frac{M^2}{4p^2p'^2\sqrt{p^2+M^2}\sqrt{p'^2+M^2}} \\ \times \Big[2^n p^2 p'^2\ \delta^{\mu\nu} + 2^n(\sqrt{p^2+M^2}\sqrt{p'^2+M^2}+M^2)N^{\mu\nu} \\ - M(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2}){\rm tr}\left[\gamma^\mu\not{p}\gamma^\nu\not{p}'\gamma^5\right]\Big], \tag{B.6}$$

and $N^{\mu\nu} = p\cdot p'\delta^{\mu\nu} - p^\mu p'^\nu - p^\nu p'^\mu$.

Similarly, ${\rm tr}\left[\gamma^\mu D^{(-)}\gamma^\nu S'^{(-)}\right]$ is given by

$${\rm tr}\left[\gamma^\mu D^{(-)}\gamma^\nu S'^{(-)}\right] \equiv \beta(p,p')\ e^{(s'+s)\sqrt{p^2+M^2}}\ e^{(s'-s)\sqrt{p'^2+M^2}}, \tag{B.7}$$

where

$$\beta(p,p') = \frac{M}{4p^2\sqrt{p^2+M^2}\sqrt{p'^2+M^2}} \\ \times \Big[2^n M(-p^2\delta^{\mu\nu}+N^{\mu\nu}) - \sqrt{p^2+M^2}\ {\rm tr}\left[\gamma^\mu\not{p}\gamma^\nu\not{p}'\gamma^5\right] \\ + p^2\sqrt{p'^2+M^2}\ {\rm tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right]\Big]. \tag{B.8}$$

${\rm tr}\left[\gamma^\mu S^{(-)}\gamma^\nu D'^{(-)}\right]$ is given by

$${\rm tr}\left[\gamma^\mu S^{(-)}\gamma^\nu D'^{(-)}\right] \equiv \gamma(p,p')\ e^{(s'+s)\sqrt{p'^2+M^2}}\ e^{(s'-s)\sqrt{p^2+M^2}}, \tag{B.9}$$

where

$$\gamma(p,p') = \frac{M}{4p'^2\sqrt{p'^2+M^2}\sqrt{p^2+M^2}} \times \Big[2^n M(-p'^2\delta^{\mu\nu}+N^{\mu\nu}) - \sqrt{p'^2+M^2}\ \mathrm{tr}\left[\gamma^\mu \not{p}\gamma^\nu \not{p}'\gamma^5\right] + p'^2\sqrt{p^2+M^2}\ \mathrm{tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right]\Big]. \tag{B.10}$$

Consequently, I is obtained as follows:

$$\begin{aligned}\mathrm{I} = &\ \mathrm{tr}\left[A_\mu(-k)A_\nu(k)\right] \\ &\times \int \frac{d^dp}{(2\pi)^d}\Bigg[-\frac{2^nM^2\delta^{\mu\nu}}{8(p^2+M^2)(p'^2+M^2)} \\ &+\frac{2^nM^2(p'^2\sqrt{p'^2+M^2}+p^2\sqrt{p^2+M^2})N^{\mu\nu}}{8p^2p'^2(p^2+M^2)(p'^2+M^2)(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})} \\ &+\frac{2^nM^2(p^2p'^2\delta^{\mu\nu}+(\sqrt{p^2+M^2}\sqrt{p'^2+M^2}+M^2)N^{\mu\nu})}{8p^2p'^2(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})^2\sqrt{p^2+M^2}\sqrt{p'^2+M^2}} \\ &-\frac{M(p^2+p'^2+M^2)\ \mathrm{tr}\left[\gamma^\mu\not{p}\gamma^\nu\not{p}'\gamma^5\right]}{8p^2p'^2\sqrt{p^2+M^2}\sqrt{p'^2+M^2}(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})} \\ &+\frac{M(p^2+p'^2+2M^2)\ \mathrm{tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right]}{8(p^2+M^2)(p'^2+M^2)(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})}\Bigg].\end{aligned} \tag{B.11}$$

The last term that includes $\mathrm{tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right]$ will be canceled with the contribution from region I′ because the net effect of interchanging $s \leftrightarrow s'$ changes the sign of γ^5 in $S^{(-)}$.

Similarly, II is given by

$$\mathrm{II} = \iint_{(\mathrm{II})} \mathrm{tr}\left[\bar{A}_\mu(-k,s')\bar{A}_\nu(k,s)\right] \int\frac{d^dp}{(2\pi)^d}\left[\mathrm{tr}\left[\gamma^\mu D^{(+-)}\gamma^\nu D'^{(-+)}\right]-\mathrm{tr}\left[\gamma^\mu S^{(-)}\gamma^\nu S'^{(-)}\right]\right] \tag{B.12}$$

$$\begin{aligned}= &\ \mathrm{tr}[A_\mu(-k)A_\nu(k)] \\ &\int\frac{d^dp}{(2\pi)^d}\Bigg[-\frac{2^nM^2\delta^{\mu\nu}}{4\sqrt{p^2+M^2}\sqrt{p'^2+M^2}(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})^2} \\ &+\frac{2^nM^2N^{\mu\nu}(p^2+p'^2+M^2+\sqrt{p^2+M^2}\sqrt{p'^2+M^2})}{4p^2p'^2\sqrt{p^2+M^2}\sqrt{p'^2+M^2}(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})^2} \\ &-\frac{M\ \mathrm{tr}\left[\gamma^\mu\not{p}\gamma^\nu\not{p}'\gamma^5\right]}{4p^2p'^2(\sqrt{p^2+M^2}\sqrt{p'^2+M^2})} \\ &+\frac{M\ \mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^5]}{4\sqrt{p^2+M^2}\sqrt{p'^2+M^2}(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})}\Bigg].\end{aligned} \tag{B.13}$$

Again, the last term will be canceled with the contribution from region II′.

III is given by

$$\mathrm{III} = \iint_{(\mathrm{III})} \mathrm{tr}\left[\bar{A}_\mu(-k,s')\bar{A}_\nu(k,s)\right] \times \int \frac{d^dp}{(2\pi)^d}\left[T^{(+)}_{\mathrm{local}}(p,p',s,s') + T_{\mathrm{bulk}}(p,p',s,s')\right], \tag{B.14}$$

where

$$T_{\mathrm{bulk}} = \mathrm{tr}\left[\gamma^\mu S^{(+)}\gamma^\nu S'^{(+)}\right] - \mathrm{tr}\left[\gamma^\mu S^{(-)}\gamma^\nu S'^{(-)}\right], \tag{B.15}$$

$$T^{(+)}_{\mathrm{local}} = \mathrm{tr}\left[\gamma^\mu D^{(+)}\gamma^\nu D'^{(+)}\right] + \mathrm{tr}\left[\gamma^\mu D^{(+)}\gamma^\nu S'^{(+)}\right] + \mathrm{tr}\left[\gamma^\mu S^{(+)}\gamma^\nu D'^{(+)}\right]. \tag{B.16}$$

Here, $\mathrm{tr}\left[\gamma^\mu D^{(+)}\gamma^\nu D'^{(+)}\right]$ is calculated similarly to Eq. (B.5):

$$\mathrm{tr}\left[\gamma^\mu D^{(+)}\gamma^\nu D'^{(+)}\right] \tag{B.17}$$

$$= \frac{-M^2 e^{-(s+s')(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})}}{4p^2p'^2\sqrt{p^2+M^2}\sqrt{p'^2+M^2}} \times \mathrm{tr}\left[\gamma^\mu\, \not{p}(i\not{p} + \sqrt{p^2+M^2}\gamma^5 + M)\gamma^\nu\, \not{p}'(i\not{p}' + \sqrt{p'^2+M^2}\gamma^5 + M)\right] \tag{B.18}$$

$$= \alpha(p,p')\, e^{-(s+s')(\sqrt{p^2+M^2}+\sqrt{p'^2+M^2})}. \tag{B.19}$$

Note that $\alpha(p,p')$ in Eq. (B.19) is equal to Eq. (B.6). We obtain similar results for $\mathrm{tr}\left[\gamma^\mu D^{(+)}\gamma^\nu S'^{(+)}\right]$ and $\mathrm{tr}\left[\gamma^\mu S^{(+)}\gamma^\nu D'^{(+)}\right]$, and $T^{(+)}_{\mathrm{local}}$ is written as Eq. (4.52).

References

[1] D. B. Kaplan, Phys. Lett. B **288**, 342 (1992) [arXiv:hep-lat/9206013] [Search INSPIRE].
[2] H. Neuberger, Phys. Lett. B **417**, 141 (1998) [arXiv:hep-lat/9707022] [Search INSPIRE].
[3] H. Neuberger, Phys. Lett. B **427**, 353 (1998) [arXiv:hep-lat/9801031] [Search INSPIRE].
[4] H. Neuberger, Phys. Rev. D **57**, 5417 (1998) [arXiv:hep-lat/9710089] [Search INSPIRE].
[5] R. Narayanan and H. Neuberger, Nucl. Phys. B **443**, 305 (1995) [arXiv:hep-th/9411108] [Search INSPIRE].
[6] R. Narayanan and H. Neuberger, Nucl. Phys. B **412**, 574 (1994) [arXiv:hep-lat/9307006] [Search INSPIRE].
[7] R. Narayanan and H. Neuberger, Phys. Rev. Lett. **71**, 3251 (1993) [arXiv:hep-lat/9308011] [Search INSPIRE].
[8] K. Jansen, Phys. Rept. **273**, 1 (1996) [arXiv:hep-lat/9410018] [Search INSPIRE].
[9] Y. Shamir, Nucl. Phys. B **406**, 90 (1993) [arXiv:hep-lat/9303005] [Search INSPIRE].
[10] V. Furman and Y. Shamir, Nucl. Phys. B **439**, 54 (1995) [arXiv:hep-lat/9405004] [Search INSPIRE].
[11] S. Aoki and H. Hirose, Phys. Rev. D **49**, 2604 (1994) [arXiv:hep-lat/9309014] [Search INSPIRE].
[12] R. Narayanan and H. Neuberger, Phys. Lett. B **302**, 62 (1993) [arXiv:hep-lat/9212019] [Search INSPIRE].
[13] M. F. L. Golterman, K. Jansen, and D. B. Kaplan, Phys. Lett. B **301**, 219 (1993) [arXiv:hep-lat/9209003] [Search INSPIRE].
[14] M. Luscher, Nucl. Phys. B **568**, 162 (2000) [arXiv:hep-lat/9904009] [Search INSPIRE].
[15] M. Luscher, Nucl. Phys. B **549**, 295 (1999) [arXiv:hep-lat/9811032] [Search INSPIRE].
[16] D. M. Grabowska and D. B. Kaplan, Phys. Rev. Lett. **116**, 211602 (2016) [arXiv:1511.03649 [hep-lat]] [Search INSPIRE].
[17] D. M. Grabowska and D. B. Kaplan, Phys. Rev. D **94**, 114504 (2016) [arXiv:1610.02151 [hep-lat]] [Search INSPIRE].

[18] C. G. Callan, Jr and J. A. Harvey, Nucl. Phys. B **250**, 427 (1985).
[19] S. Deser, R. Jackiw, and S. Templeton, Annals Phys. **140**, 372 (1982); **185**, 406 (1988) [erratum] [Annals Phys. **281**, 409 (2000)].
[20] H. Suzuki and O. Morikawa, Poster session "Lorentz symmetry violation in the fermion number anomaly with the chiral overlap operator" at KEK Theory Workshop 2016.
[21] K. Okumura and H. Suzuki, Prog. Theor. Exp. Phys. **2016**, 123B07 (2016) [arXiv:1608.02217 [hep-lat]] [Search INSPIRE].
[22] H. Makino and O. Morikawa, Prog. Theor. Exp. Phys. **2016**, 123B06 (2016) [arXiv:1609.08376 [hep-lat]] [Search INSPIRE].
[23] H. Fukaya, T. Onogi, S. Yamamoto, and R. Yamamura, PoS LATTICE **2016**, 330 (2017) [arXiv:1612.00096 [hep-lat]] [Search INSPIRE].
[24] R. Narayanan and H. Neuberger, J. High Energy Phys. **0603**, 064 (2006) [arXiv:hep-th/0601210] [Search INSPIRE].
[25] M. Lüscher, J. High Energy Phys. **1008**, 071 (2010); **1403**, 092 (2014) [erratum] [arXiv:1006.4518 [hep-lat]] [Search INSPIRE].
[26] M. Lüscher and P. Weisz, J. High Energy Phys. **1102**, 051 (2011) [arXiv:1101.0963 [hep-th]] [Search INSPIRE].
[27] S. L. Adler, Phys. Rev. **177**, 2426 (1969).
[28] J. S. Bell and R. Jackiw, Nuovo Cimento A **60**, 47 (1969).