Non-integrability of the spacial n-center problem

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Abstract

We prove the non-integrability of the spacial n-center problem. In order to prove it, we focus on the singularity of the differential equations extended to the complex space and then apply the Morales-Ramis theory to it. We also show the non-integrability of the spacial restricted $n+1$-body problem.

1 Introduction

Let $H : \mathcal{D} \to \mathbb{R}$ be a smooth function where $\mathcal{D}$ is an open set in $\mathbb{R}^{2k}$. The Hamiltonian system is represented by the ordinary differential equations

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}(\boldsymbol{q}, \boldsymbol{p}), \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}(\boldsymbol{q}, \boldsymbol{p}) \qquad (j = 1, \ldots, k) \tag{1}$$

where $(\boldsymbol{q}, \boldsymbol{p}) = (q_1, \ldots, q_k, p_1, \ldots, p_k) \in \mathcal{D}$. The function H is called the Hamiltonian and the natural number k is called the degrees of freedom.

A function $F : \mathcal{D} \to \mathbb{R}$ is called the first integral of (1) if F is conserved along each solution of (1). The Poisson bracket of two functions $F, G : \mathcal{D} \to \mathbb{R}$ is the function defined by

$$\{F, G\} = \sum_{k=1}^{k} \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} \right).$$

A function $F : \mathcal{D} \to \mathbb{R}$ is a first integral of (1) if and only if $\{F, H\}$ is identically zero. Hamiltonian system (1) is called integrable if there are k first integrals $F_1(= H), F_2, \ldots, F_k$ such that $dF_1, \ldots, dF_k$ are linearly independent in an open dense set of $\mathcal{D}$ and that $\{F_i, F_j\}$ is identically zero for any $i, j = 1, \ldots, k$.

The behavior of the orbits of integrable systems can be understood as quasi-periodic orbits on k-dimensional tori (see [1, Chapter 10]) while the dynamics of

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the non-integrable Hamiltonian systems are thought to be chaotic. Therefore it is an important subject to determine whether a given Hamiltonian is integrable or non-integrable.

This subject have been studied for centuries. Several approaches have been attempted for proving the integrability of some Hamiltonians. Noether theorem states that if a Hamiltonian has some symmetry, it has some first integrals. For example, from the fact that the central force systems have the rotating symmetry, the angular momentum is a first integral. As another method, if the Hamilton-Jacobi equation can be solved, the Hamiltonian can be represented as a function which depends only on the momentum variables. For example, the Hamilton-Jacobi equation of the two-center problem is separable, and hence can be solved. Therefore the two-center problem is integrable. As an example that the Hamilton-Jacobi equation is not separable, but that the Hamiltonian is integrable, Toda lattice is well known.

On the other hand, some methods for showing the non-integrability have been developed. Bruns [2] proved that in the 3-body problem there is no additional first integral which is represented by an algebraic function. After that, Poincaré [10] proved that for the perturbed Hamiltonian systems, there is no analytic first integral which also depends analytically on a parameter. Then by applying it to the restricted 3-body problem, he proved the non-existence of an analytic first integral depending analytically on a mass parameter.

Another theory in this field was originated by Kovalevskaya [6]. By focusing on singularities, she discovered new integrable parameters for the rigid body model. As a development of her approach, Ziglin [12, 13] established the theory of the monodromy group for proving the non-integrability. By applying the Ziglin analysis, Yoshida [11] provided criteria for the non-integrability of the homogeneous Hamiltonian systems. Morales-Ruiz and Ramis [8, 9] established a stronger theory by applying the differential Galois theory (Picard-Vessiot theory). Maciejewski and Przybylska [7] proved the non-integrability of the three-body problem for any fixed masses by using the Morales-Ramis theory. In order to prove it, they focused on the homothetic solutions and analyzed the variational equations along it.

In this paper, we show the non-integrability of the spatial n-center problem. Fix n positive contants m_k and n distinct points $\boldsymbol{c}_k \in \mathbb{R}^d$, and let

$$U(\boldsymbol{q}) = -\sum_{k=1}^{n} \frac{m_k}{|\boldsymbol{q} - \boldsymbol{c}_k|} = -\sum_{k=1}^{n} \frac{m_k}{\sqrt{(\boldsymbol{q} - \boldsymbol{c}_k) \cdot (\boldsymbol{q} - \boldsymbol{c}_k)}}.$$

The n-center problem is given by the Hamiltonian system with Hamiltonian

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}|\boldsymbol{p}|^2 + U(\boldsymbol{q}).$$

As we wrote above, the planar two-center problem is integrable(see for example [1]). The spacial two-center problem is also integrable. Bolotin [3] proved the non-integrability of the planar n-center problem for $n \geq 3$ by using geometric methods. The purpose of this paper is to study the non-integrability of the spacial n-center problem. Our appoach is based on the differential Galois theory, and is quite different from Bolotin's one.

This paper is organized as follows. In the next section, we focus on the singularities and extend the differential equation to complex differential equations, and then show our main theorem. In Section 3, we provide one example. In Section 4 we show the non-integrability of the restricted $n+1$-body problem.

2 Singularity analysis

The property of the singularities $\boldsymbol{c}_k$ is like one of the Kepler problem, and the singularities are isolated. Hence it is difficult to show the non-integrability by focusing on one of the singularities. But if we extend the domain $\mathbb{R}^3$ to $\mathbb{C}^3$, the sets of the singularities

$$S_k = \{(x, y, z) \in \mathbb{C}^3 \mid (x - a_k)^2 + (y - b_k)^2 + (z - c_k)^2 = 0\} \qquad (i = 1, \dots, k)$$

are 2-dimensional complex varieties and some of S_k can intersect.

Let $\boldsymbol{e} = (e_x, e_y, e_z) \in S_1 \cap S_2 \cap \dots \cap S_l$ and $\boldsymbol{e} \notin S_k (k = l+1, \dots, n)$. The kinetic part $\frac{1}{2}(p_x^2 + p_y^2 + p_z^2)$ of the Hamiltonian can be regarded as a holomorphic function on $\mathbb{C}^3$. Since the potential function

$$U = -\sum_{k=1}^{n} \frac{m_k}{\sqrt{(x - a_k)^2 + (y - b_k)^2 + (z - c_k)^2}}$$

includes square root, U is not meromorphic. We can not apply the Morales-Ramis theory directly. But Combot [4] established the extension so as to apply for such Hamiltonian systems.

The scaled Hamiltonian is defined by

$$H_0(\boldsymbol{q}, \boldsymbol{p}) = \lim_{\lambda \to +0} \lambda^2 H(\lambda^4 \boldsymbol{q} + \boldsymbol{e}, \lambda^{-1} \boldsymbol{p}).$$

It can be represented by

$$H_0(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}|\boldsymbol{p}|^2 - \sum_{k=1}^{l} \frac{m_k}{\sqrt{2\boldsymbol{q} \cdot (\boldsymbol{e} - \boldsymbol{c}_k)}}.$$

Here the inner product stands for the real one: $\boldsymbol{x} \cdot \boldsymbol{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 (x_1, \dots, y_3 \in \mathbb{C})$. Note that this is different from the Hermitian inner product.

Proposition. *If H is rationally integrable, then H_0 is rationally integrable*[1].

Proof. Assume that there is a rational function $F : \mathbb{C}^6 \to \mathbb{C}$ such that $\{F, H\} = 0$. If $\lambda = 0$ is a singularity of $F(\lambda^4 \boldsymbol{q} + \boldsymbol{e}, \lambda^{-1} \boldsymbol{p})$, this is a pole, since F is

[1]" rationally integrable" means that the Hamiltonian with k degrees of freedom is integrable such that the first integrals $F_1, \dots, F_n$ can be taken as rational functions on $\mathbb{C}^n$.

rational. Hence the Laurent extension of $F(\lambda^4 \boldsymbol{q} + \boldsymbol{e}, \lambda^{-1}\boldsymbol{p})$ with respect to λ can be written as follows:

$$F(\lambda^4 \boldsymbol{q} + \boldsymbol{e}, \lambda^{-1}\boldsymbol{p}) = \sum_{k=K}^{\infty} \lambda^k F_{k-K}(\boldsymbol{q}, \boldsymbol{p}).$$

If ∇F_0 and ∇H_0 are linearly dependent, we consider $F - \lambda^{K+2} H$ instead of F. Since ∇F and ∇H are linearly independent, we can assume that ∇F_0 and ∇H_0 are linearly independent by repeating it. Since

$$\{F, H\} = \lambda^{k-2}\{F_0, H_0\} + O(\lambda^{k-1}) \qquad (\lambda \to +0),$$

we get

$$\{F_0, H_0\} = 0.$$

Therefore H_0 has independent from H_0 and rational first integral F_0. □

The canonical differential equations of H_0 are

$$\frac{d^2 \boldsymbol{q}}{dt^2} = \sum_{k=1}^{l} \frac{m_k}{(2\boldsymbol{q} \cdot (\boldsymbol{e} - \boldsymbol{c}_k))^{3/2}} (\boldsymbol{e} - \boldsymbol{c}_k). \tag{2}$$

We study them as complex differential equations. Suppose that $\boldsymbol{d} = (d_x, d_y, d_z)$ satisfies

$$C\boldsymbol{d} = \sum_{k=1}^{l} \frac{m_k}{(2(\boldsymbol{e} - \boldsymbol{c}_k) \cdot \boldsymbol{d})^{3/2}} (\boldsymbol{e} - \boldsymbol{c}_k)$$

and that $g(t)$ satisfies

$$\frac{d^2 g}{dt^2} = C g^{-3/2}.$$

Then $\boldsymbol{q} = g(t)(d_x, d_y, d_z)$ satisfies (2).

The variational equations along this solution are

$$\frac{d^2 \boldsymbol{X}}{dt^2} = g^{-5/2} A \boldsymbol{X}$$

where

$$A = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix} = \sum_{k=1}^{l} \frac{3m_k}{(2(\boldsymbol{e} - \boldsymbol{c}_k) \cdot \boldsymbol{d})^{5/2}} {}^t(\boldsymbol{e} - \boldsymbol{c}_k)(\boldsymbol{e} - \boldsymbol{c}_k).$$

Since $\left(\frac{dg}{dt}\right)^2 + 2Cg^{-1/2}$ is conserved, we fix the value at h:

$$\frac{1}{2}\left(\frac{dg}{dt}\right)^2 + 2Cg^{-1/2} = h.$$

By letting $w = \frac{2C}{hg(t)^{1/2}}$, the variational equations become

$$w(1-w)\frac{d^2\boldsymbol{X}}{dw^2} + \left(3 - \frac{7}{2}w\right)\frac{d\boldsymbol{X}}{dw} = C^{-1}A\boldsymbol{X}.$$

Assume that A is diagonalizable. Let ρ_1, ρ_2, ρ_3 be the eigenvalues of A. By diagonalizing the matrix A, each component of the differential equations is represented by

$$w(1-w)\frac{d^2\xi}{dw^2} + \left(3 - \frac{7}{2}w\right)\frac{d\xi}{dw} - C^{-1}\rho_k\xi = 0. \tag{3}$$

This is identical to the Gaussian hypergeometric equation

$$w(1-w)\frac{d^2\xi}{dw^2} + (\gamma - (\alpha+\beta+1)w)\frac{d\xi}{dw} - \alpha\beta\xi = 0 \tag{4}$$

where $\gamma = 3$, $\alpha + \beta = \frac{5}{2}$ and $\alpha\beta = \frac{\rho_k}{C}$. Here we introduce Kimura's result on the solvability of the Gaussian hypergeometric equation.

Proposition (Kimura [5])**.** *The hypergeometric equation* (4) *is solvable if and only if $\lambda = 1-\gamma, \mu = \gamma - \alpha - \beta, \nu = \beta - \alpha$ satisfies one of the following three property:*

- *at least one of $\lambda \pm \mu \pm \nu$ is odd integer,*
- *$(\pm\lambda, \pm\mu, \pm\nu)$ are in Schwarz table (in an arbitrary order) modulo $\mathbb{Z}$:*

$$\begin{gathered}\left(\frac{1}{2},\frac{1}{3},\frac{1}{3}\right),\left(\frac{2}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{2},\frac{1}{3},\frac{1}{4}\right),\left(\frac{2}{3},\frac{1}{4},\frac{1}{4}\right),\left(\frac{1}{2},\frac{1}{3},\frac{1}{5}\right),\\ \left(\frac{2}{5},\frac{1}{3},\frac{1}{3}\right),\left(\frac{2}{3},\frac{1}{5},\frac{1}{5}\right),\left(\frac{1}{2},\frac{2}{5},\frac{1}{5}\right),\left(\frac{3}{5},\frac{1}{3},\frac{1}{5}\right),\left(\frac{2}{5},\frac{2}{5},\frac{2}{5}\right),\\ \left(\frac{2}{3},\frac{1}{3},\frac{1}{5}\right),\left(\frac{4}{5},\frac{1}{5},\frac{1}{5}\right),\left(\frac{1}{2},\frac{2}{5},\frac{1}{3}\right),\left(\frac{3}{5},\frac{2}{5},\frac{1}{3}\right)\end{gathered}$$

- *at least two of λ, μ, ν belong to $\frac{1}{2} + \mathbb{Z}$.*

In our case, λ, μ and ν are

$$\begin{aligned}\lambda &:= 1 - \gamma = -2,\\ \mu &:= \gamma - \alpha - \beta = \frac{1}{2},\\ \nu &:= \beta - \alpha = \pm\sqrt{(\alpha+\beta)^2 - 4\alpha\beta} = \pm\sqrt{\frac{25}{4} - \frac{4\rho_k}{C}}.\end{aligned}$$

By checking the list in this proposition, it turns out that (3) is integrable if and only if

$$\nu_k = \sqrt{\frac{25}{4} - \frac{4\rho_k}{C}} \tag{5}$$

belongs to $\frac{1}{2}+\mathbb{Z}$ for $k=1,2,3$. The Morales-Ramis theory [8, 9] states that if the Hamiltonian system is integrable, the variational equations along any particular solution are integrable. Their theorem can be applied to meromorphic Hamiltonians. Combot[4] extended their theorem for algebraic potential systems like the n-body problem, and hence the theorem works for our Hamiltonian H_0. Consequently we have proven the following:

Theorem 1. *If the n-center problem is rationally integrable, ν_k defined by* (5) *belongs to $\frac{1}{2}+\mathbb{Z}$ for each $k=1,2,3$.*

3 Example

We give an example here. Assume that $m_k = 1 (k = 1, \ldots, l)$ and $(a_k, b_k)(k = 1, \ldots, l)$ forms regular l-gon:

$$a_k = \cos\frac{2\pi k}{l}, \quad b_k = \sin\frac{2\pi k}{l} \qquad (k = 1, \ldots, l).$$

Let (e_x, e_y, e_z) and (d_x, d_y, d_z) be

$$(e_x, e_y, e_z) = (0, 0, \pm i), \quad (d_x, d_y, d_z) = \left(0, 0, -\frac{i}{2}\left(-\frac{2l}{c}\right)^{2/5}\right).$$

Next we compute $A_{11}, \ldots, A_{33}$. Assume that $l \geq 3$. We obtain

$$\begin{aligned}
A_{11} &= \frac{3}{(2id_z)^{5/2}}\sum_{k=1}^{l} a_k^2 = \frac{3}{(2id_z)^{5/2}}\sum_{k=1}^{l}\cos^2\frac{2\pi k}{l} = \frac{3}{(2id_z)^{5/2}}\sum_{k=1}^{l}\frac{\cos\frac{4\pi k}{l}+1}{2} \\
&= \frac{3l}{2(2id_z)^{5/2}} \\
A_{12} &= \frac{3}{(2id_z)^{5/2}}\sum_{k=1}^{l}\cos\frac{2\pi k}{l}\sin\frac{2\pi k}{l} = \frac{3}{(2id_z)^{5/2}}\sum_{k=1}^{l}\frac{1}{2}\sin\frac{4\pi k}{l} = 0
\end{aligned}$$

since

$$\sum_{k=1}^{l}\cos\frac{4\pi k}{l} + i\sin\frac{4\pi k}{l} = \sum_{k=1}^{l}\exp\frac{4\pi i k}{l} = \frac{\exp\frac{4\pi i}{l}(1-\exp\frac{4\pi i l}{l})}{1-\exp\frac{4\pi i}{l}} = 0.$$

Similarly, we get

$$A_{13} = A_{21} = A_{31} = A_{32} = 0, \quad A_{22} = \frac{3l}{2(2id_z)^{5/2}}, \quad A_{33} = -\frac{3l}{(2id_z)^{5/2}}.$$

Therefore we have

$$\frac{A_{11}}{C} = \frac{A_{22}}{C} = -\frac{3}{4}, \quad \frac{A_{33}}{C} = \frac{3}{2}.$$

Since (A_{ij}) is diagonal matrix and the eigenvalues of A are $\rho_k = A_{kk}$. Hence we have

$$\nu_1 = \nu_2 = \sqrt{\frac{25}{4} - \frac{4\rho_1}{C}} = \frac{\sqrt{37}}{2}, \quad \nu_3 = \sqrt{\frac{25}{4} - 6} = \frac{1}{2}$$

Consequently the n-center problem in this setting is not integrable.

Remark. *In the case of $l = 2$, we get*

$$A_{11} = \frac{6}{(2id_z)^{5/2}}, \quad A_{22} = 0, \quad A_{33} = -\frac{6}{(2id_z)^{5/2}}.$$

Therefore we have

$$\frac{A_{11}}{C} = -\frac{3}{2}, \quad \frac{A_{22}}{C} = 0, \quad \frac{A_{33}}{C} = \frac{3}{2}.$$

Hence we get

$$\nu_1 = \frac{7}{2}, \quad \nu_2 = \frac{5}{2}, \quad \nu_3 = \frac{1}{2}.$$

These are all $j + \frac{1}{2}$ type number. This is reasonable because the 2-center problem is integrable.

4 The restricted $n + 1$-body problem

Consider the motion of particles under the gravitational attraction. Assume that n particles with masses $m_k (k = 1, \ldots, n)$ move along circles on a plane with a same period around the origin

$$(a_k \cos t - b_k \sin t, b_k \cos t + a_k \sin t, 0),$$

and consider the spatial motion of a massless particle. The massless particle is attracted by the other n particles. Studying the motion of the massless particle is called the (spacial circular) restricted $n + 1$-body problem. The restricted $n + 1$-body problem is governed by the Hamiltonian system with Hamiltonian

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + p_x y - p_y x - \sum_{k=1}^{n} \frac{m_k}{\sqrt{(x - a_k)^2 + (y - b_k)^2 + z^2}}.$$

Here (x, y, z) are the rotating coordinates with respect to z-axis. In this coordinates, n particles are fixed.

The scaled Hamiltonian is defined by

$$H_0(\boldsymbol{q}, \boldsymbol{p}) = \lim_{\lambda \to +0} \lambda^2 H(\lambda^4 \boldsymbol{q} + \boldsymbol{e}, \lambda^{-1} \boldsymbol{p}),$$

which is

$$H_0(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{2}|\boldsymbol{p}|^2 - \sum_{k=1}^{n} \frac{m_k}{\sqrt{2\boldsymbol{q}\cdot(\boldsymbol{e}-\boldsymbol{c}_k)}}.$$

Similarly if H is integrable, so is H_0. We can also apply our proof to the restricted $n+1$-body problem.

Theorem 2. *If the restricted $n+1$-body problem is rationally integrable, ν_k defined by* (5) *belongs to $\frac{1}{2}+\mathbb{Z}$.*

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