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p-th power relations and Euler-Carlitz relations among multizeta values

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Abstract

In this paper, we study p-th power relations and Euler-Carlitz relations among multizeta values in characteristic p. By definition, two multizeta values have the p-th power relation if their indices map to each other multiplying by some power of p. The multizeta values of depth one at “even” integers satisfy Euler-Carlitz relations which are analogues of the relations among the Riemann zeta values at positive even integers. We prove that all algebraic relations among given multizeta values come from p-th power relations and Euler-Carlitz relations if their indices satisfy some conditions.

§1. Introduction

Let \( \mathbf{n} = (n_1, \ldots, n_d) \in (\mathbb{Z}_{\geq 1})^d \) be a \( d \)-tuple \( (d \geq 1) \) of positive integers such that \( n_1 \geq 2 \). The sum
\[
\zeta_{\mathbb{Z}}(\mathbf{n}) := \sum_{m_1 > \cdots > m_d > 0} \frac{1}{m_1^{n_1} \cdots m_d^{n_d}} \in \mathbb{R}
\]
is called the multiple zeta value (MZV) and studied by many mathematicians. Many relations over \( \mathbb{Q} \) among MZV are known. For example, Euler showed that
\[
\zeta_{\mathbb{Z}}(n) \in (2\pi \sqrt{-1})^n \cdot \mathbb{Q}^\times
\]
for each positive even integer \( n \geq 2 \). We also have the harmonic product formula. The simplest case is as follows:
\[
\zeta_{\mathbb{Z}}(n_1) \zeta_{\mathbb{Z}}(n_2) = \zeta_{\mathbb{Z}}(n_1, n_2) + \zeta_{\mathbb{Z}}(n_2, n_1) + \zeta_{\mathbb{Z}}(n_1 + n_2).
\]
We also want to know the linear/algebraic independence among given MZVs. However, we do not even know whether $\zeta(n)$ is transcendental over $\mathbb{Q}$ for each positive odd integer $n \geq 3$. In general, such problems seem very difficult.

Next, we consider the positive characteristic case. We fix a prime number $p$ and its power $q$. Let $\theta$ be a variable, $A := \mathbb{F}_q[\theta]$ the one variable polynomial ring over $\mathbb{F}_q$, $K := \mathbb{F}_q(\theta)$ the fraction field of $A$, $K_{\infty} := \mathbb{F}_q((\theta^{-1}))$ the $\infty$-adic completion of $K$, $\mathbb{C}_\infty$ the $\infty$-adic completion of an algebraic closure of $K_{\infty}$, and $\overline{K}$ the algebraic closure of $K$ in $\mathbb{C}_\infty$. Let $\mathbf{n} = (n_1, \ldots, n_d) \in (\mathbb{Z}_{\geq 1})^d$ be a $d$-tuple ($d \geq 1$) of positive integers. Such an $\mathbf{n}$ is called an index of weight $\text{wt}(\mathbf{n}) := \sum n_i$ and depth $\text{dep}(\mathbf{n}) := d$. For an index $\mathbf{n}$, Thakur ([9, Section 5.10]) defined the multizeta value in characteristic $p$ by

$$\zeta(\mathbf{n}) := \sum_{a_1, \ldots, a_d \in A: \text{monic}}^{\deg(a_1) > \cdots > \deg(a_d) \geq 0} \frac{1}{a_1^{n_1} \cdots a_d^{n_d}} \in K_{\infty}.$$ 

We are also interested in determining all relations over $\overline{K}$ among given MZVs. For an index $\mathbf{n} = (n_1, \ldots, n_d)$ and an integer $e \in \mathbb{Z}$, we set

$$p^e \mathbf{n} := (p^e n_1, \ldots, p^e n_d) \in \mathbb{Z}[1/p]^d.$$ 

If $p^e \mathbf{n} \in \mathbb{Z}^d$, the $p$-th power relation

$$\zeta(p^e \mathbf{n}) = \zeta(\mathbf{n})^p$$

follows immediately from the definition of $\zeta(\mathbf{n})$. The MZVs of depth one are defined by Carlitz ([4]) and called the Carlitz zeta values. Carlitz showed the relation

$$\zeta(n) = \overline{\pi}^n \cdot \frac{B_n}{\Gamma_{n+1}}$$

for each positive integer $n \geq 1$ which is divisible by $q - 1$, where

$$\overline{\pi} := (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1} \in (-\theta)^{\frac{1}{q-1}} K_{\infty}^\times$$

is the Carlitz period, $B_n \in A$ is the Bernoulli-Carlitz number and $\Gamma_{n+1} \in A$ is the factorial of Carlitz (see Section 2). These relations are called the Euler-Carlitz relations. These are analogues of Euler’s relations of the special zeta values at positive even integers. We say that a positive integer $n \geq 1$ is “even” (resp. “odd”) if $n$ is divisible (resp. not divisible) by $q - 1$. After works of Wade ([10]) and Yu ([11], [12]), finally Chang and Yu ([5, Corollary 4.6]) proved that all relations over $\overline{K}$ among the Carlitz zeta values come from $p$-th power relations and Euler-Carlitz relations. This means that if $n_1, \ldots, n_d \geq 1$ are positive “odd” integers such that $n_i/n_j$ is not an integral power of
Let $n = (n_1, \ldots, n_d)$ be an index.

1. We set
   \[ \text{Sub}'(n) := \{(n_{i_1}, \ldots, n_{i_r}) | 1 \leq r \leq d, 1 \leq i_1 < \cdots < i_r \leq d \} \]
   Thus we have
   \[ \text{Sub}'(n) \supset \text{Sub}(n), \quad \# \text{Sub}'(n) \leq 2^d - 1 \quad \text{and} \quad \# \text{Sub}(n) \leq \frac{d(d+1)}{2}. \]

2. For each $1 \leq j < i \leq d + 1$, we set
   \[ n_{ij} := (n_j, n_{j+1}, \ldots, n_{i-1}) \]
   Thus we have
   \[ \text{Sub}(n) = \{n_{ij} | 1 \leq j < i \leq d + 1 \}. \]

3. Let $n'$ be another index. We say that $n$ and $n'$ are equivalent and denote by $n \sim n'$ if there exists an integer $e \in \mathbb{Z}$ such that $n = p^e n'$, or both $n, n'$ are of depth 1 and $n = (m), n' = (m')$ for some $m, m' \in (q-1)\mathbb{Z}$ (hence $\text{dep}(n) = \text{dep}(n')$ if $n \sim n'$). For positive integers $m$ and $m'$, we write $m \sim m'$ if the indices $(m)$ and $(m')$ of depth one are equivalent.

4. Let $S$ be a set of indices. We denote by $S/\sim$ the quotient set of $S$ by the equivalence relation $\sim$.

The purpose of this paper is to generalize the result [7, Theorem 1.1] to the following three directions:

- $n_i/n_j$ may be an integral power of $p$,
- $n_i$ may be “even”,
- treat elements $\zeta(m)$ for $m \in \text{Sub}'(n)$. 

We expect that if the given indices satisfy some “good” conditions, then all algebraic relations over $\overline{K}$ among the multizeta values at such points come from $p$-th power relations and Euler-Carlitz relations. In fact, in this paper, we prove the following theorems:

**Theorem 1.2.** Let $\mathbf{n} = (n_1, \ldots, n_d)$ be an index such that the $n_i$’s are “odd” and distinct from each other. Assume that there exists exactly one pair $j_1 < j_2$ such that $n_{j_1} \sim n_{j_2}$. We set

$$S := \{ m \mid m \in \text{Sub}'(\mathbf{n}), (n_{j_1}, n_{j_2}) \notin \text{Sub}(m) \}.$$  

Then we have

$$ \text{tr.deg}_{\overline{K}}(\overline{\pi}, \zeta(m) \mid m \in S) = \#((\{(q-1)\} \cup S)/\sim). $$

Note that the condition $(n_{j_1}, n_{j_2}) \notin \text{Sub}(m)$ means that $m$ is not an index of the form $m = (\ldots, n_{j_1}, n_{j_2}, \ldots)$.

As a consequence of Theorem 1.2, we have the following corollary:

**Corollary 1.3.** Let $\mathbf{n} = (n_1, \ldots, n_d)$ be an index of positive “odd” integers such that $n_i/n_j$ is not an integral power of $p$ for each $i \neq j$. Then we have

$$ \text{tr.deg}_{\overline{K}}(\overline{\pi}, \zeta(m) \mid m \in \text{Sub}'(\mathbf{n})) = 2^d. $$

We also have the following theorem:

**Theorem 1.4.** Let $\mathbf{n} = (n_1, \ldots, n_d)$ be an index such that the $n_i$’s are “odd” and distinct from each other. Assume that there exists exactly one pair $j_1 \neq j_2$ such that $n_{j_1} \sim n_{j_2}$. Then we have

$$ \text{tr.deg}_{\overline{K}}(\overline{\pi}, \zeta(m) \mid m \in \text{Sub}(\mathbf{n})) = \#((\{(q-1)\} \cup \text{Sub}(\mathbf{n})) / \sim) = \frac{d(d+1)}{2}. $$

**Remark.** We do not know in general that when

$$ \overline{\pi}, \zeta(n) $$

and

$$ \zeta((q-1)m, n) $$

(or $\overline{\pi}, \zeta(n)$ and $\zeta(n, (q-1)m)$)

are algebraically independent over $\overline{K}$, where $n$ is “odd”. Thus we do not treat “even” integers in Theorems 1.2 and 1.4. When we treat “even” integers, we need to assume that the above elements are already algebraically independent over $\overline{K}$ as in Theorem 1.5.

For a set $S$ of indices, we define a set $[S]$ by

$$ [S] := \{ m \mid \text{index} \mid m \sim \mathbf{n} \text{ for some } \mathbf{n} \in S \}. $$

Theorems 1.2 and 1.4 follow from the following theorem:
Theorem 1.5. Let \( n^{(m)} = (n_1^{(m)}, \ldots, n_{d_m}^{(m)}) \) \((1 \leq m \leq k, k \geq 2)\) be indices. If the following conditions (1) \(\sim\) (5) hold, then we have

\[
\text{tr.deg}_K \overline{K}(\pi, \zeta(n)|n \in \bigcup_{m=1}^k \text{Sub}(n^{(m)})) = \# \left( \left\{ (q-1) \right\} \cup \bigcup_{m=1}^k \text{Sub}(n^{(m)}) \right) / \sim .
\]

(1) \(\text{dep}(n^{(k)}) = d_k \geq 2\).

(2) \(n^{(k)} \neq (n_1^{(k)}, n_1^{(k)}, \ldots, n_1^{(k)})\).

(3) \(\text{Sub}(n^{(k)}) \sim \bigcup_{m=1}^{k-1} \text{Sub}(n^{(m)}) = \{n^{(k)}\}\).

(4) \(\text{tr.deg}_K \overline{K}(\pi, \zeta(n)|n \in \bigcup_{m=1}^{k-1} \text{Sub}(n^{(m)})) = \#(\{(q-1)\} \cup \bigcup_{m=1}^{k-1} \text{Sub}(n^{(m)}) / \sim)\).

(5) In the following, the \((m, i, j)\) runs over all triples of integers such that

\[1 \leq m \leq k - 1, \quad 1 \leq j < i \leq d_m + 1, \quad i - j = d_k,\]

One of the following four conditions holds:

(5-1)

\[n_1^{(k)} \neq n_{i-1}^{(m)} \text{ or } n_{d_k+1,2}^{(k)} = (n_2^{(k)}, \ldots, n_{d_k}^{(k)}) \neq n_{i-1,j}^{(m)} = (n_j^{(m)}, \ldots, n_{i-2}^{(m)})\]

and

\[n_1^{(k)} \neq n_j^{(m)} \text{ or } n_{d_k+1,2}^{(k)} = (n_2^{(k)}, \ldots, n_{d_k}^{(k)}) \neq n_{i-1,j}^{(m)} = (n_j^{(m)}, \ldots, n_{i-1}^{(m)})\]

for each \((m, i, j)\) and \(n_1^{(k)}, n_{d_k+1,2}^{(k)} \neq q - 1\).

(5-1)’

\[n_1^{(k)} \neq n_{i-1}^{(m)} \text{ or } n_{d_k+1,1}^{(k)} = (n_1^{(k)}, \ldots, n_{d_k-1}^{(k)}) \neq n_{i-1,j}^{(m)} = (n_j^{(m)}, \ldots, n_{i-2}^{(m)})\]

and

\[n_1^{(k)} \neq n_j^{(m)} \text{ or } n_{d_k+1,1}^{(k)} = (n_1^{(k)}, \ldots, n_{d_k-1}^{(k)}) \neq n_{i-1,j}^{(m)} = (n_j^{(m)}, \ldots, n_{i-1}^{(m)})\]

for each \((m, i, j)\) and \(n_{d_k+1,1}^{(k)}, n_{d_k+1,2}^{(k)} \neq q - 1\).

(5-2) There exists exactly one triple \((m_0, i_0, j_0)\) such that

\[n_1^{(k)} \sim n_{i_0-1}^{(m_0)}, \quad n_{d_k}^{(k)} \sim n_{j_0}^{(m_0)}, \quad n_{d_k+1,2}^{(k)} \sim n_{i_0,j_0+1}^{(m_0)}; \quad n_{d_k+1,2}^{(k)} \sim n_{i_0-1,j_0}^{(m_0)}; \quad n_{d_k}^{(k)} \neq n_{i+2}^{(k)} \text{ for some } i \text{ (resp. } n_1^{(k)} \neq (n_1^{(m_0)}, n_{i_0}^{(m_0)})) \text{ if } d_k \geq 3 \text{ (resp. } d_k = 2)\), and for other \((m, i, j)\)’s

\[n_1^{(k)}, n_{d_k}^{(k)} \neq n_{i-1}^{(m)} \text{ or } n_{d_k+1,1}^{(k)}, n_{d_k+1,2}^{(k)} \neq n_{i-1,j}^{(m)} \]
and

\[ n^{(k)}_1, n^{(k)}_d \neq n^{(m)}_j, \quad \text{or} \quad \# n^{(k)}_{d_k,1}, \# n^{(k)}_{d_k+1,2} \neq \# n^{(m)}_{i,j+1}, \]

and \( n^{(k)}_1, n^{(k)}_d \neq q-1, \) and \( n^{(k)}_{d_k,1} \neq n^{(k)}_{d_k+1,2}. \)

\( (5-2)' \) There exists exactly one triple \((m_0, i_0, j_0)\) such that

\[ n^{(k)}_1 \sim n^{(m_0)}_{j_0}, \quad n^{(k)}_d \sim n^{(m_0)}_{i_0-1}, \quad \# n^{(k)}_{d_k,1} \sim \# n^{(m_0)}_{i_0-1,j_0}, \quad \# n^{(k)}_{d_k+1,2} \sim \# n^{(m_0)}_{i_0,j_0+1}, \]

and for other \((m, i, j)\)’s

\[ n^{(k)}_1, n^{(k)}_d \neq \# n^{(m)}_{i-1,j}, \quad \text{or} \quad \# n^{(k)}_{d_k,1}, \# n^{(k)}_{d_k+1,2} \neq \# n^{(m)}_{i,j+1}, \]

and \( n^{(k)}_1, n^{(k)}_d \neq q-1, \) and \( n^{(k)}_{d_k,1} \neq n^{(k)}_{d_k+1,2}. \)

**Proof of Theorem 1.2.** We fix an order of the set \( S = \{ n^{(1)}, n^{(2)}, \ldots \} \) such that \( \deg(n^{(1)}) \leq \deg(n^{(2)}) \leq \ldots \). For each \( 1 \leq k \leq \# S \), we show the equality

\[ \text{tr.deg}_K \overline{K}(\pi, \zeta(n^{(1)}), \ldots, \zeta(n^{(k)})) = \#((q-1), n^{(1)}, \ldots, n^{(k)})/\sim \]

by induction on \( k \). If \( \deg(n^{(k)}) = 1 \), then the equality comes from the result of Chang and Yu ([5, Corollary 4.6]). Let \( \deg(n^{(k)}) \geq 2 \), then it is clear that the conditions (1), (2) and (3) of Theorem 1.5 hold. By the induction hypothesis, the condition (4) also holds. When \( n^{(k)}_1 \notin \{ n_{j_1}, n_{j_2} \} \), the condition (5-1) holds if \( d_k \geq 3 \), and the condition (5-1), (5-2) or (5-2)’ holds if \( d_k = 2 \). Similarly, when \( n^{(k)}_d \notin \{ n_{j_1}, n_{j_2} \} \), the condition (5-1)’ holds if \( d_k \geq 3 \), and the condition (5-1), (5-2) or (5-2)’ holds if \( d_k = 2 \). When \( n^{(k)}_1 \sim n^{(k)}_d \) (this means that \( n^{(k)}_1 = n_{j_1} \) and \( n^{(k)}_d = n_{j_2} \), then we have \( d_k \geq 3 \) by the definition of \( S \), and the conditions (5-1) and (5-1)’ hold. In any case, the condition (5) of Theorem 1.5 holds, and hence the equality (1.1) follows from Theorem 1.5.

**Proof of Theorem 1.4.** By Theorem 1.2, we may assume that \( j_2 = j_1 + 1 \). The proof is similar to that of Theorem 1.2. We fix an order on \( S \) as before, and show the equality (1.1) by induction. Let \( \deg(n^{(k)}) \geq 2 \). Then the conditions (1), (2) and (3) of Theorem 1.5 hold clearly, and the condition (4) follows from the induction hypothesis. In this case, the conditions (5-1) and (5-1)’ hold.

The next proposition does not follow from Theorem 1.5, but we can show this by similar arguments of the proof of Theorem 1.5.

**Proposition 1.6.** Let \( n = (n_1, n_2, n_3) \) be an index of depth three. If the \( n_i \)'s are “odd” and distinct from each other, then we have

\[ \text{tr.deg}_K \overline{K}(\pi, \zeta(m)|m \in \text{Sub}(n)) = \#((q-1) \cup \text{Sub}(n))/\sim. \]
In Section 2, we define notations which are used in this paper and briefly review Papanikolas’ theory of pre-t-motives. In Section 3 (resp. 4), we study “lifts” of p-th power (resp. Euler-Carlitz) relations. To apply Papanikolas’ theory to MZVs which have p-th power or Euler-Carlitz relations, we need their lifts. In Section 5, we prove Theorem 1.5 and Proposition 1.6. The proofs are refinements of the proofs in [7].

§2. Preliminaries

We continue to use the notations of the Introduction. Let $t$ be a new variable independent from $\theta$. We fix an $\infty$-adic valuation $|\cdot|_{\infty}$ on $\mathbb{C}_{\infty}$. Let $\mathbb{T} := \{f \in \mathbb{C}_{\infty}[t]| f \text{ converges on } |t|_{\infty} \leq 1\}$ be the Tate algebra over $\mathbb{C}_{\infty}$ and $\mathbb{L}$ the fraction field of $\mathbb{T}$. For a formal Laurent series $f = \sum_{i}a_{i}t^{i} \in \mathbb{C}_{\infty}((t))$, and an integer $n \in \mathbb{Z}$, we define the $n$-fold twisting of $f$ by $f^{(n)} := \sum_{i}a_{i}^{q^{n}}t^{i}$. The fields $\overline{K}(t) \subset \mathbb{L}$ are stable under the action $f \mapsto f^{(n)}$ for each $n \in \mathbb{Z}$ and their fixed parts under the action $f \mapsto f^{(-1)}$ are $\mathbb{F}_{q}(t)$. Let $\|f\|_{\infty} := \max\{|a_{i}|_{\infty}\}$ denote the Gauss norm of $f$.

The formal power series

$$\Omega(t) := (-\theta)^{-\frac{q}{q-1}} \prod_{i=1}^{\infty}(1-\frac{t}{\theta^{q^{i}}}) \in \overline{K}_{\infty}[t]$$

is an entire function and it is an element of $\mathbb{T}^{\times}$. Clearly, it satisfies

$$\Omega(\theta) = \frac{1}{\pi} \text{ and } \Omega^{(-1)} = (t-\theta)\Omega.$$ 

Since $\Omega(t)$ has infinitely many zeros, it is transcendental over $\overline{K}(t)$.

Let $D_{0} := 1$ and $D_{i} := \prod_{j=0}^{i-1}(\theta^{q^{j}}-\theta^{q^{j}})$ for $i \geq 1$. For an integer $n \geq 0$ with $q$-adic expansion $n = \sum_{i}n_{i}q^{i}$ ($0 \leq n_{i} < q$), the factorial of Carlitz $\Gamma_{n+1} \in A$ is defined by

$$\Gamma_{n+1} := \prod_{i}D_{i}^{n_{i}}.$$ 

We set $D_{n}(t)$ (resp. $\Gamma_{n+1}(t)$) to be the inverse image of $D_{n}$ (resp. $\Gamma_{n+1}$) by the $\mathbb{F}_{q}$-isomorphism $\mathbb{F}_{q}[t] \xrightarrow{\cong} A; t \mapsto \theta$. For an integer $n \geq 0$, the Bernoulli-Carlitz number $B_{n} \in A$ is defined by

$$\sum_{n=0}^{\infty} B_{n} \prod_{i=0}^{n-1} \left(1 - \sum_{i=0}^{\infty} \frac{\prod_{j=1}^{i}(t^{q^{j}}-\theta^{q^{j}})}{D_{i}(t)} \frac{z^{q^{i}}}{D_{i}(t)}\right)^{-1}.$$ 

For each integer $n \geq 0$, Anderson and Thakur ([2, 3.7.1]) defined a polynomial $H_{n} \in A[t]$ by

$$\sum_{n=0}^{\infty} \frac{H_{n}}{\Gamma_{n+1}(t)} z^n = \left(1 - \sum_{i=0}^{\infty} \frac{\prod_{j=1}^{i}(t^{q^{j}}-\theta^{q^{j}})}{D_{i}(t)} \frac{z^{q^{i}}}{D_{i}(t)}\right)^{-1}.$$ 

We also set

\[ S_i(n) := \sum_{a \in A: \text{monic}} \frac{1}{a^n} \]

for each \( n \geq 1 \) and \( i \geq 0 \). These satisfy

\[ |H_{n-1}|_{\infty} < |\theta|_{\infty}^{\frac{nq}{1-q}} \]

and

\[ (H_{n-1}\Omega^n)^{(i)}(\theta) = \frac{r^n S_i(n)}{\pi^n} \]

for each \( n \geq 1 \) and \( i \geq 0 \) (see [2, 3.7.4], [3, 2.4.1]).

For an index \( \underline{n} = (n_1, \ldots, n_d) \), the formal power series

\[ L_{\underline{n}}(t) := \sum_{i_1 > \cdots > i_d \geq 0} \frac{H_{n_1}^{(i_1)} \cdots H_{n_d}^{(i_d)}}{(t-\theta)^{n_1+\cdots+n_d}} \]

converges on \( |t|_{\infty} < |\theta|_{\infty}^{q} \) and it is an element of \( T \). Clearly, it satisfies

\[ L_{\underline{n}}(-1) = \frac{H_{n_d-1}^{(-1)} \cdots H_{n_d-1}^{(-1)}}{(t-\theta)^{n_1+\cdots+n_{d-1}}} L_{n_d1} + \frac{L_{n_d}}{(t-\theta)^{n_1+\cdots+n_d}} , \]

where we set \( L_{\underline{n}11} = L_{\emptyset} := 1 \) when \( d = 1 \). Anderson and Thakur ([3, 2.5.6]) showed that

\[ L_{\underline{n}}(\theta) = \Gamma_{n_1} \cdots \Gamma_{n_d} \zeta(n) . \]

Next, we recall Papanikolas’ theory of pre-t-motives. We do not give the complete details, but see [8] for more on this theory. See also [6, Section 2], [7, Section 3].

A pre-t-motive \( M \) is a finite dimensional \( \overline{K}(t) \)-vector space equipped with a bijective additive map \( \varphi: M \to M \) such that \( \varphi(fm) = f^{(-1)} \varphi(m) \) for \( f \in \overline{K}(t) \) and \( m \in M \). We always assume that \( M \) is rigid analytically trivial. Thus such \( M \) is determined by the matrix \( \Phi \in \text{GL}_r(\overline{K}(t)) \) (\( r := \dim M \)) representing the \( \varphi \)-action with respect to a fixed basis, such that

\[ \Psi^{(-1)} = \Phi \Psi \]

for some matrix \( \Psi \in \text{GL}_r(\mathbb{L}) \). The Betti realization \( \omega(M) \) is defined and is functorial on \( M \) (see [8, 3.4 and 3.5]). The space \( \omega(M) \) is an \( \mathbb{F}_q(t) \)-vector space and its dimension over \( \mathbb{F}_q(t) \) is equal to the dimension of \( M \) over \( \overline{K}(t) \). The category of (rigid analytically trivial) pre-t-motives forms a neutral Tannakian category over \( \mathbb{F}_q(t) \) with fiber functor \( \omega \). We denote by \( G_M \) the fundamental group of the Tannakian subcategory generated by \( M \). When we fix a basis of \( M \) and choose a matrix \( \Psi \) as above, we also define

\[ G_{\Psi} := \text{Spec}(\mathbb{F}_q(t)[X, 1/\det X]/\text{Ker } \nu) \subset \text{GL}_{r, \mathbb{F}_q(t)} , \]

where \( X = (X_{ij}) \) is a matrix of \( r \times r \) variables and \( \nu \) is the \( \mathbb{F}_q(t) \)-morphism defined by

\[ \nu: \mathbb{F}_q(t)[X, 1/\det X] \to \mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L} ; \ X_{ij} \mapsto \tilde{\Psi}_{ij} . \]
Here we set \( \overline{\Psi} := \Psi_1^{-1}\Psi_2 \in \text{GL}_r(\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L}) \), where \( \Psi_1 \) (resp. \( \Psi_2 \)) is the matrix defined by \( (\Psi_1)_{ij} := \Psi_{ij} \otimes 1 \) (resp. \( (\Psi_2)_{ij} := 1 \otimes \Psi_{ij} \)). Papanikolas ([8, Theorem 4.2.11]) showed that the scheme \( G_\Psi \) is a closed subgroup scheme of \( \text{GL}_r(\overline{\mathbb{F}}_q(t)) \). Moreover, he proved the following theorem:

**Theorem 2.1** ([8, Theorems 4.3.1, 4.5.10, 5.2.2]). There exists a natural isomorphism \( G_\Psi \cong G_M \) and the equality

\[
\dim G_\Psi = \text{tr.deg}_{\overline{K}(t)} K(t)(\Psi_{ij}|i, j)
\]

holds. Moreover, this value is equal to

\[
\text{tr.deg}_{\overline{K}} K(\Psi_{ij}(\theta)|i, j)
\]

if \( \Phi \in \text{Mat}_r(\overline{K}[t]) \), \( \det \Phi = c(t-\theta)^n \) for some \( n \in \mathbb{Z}_{\geq 0} \) and \( c \in \overline{K}^\times \), each entry of \( \Psi \) is entire and \( \Psi \in \text{GL}_r(\mathbb{T}) \).

**Remark.** The last part of Theorem 2.1 is proved by using a very deep result in [1, Theorem 3.1.1], which is called the ABP-criterion. We use the last part of Theorem 2.1 to prove our theorems.

**Example 2.2.** Let \( n = (n_1, \ldots, n_d) \) be an index. Let \( M[n] \) be the pre-t-motive defined by the \((d+1) \times (d+1)\)-matrix

\[
\Phi[n] := \begin{bmatrix}
(t-\theta)^{n_1+\cdots+n_d} & 0 & 0 & \cdots & 0 \\
H_{n_1-1}^{(-1)}(t-\theta)^{n_1+\cdots+n_d} & (t-\theta)^{n_2+\cdots+n_d} & 0 & \cdots & 0 \\
0 & H_{n_2-1}^{(-1)}(t-\theta)^{n_2+\cdots+n_d} & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & (t-\theta)^{n_d} & 0 \\
0 & \cdots & 0 & H_{n_d-1}^{(-1)}(t-\theta)^{n_d} & 1
\end{bmatrix}
\]

We also set

\[
\Psi[n] := \begin{bmatrix}
\Omega^{n_1+\cdots+n_d} & 0 & 0 & \cdots & 0 \\
\Omega^{n_1+\cdots+n_d} L_{n_2} & \Omega^{n_2+\cdots+n_d} & \ddots & \ddots & 0 \\
\Omega^{n_1+\cdots+n_d} L_{n_2} & \Omega^{n_2+\cdots+n_d} L_{n_3} & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\Omega^{n_1+\cdots+n_d} L_{n_d+1} & \cdots & \cdots & \cdots & \Omega^{n_d} L_{n_d+1, d} & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \Omega^{n_d} L_{n_d+1, d}
\end{bmatrix}
\]

Then we have \( \Psi[n]^{(-1)} = \Phi[n]\Psi[n] \). By Theorem 2.1, we have \( G_{\Psi[n]} \cong G_{M[n]} \) and

\[
\dim G_{\Psi[n]} = \text{tr.deg}_{\overline{K}(t)} K(t)(\Omega, L_m|m \in \text{Sub}(n)) = \text{tr.deg}_{\overline{K}} K(\pi, \zeta(m)|m \in \text{Sub}(n)).
\]
The matrix $\Psi[n] = (\Psi[n]_{ij})$ is calculated as

$$\Psi[n]_{ij} = (\Omega^{-1} \otimes \Omega)^{n_1 + \cdots + n_d} \sum_{s=j}^{i} \sum_{r=0}^{i-s} (-1)^{r} \sum_{s=i_0 < i_1 < \cdots < i_{r-1} < i_r = i} \cdots \sum_{s=i_0 < i_1 < \cdots < i_{r-1} < i_r = i} L_{i_1 i_0} \otimes \Omega^{n_j + \cdots + n_i-1} L_{s j}$$

for each $j < i$, where we denote $L_{k \ell} := L_{\underline{k \ell}}$.

**Example 2.3.** For $k = 1, 2$, let $\Phi_k \in \text{GL}_{r_k}(\overline{K}(t))$ and $\Psi_k \in \text{GL}_{r_k}(L)$ be matrices such that $\Psi_k^{(-1)} = \Phi_k \Psi_k$, and let $M_k$ be the pre-$t$-motive defined by $\Phi_k$. Since $M_k$ is a direct factor of $M_1 \oplus M_2$, there exists a surjective map

$$G_{\Psi_1 \oplus \Psi_2} \cong G_{M_1 \oplus M_2} \rightarrow G_{M_k} \cong G_{\Psi_k}$$

by Tannakian duality and Theorem 2.1. This coincides with the restriction of the $k$-th projection $\text{GL}_{r_1} \times \text{GL}_{r_2} \rightarrow \text{GL}_{r_k}$.

§3. Lifts of $p$-th power relations

In this section, we study $p$-th power relations among $L_\underline{n}$’s, which are lifts of $p$-th power relations among MZVs. For an index $\underline{n} = (n_1, \ldots, n_d)$, we use the notations

$$\Omega^n := \Omega^{n_1 + \cdots + n_d} \quad \text{and} \quad \underline{n}^' := (n_1, \ldots, n_{d-1}).$$

**Lemma 3.1.** For each positive integer $n \geq 1$ and each non-negative integer $e \geq 0$, we have

$$\frac{H_{p^e n - 1}}{\Gamma_{p^e n}(t)} = \left(\frac{H_{n-1}}{\Gamma_n(t)}\right)^{p^e}.$$

**Proof.** We have

$$\left(\frac{H_{s-1}}{\Gamma_s(t)}(\theta^{q^{-i}})\right)^{q^i} = \left(\frac{H_{s-1}}{\Gamma_s(t)}(\theta)\right)^{(i)} = \frac{S_i(s)}{\pi^s(\Omega^s)^{(i)}(\theta)}.$$

Therefore

$$\left(\frac{H_{p^e n - 1}}{\Gamma_{p^e n}(t)}(\theta^{q^{-i}})\right)^{q^i} = \frac{S_i(p^n)}{\pi^{p^n}(\Omega^{p^n})^{(i)}(\theta)} = \left(\frac{S_i(n)}{\pi^n(\Omega^n)^{(i)}(\theta)}\right)^{p^e} = \left(\frac{H_{n-1}}{\Gamma_n(t)}(\theta^{q^{-i}})\right)^{p^e} \left(\frac{H_{n-1}}{\Gamma_n(t)}(\theta^{q^{-i}})\right)^{q^i}$$

for each $i \geq 0$. Thus we have

$$\frac{H_{p^e n - 1}}{\Gamma_{p^e n}(t)}(\theta^{q^{-i}}) = \left(\frac{H_{n-1}}{\Gamma_n(t)}\right)^{p^e}(\theta^{q^{-i}})$$
for each $i \geq 0$.

We set
\[ \gamma_{e,n} := \frac{H_{p^e n-1}}{H_{n-1}^{p^e}} = \frac{\mathrm{r}_{p^e n}^t(t)}{\mathrm{r}_{n}^t(t)^{p^e}} \in \mathbb{F}_q(t)^\times \]
and
\[ \gamma_{e,n} := \prod_{i=1}^{d} \gamma_{e,n_i} \]
for any index $\underline{n} = (n_1, \ldots, n_d)$. When $p^{-e} \underline{n} \in \mathbb{Z}^d$, we also set
\[ \gamma_{-e,n} := \gamma_{e,p\underline{n}}^{-p^{-e}}. \]

The next lemma gives a $p$-th power relation among $L_{\underline{n}}$ and $L_{p^e \underline{n}}$, which is a lift of the $p$-th power relation $\zeta(p^e \underline{n}) = \zeta(\underline{n})^{p^e}$.

**Lemma 3.2.** For each index $\underline{n}$ of depth $d$ and each integer $e \in \mathbb{Z}$ such that $p^e \underline{n} \in \mathbb{Z}^d$, we have
\[ L_{p^e \underline{n}} = \gamma_{e,\underline{n}} L_{\underline{n}}^{p^e}. \]

**Proof.** By the definition of $\gamma_{e,\underline{n}}$ for negative integers $e$, we may assume that $e \geq 0$. We prove this equality by induction on $d$. When $d = 0$, it is clear. We take $d \geq 1$ and assume that the above equality holds for any indices whose depths are lower than $d$. Then we have
\[
(\Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e})^{(-1)}
= H_{p^e n_d-1}^{(-1)}(t-\theta)^{p^e n_d} - \gamma_{e,n_d} (H_{n_d-1}^{(-1)}(t-\theta)^{n_d})^{p^e}
= \Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e} + (t-\theta)^{p^e n_d} \Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e} - \gamma_{e,n_d} (H_{n_d-1}^{(-1)})^{p^e} L_{\underline{n}}^{p^e}
= \Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e} + (t-\theta)^{p^e n_d} \Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e} - \gamma_{e,n_d} (H_{n_d-1}^{(-1)})^{p^e} L_{\underline{n}}^{p^e}
= \Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e} + (t-\theta)^{p^e n_d} \Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e} - \gamma_{e,n_d} (H_{n_d-1}^{(-1)})^{p^e} L_{\underline{n}}^{p^e}
= \Omega^{p^e \underline{n}} L_{p^e \underline{n}} - \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e}.
\]
Thus
\[ \Omega^{p^e \underline{n}} L_{p^e \underline{n}} = \gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e} + c \]
for some $c \in \mathbb{F}_q(t)$. Since we have
\[
(\gamma_{e,\underline{n}} (\Omega^{\underline{n}} L_{\underline{n}})^{p^e})(\theta) = \frac{\Gamma_{p^e n_1} \cdots \Gamma_{p^e n_d}}{\Gamma_{n_1} \cdots \Gamma_{n_d}} \left( \frac{\Gamma_{n_1} \cdots \Gamma_{n_d} \zeta(\underline{n})}{\pi^{n_1+\cdots+n_d}} \right)^{p^e}
= \frac{\Gamma_{p^e n_1} \cdots \Gamma_{p^e n_d} \zeta(p^e \underline{n})}{\pi^{p^e n_1+\cdots+p^e n_d}}
= (\Omega^{p^e \underline{n}} L_{p^e \underline{n}})(\theta),
\]
we conclude \( c(\theta) = 0 \), and hence \( c = 0 \). □

**Lemma 3.3.** Let \( n = (n_1, \ldots, n_d) \) be an index and \( \Psi := \Psi[n] \in \text{GL}_{d+1}(\mathbb{L}) \) the matrix defined in Example 2.2. Take \( 1 \leq j < i \leq d+1 \) and \( 1 \leq \ell < k \leq d+1 \) such that \( n_{ij} = p^e n_{k\ell} \) for some integer \( e \in \mathbb{Z} \). Then the equality

\[
\Psi_{ij}/\Psi_{ii} = \gamma_{e, n_{k\ell}} (\Psi_{k\ell}/\Psi_{kk})^{p^e}
\]

holds.

**Proof.** We may assume that \( e \geq 0 \). By Example 2.2, we have

\[
\Psi_{ij}/\Psi_{ii} = \sum_{s=j}^{i} \sum_{r=0}^{i-s} (-1)^r \sum_{s=i_0 < i_1 < \cdots < i_{r-1} = i} L_{n_{i_1i_0}} \cdots L_{n_{i_{r}i_{r-1}}} \otimes \Omega^{n_{ij}} L_{n_{j}}
\]

\[
= \sum_{s=\ell}^{k} \sum_{r=0}^{k-s} (-1)^r \sum_{s=i_0 < i_1 < \cdots < i_{r-1} = i} L_{p^e n_{i_1i_0}} \cdots L_{p^e n_{i_{r}i_{r-1}}} \otimes \Omega^{p^e n_{k\ell}} L_{p^e n_{s\ell}}
\]

\[
= \sum_{s=\ell}^{k} \sum_{r=0}^{k-s} \gamma_{e, n_{k\ell}} \gamma_{e, n_{i_{1}i_{0}}} \cdots \gamma_{e, n_{i_{r}i_{r-1}}} (L_{p^e n_{i_1i_0}} \cdots L_{p^e n_{i_{r}i_{r-1}}} \otimes \Omega^{p^e n_{k\ell}} L_{p^e n_{s\ell}})^{p^e}
\]

\[
= \gamma_{e, n_{k\ell}} (\Psi_{k\ell}/\Psi_{kk})^{p^e}.
\]

□

§ 4. Lifts of Euler-Carlitz relations

In this section, we study Euler-Carlitz relations among \( \Omega \) and \( L_n \)'s, which are lifts of Euler-Carlitz relations among \( \widetilde{\pi} \) and Carlitz zeta values at positive “even” integers. Let \( n \geq 1 \) be a positive “even” integer. By [6, Remark 3.3], there exist \( c_n \in \mathbb{F}_q(t)^\times \) and \( f_n \in \overline{K}(t) \) such that

\[
\Omega^n L_n - c_n = f_n \Omega^n.
\]

This gives a lift of the Euler-Carlitz relation at \( n \). The \( c_n \) is determined by

\[
c_n(\theta) = \frac{\Gamma_n \zeta(n)}{\overline{\pi}^n} = \frac{\Gamma_n B_n}{\Gamma_{n+1}} \in K^\times.
\]
Let \( n = (n_1, \ldots, n_d) \) be an index and take \( j \) such that \( n_j \) is “even”. Then we have

\[
\tilde{\Psi}_{j+1,j}/\tilde{\Psi}_{j+1,j+1} = c_{n_j}(1 - (\Omega^{-1} \otimes \Omega)^{n_j}),
\]
where \( \Psi := \Psi[n] \) is the matrix defined in Example 2.2.

§5. Proofs

In this section, we always assume that algebraic groups are defined over \( \mathbb{F}_q(t) \). We need the following lemma. This can be proved easily and we omit the proof.

**Lemma 5.1.** Let \( V \subset \mathbb{G}_a^r \) be an algebraic subgroup of dimension zero. Let \( m_1, \ldots, m_r \in \mathbb{Z} \) be non-zero integers. Assume that \( V \) is stable under the \( \mathbb{G}_m \)-action on \( \mathbb{G}_a^r \) defined by

\[
a.(x_1, \ldots, x_r) = (a^{m_1} x_1, \ldots, a^{m_r} x_r) \quad (a \in \mathbb{G}_m, (x_i) \in \mathbb{G}_a^r)
\]

Then \( V(\overline{\mathbb{F}_q(t)}) \) is trivial.

From now on, we identify group schemes over \( \mathbb{F}_q(t) \) with the sets of \( \overline{\mathbb{F}_q(t)} \)-valued \((m)\) points of them. We use letters \( a, x_{ij}, \ldots \) for coordinate variables of algebraic groups.

**Proof of Theorem 1.5.** For \( 1 \leq \ell \leq k \), let \( G^{\leq \ell} \) be the algebraic group defined by

\[
\left[
a^{n_{1}^{(m)}+} & \cdots & n_{d_{m}}^{(m)} &  &  & \\
x_{21}^{(m)} &  &  & \ddots &  & \\
 &  &  &  & a^{n_{d_{m}}^{(m)}} & \\
x_{d_{m}+1,1}^{(m)} & \cdot &  &  & \cdot & 1
\right] \in \mathbb{G}_m \times \prod_{m=1}^\ell \text{GL}_{d_{m}+1}
\]

By Theorem 2.1 and the condition (4), this inclusion is actually an equality for \( 1 \leq \ell \leq k - 1 \). It suffices to show that this inclusion is actually an equality for \( \ell = k \).

We already have

\[
\dim G^{\leq k-1} \leq \dim G^{\leq k} \leq \dim G^{\leq k-1} + 1
\]

by the condition (3) and it suffices to show that the second inequality is an equality. Let

\[
\psi: G^{\leq k} \to G^{\leq k-1} \quad \text{and} \quad \pi^{\leq \ell}: G^{\leq \ell} \to G_{\Omega} \cong \mathbb{G}_m \ (1 \leq \ell \leq k)
\]
be the surjections obtained as in Example 2.3. We set $V^{\leq \ell} := \ker \pi^{\leq \ell}$ to be the unipotent radical of $G^{\leq \ell}$. Then we have the following commutative diagram

$$
\begin{array}{cccc}
1 & \rightarrow & V^{\leq k} & \rightarrow \ G^{\leq k} \ \\
\downarrow \psi' & & \downarrow \psi & \rightarrow \ G_m \rightarrow \ 1 \\
1 & \rightarrow & V^{\leq k-1} & \rightarrow \ G^{\leq k-1} \ \\
\end{array}
$$

with exact rows, where $\psi'$ is the restriction of $\psi$ to $V^{\leq k}$. The morphism $\psi'$ is surjective. It is clear that $\ker \psi'$ is a normal subgroup of $G^{\leq k}$. The conjugate action $X \mapsto A^{-1}XA$ of $G^{\leq k}$ on $\ker \psi'$ factors through the action of $\mathbb{G}_m$ on $\ker \psi'$ by

$$a.(\ldots, 0, x^{(k)}_{d_k+1,1}, 0, \ldots) = (\ldots, 0, a^{n_{1}^{(k)}+\cdots+n_{d_k}^{(k)}}x^{(k)}_{d_k+1,1}, 0, \ldots).$$

Now, we assume that $\dim G^{\leq k} = \dim G^{\leq k-1}$ and we shall induce a contradiction. Since $\dim \ker \psi' = 0$, the group $\ker \psi'$ is trivial by Lemma 5.1. Let $X = (x_{ij}^{(m)})$ (resp. $A = (a_{ij}^{(m)})$) be any element of $V^{\leq k}$ such that $x_{ij}^{(m)} = 0$ if $i-j \neq 1$ (resp. $a_{ij}^{(m)} = 0$ if $i-j \neq d_k - 1$) and $(m, i, j) \neq (k, d_k + 1, 1)$. For each $1 \leq m \leq k$, we set $x^{(m)}_j := x^{(m)}_{j+1,j}$ (1 $\leq j \leq d_m$) and $a^{(m)}_j := a^{(m)}_{d_k-1+j,j}$ (1 $\leq j \leq d_m - d_k + 2$). We also set $d := d_k - 1$ (which is $\geq 1$ by the condition (1)). Then the $m$-th component of $A^{-1}XA$ is equal to

$$
\begin{bmatrix}
1 \\
x_1^{(m)} \\
0 \\
\vdots \\
0 \\
0 \\
0 \\
\vdots \\
0 \\
-a_{d+2}^{(m)}(a_1^{(m)}x^{(m)}_{d+1} - a_2^{(m)}x_1^{(m)}) \\
0 \\
\vdots \\
0 \\
-a_{d+3}^{(m)}(a_2^{(m)}x^{(m)}_{d+2} - a_3^{(m)}x_2^{(m)}) \\
\vdots \\
0 \\
\vdots \\
0 \\
a_{d+2}^{(m)}a_1^{(m)}x^{(m)}_{d+1} - a_2^{(m)}x_1^{(m)} \\
0 \\
\vdots \\
\vdots \\
a_{d+3}^{(m)}a_2^{(m)}x^{(m)}_{d+2} - a_3^{(m)}x_2^{(m)} \\
\vdots \\
\vdots
\end{bmatrix}
$$
for each $1 \leq m < k$. Thus the $(i,j)$-th entry of the $m$-th (resp. $k$-th) component of $A^{-1}XA$ is

$$
\begin{cases}
1 & (i-j = 0) \\
x_{j} & (i-j = 1) \\
\left(-1\right)^{r-1} \prod_{s=1}^{r-1} a_{sd_j+1}^{(m)} & (i-j = rd+1, \ r \geq 1) \\
\left(\text{resp. } d_{1}, d_{k} \right) & (\text{resp. } i-j = d+1 = d_{k}) \\
0 & \text{(otherwise)}
\end{cases}
$$

Therefore, if the equalities

$$
(5.1) \quad a_{i-1,j}^{(m)}x_{i,i-1}^{(m)} - a_{i,j+1}^{(m)}x_{i,i-1}^{(m)} = 0 \quad (1 \leq m < k, \ 1 \leq j < i \leq d_{m}+1, \ i - j = d_{k})
$$

hold, then $X^{-1}A^{-1}XA \in \text{Ker} \psi' = \{1\}$ and the equality

$$
(5.2) \quad a_{d_{k}1}^{(k)}x_{d_{k}+1,d_{k}}^{(k)} - a_{d_{k}+1,2}^{(k)}x_{21}^{(k)} = 0
$$

must hold. We show that this implication induces a contradiction and hence we have $\dim G^{\leq k} = \dim G^{\leq k-1} + 1$.

First, we assume that the condition (5-1) holds. If $n_{d_{k},1}^{(k)} \neq n_{d_{k}+1,2}^{(k)}$, we can take

$$
a_{d_{k}+1,2}^{(k)}x_{21}^{(k)} \neq 0, \quad a_{i-1,j}^{(m)}x_{i,i-1}^{(m)} = a_{i,j+1}^{(m)}x_{i,i-1}^{(m)} = 0
$$

for each $(m, i, j)$. Then the equalities (5.1) hold and hence the equality (5.2) also holds. However, it becomes $a_{d_{k}+1,2}^{(k)}x_{21}^{(k)} = 0$. This is a contradiction. If $n_{d_{k},1}^{(k)} \sim n_{d_{k}+1,2}^{(k)}$ (and hence $n_{1}^{(k)} \sim n_{d_{k}}^{(k)}$), there exists an integer $e$ such that $n_{d_{k},1}^{(k)} = p^{e}n_{d_{k}+1,2}^{(k)}$. Then we can take

$$
a_{i-1,j}^{(m)}x_{i,i-1}^{(m)} = a_{i,j+1}^{(m)}x_{i,i-1}^{(m)} = 0
$$

for each $(m, i, j)$ and the equality (5.2) becomes

$$
\gamma_{e,n_{d_{k}+1,2}}^{(k)} (a_{d_{k}+1,2}^{(k)})^{p^{e}}x_{d_{k}+1,d_{k}}^{(k)} - a_{d_{k}+1,2}^{(k)}x_{21}^{(k)} = 0
$$

for each $a_{d_{k}+1,2}^{(k)}$ and $x_{21}^{(k)}$. Then $e$ must be zero. However, since $n_{d_{k},1}^{(k)} \neq (n_{1}^{(k)}, \ldots, n_{1}^{(k)})$ by the condition (2), $e$ is non-zero. This is a contradiction.

Similarly, when the condition (5-1)' holds, we obtain a contradiction.

Next, we assume that the condition (5-2) holds. Then we can take

$$
x_{21}^{(k)} \neq 0, \quad a_{i-1,j}^{(m)}x_{i,i-1}^{(m)} = a_{i,j+1}^{(m)}x_{i,i-1}^{(m)} = 0
$$

for each $(m, i, j) \neq (m_{0}, i_{0}, j_{0})$ and $a_{d_{k},1}^{(k)}, a_{d_{k}+1,2}^{(k)}$ as any elements. There exist integers $e_{1}, e_{2}, e_{3}, e_{4} \in \mathbb{Z}$ such that

$$
n_{i_{0}-1,j_{0}}^{(m_{0})} = p^{e_{1}}n_{d_{k}+1,2}^{(k)}, \quad n_{i_{0},j_{0}+1}^{(m_{0})} = p^{e_{2}}n_{d_{k},1}^{(k)}, \quad n_{j_{0}}^{(m_{0})} = p^{e_{3}}n_{d_{k}}^{(k)}, \quad n_{i_{0}-1}^{(m_{0})} = p^{e_{4}}n_{1}^{(k)}.
Then the \((m_0, i_0, j_0)\)-th equality of the equalities (5.1) becomes
\[
\gamma_{e_1,n_{d_k+1,2}^{(k)}}(a_{d_k+1,2}^{(k)})^{p^e_1} \gamma_{e_2,n_{d_k}^{(k)}}(x_{d_k+1,d_k}^{(k)})^{p^e_3} = 0.
\]
We take any \(a_{d_k+1,2}^{(k)}\) and set
\[
a_{d_k,1}^{(k)} := \left(\frac{\gamma_{e_1,n_{d_k+1,2}^{(k)}}(a_{d_k+1,2}^{(k)})^{p^e_1}}{\gamma_{e_2,n_{d_k}^{(k)}}(x_{d_k+1,d_k}^{(k)})^{p^e_3}}\right)^{p^{-e_2}} (a_{d_k+1,2}^{(k)})^{p^e_1-e_2}.
\]
Then the equalities (5.1) hold and the equality (5.2) becomes
\[
\left(\frac{\gamma_{e_1,n_{d_k+1,2}^{(k)}}(a_{d_k+1,2}^{(k)})^{p^e_1}}{\gamma_{e_2,n_{d_k}^{(k)}}(x_{d_k+1,d_k}^{(k)})^{p^e_3}}\right)^{p^{-e_2}} (a_{d_k+1,2}^{(k)})^{p^e_1-e_2} x_{d_k+1,d_k}^{(k)} - a_{d_k+1,2}^{(k)} x_{21}^{(k)} = 0
\]
and holds for each \(a_{d_k+1,2}^{(k)}\). This implies \(e_1 = e_2\). Then we have \(n_\ell^{(k)} \sim n_{\ell+2}^{(k)}\) for each \(\ell\) (resp. \(n^{(k)} \sim (n_2^{(m_0)}, n_1^{(m_0)})\)) if \(d_k \geq 3\) (resp. \(d_k = 2\)). In any case, we obtain a contradiction.

Similarly, when the condition (5-2)’ holds, we have \(n^{(k)} = (n_1^{(k)}, \ldots, n_1^{(k)})\) (resp. \(n^{(k)} \sim n^{(m_0)}\)) if \(d_k \geq 3\) (resp. \(d_k = 2\)). In any case, we obtain a contradiction. \(\square\)

**Proof of Proposition 1.6.** We use the notations in the proof of Theorems 1.5. We set
\[
\begin{align*}
n^{(1)} := n_1, & \quad n^{(2)} := n_2, \quad n^{(3)} := n_3, \\
n^{(4)} := (n_1, n_2), & \quad n^{(5)} := (n_2, n_3), \quad n^{(6)} := (n_1, n_2, n_3).
\end{align*}
\]
By Theorem 1.5 and the result of Chang and Yu ([5, Corollary 4.6]), we have
\[
\text{tr.deg}_K \mathcal{K}(\pi, \zeta(n^{(m)}))|1 \leq m \leq 4) = \#(\{(q-1)\} \cup \{n^{(m)}|1 \leq m \leq 4\})/\sim.
\]
If we prove
\[
\text{tr.deg}_K \mathcal{K}(\pi, \zeta(n^{(m)}))|1 \leq m \leq 5) = \#(\{(q-1)\} \cup \{n^{(m)}|1 \leq m \leq 5\})/\sim,
\]
then Proposition 1.6 follows from Theorem 1.5.

Assume that \(n^{(5)} \not\sim n^{(4)}\) and \(\dim G \leq 5 = \dim G \leq 4\). In this case, the equality \(a_{21}^{(4)} x_{32}^{(4)} - a_{32}^{(4)} x_{21}^{(4)} = 0\) implies the equality \(a_{21}^{(5)} x_{32}^{(5)} - a_{32}^{(5)} x_{21}^{(5)} = 0\). We may assume that \(n_1 \sim n_2 \sim n_3\), otherwise we obtain a contradiction from [7, Theorem 1.1] and Theorem 1.4. We set
\[
\begin{align*}
a_1 := a_{21}^{(4)}, & \quad a_2 := a_{32}^{(4)} = a_{21}^{(5)}, \quad a_3 := a_{32}^{(5)}, \\
a_1 := x_{21}^{(4)}, & \quad x_2 := x_{32}^{(4)} = x_{21}^{(5)}, \quad x_3 := x_{32}^{(5)}.
\end{align*}
\]
For each $j$, we have $n_j = p^{e_j} n$ for some $n \geq 1$ and $e_j \geq 0$ with $\min\{e_j\} = 0$. We set $a := a_{j_0}$ and $x := x_{j_0}$ for some $j_0$ such that $e_{j_0} = 0$. Thus we have $a_j = \gamma_{e_j,n} a^{p^{e_j}}$ and $x_j = \gamma_{e_j,n} x^{p^{e_j}}$ for each $j$. Then
\[
a^{p^e_1} x^{p^e_2} - a^{p^e_2} x^{p^e_1} = 0 \implies a^{p^e_2} x^{p^e_3} - a^{p^e_3} x^{p^e_2} = 0
\]
for any $a, x \in \overline{\mathbb{F}_q(t)}$. Since $e_1 \neq e_2$, we conclude that $e_1 - e_2$ divides $e_2 - e_3$. By symmetric arguments, since $e_2 \neq e_3$, we conclude that $e_3 - e_2$ divides $e_2 - e_1$. This means that $e_1 - e_2 = \pm(e_2 - e_3)$. However this is a contradiction because we assume that $\overline{n}^{(5)} \not\subseteq \overline{n}^{(4)}$ and $e_1 \neq e_3$.

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\textbf{References}


