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p-th power relations and Euler-Carlitz relations among multizeta values

By

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Abstract

In this paper, we study p-th power relations and Euler-Carlitz relations among multizeta values in characteristic p. By definition, two multizeta values have the p-th power relation if their indices map to each other multiplying by some power of p. The multizeta values of depth one at "even" integers satisfy Euler-Carlitz relations which are analogues of the relations among the Riemann zeta values at positive even integers. We prove that all algebraic relations among given multizeta values come from p-th power relations and Euler-Carlitz relations if their indices satisfy some conditions.

§ 1. Introduction

Let $\underline{n} = (n_1, \dots, n_d) \in (\mathbb{Z}_{\geq 1})^d$ be a d-tuple ($d \geq 1$) of positive integers such that $n_1 \geq 2$. The sum

$$\zeta_{\mathbb{Z}}(\underline{n}) := \sum_{m_1 > \cdots > m_d > 0} \frac{1}{m_1^{n_1} \cdots m_d^{n_d}} \in \mathbb{R}$$

is called the *multiple zeta value* (MZV) and studied by many mathematicians. Many relations over $\mathbb{Q}$ among MZV are known. For example, Euler showed that

$$\zeta_{\mathbb{Z}}(n) \in (2\pi\sqrt{-1})^n \cdot \mathbb{Q}^\times$$

for each positive even integer $n \geq 2$. We also have the *harmonic product formula*. The simplest case is as follows:

$$\zeta_{\mathbb{Z}}(n_1)\zeta_{\mathbb{Z}}(n_2) = \zeta_{\mathbb{Z}}(n_1, n_2) + \zeta_{\mathbb{Z}}(n_2, n_1) + \zeta_{\mathbb{Z}}(n_1 + n_2).$$

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We also want to know the linear/algebraic independence among given MZVs. However, we do not even know whether $\zeta_{\mathbb{Z}}(n)$ is transcendental over $\mathbb{Q}$ for each positive odd integer $n \geq 3$. In general, such problems seem very difficult.

Next, we consider the positive characteristic case. We fix a prime number p and its power q. Let θ be a variable, $A := \mathbb{F}_q[\theta]$ the one variable polynomial ring over $\mathbb{F}_q$, $K := \mathbb{F}_q(\theta)$ the fraction field of A, $K_\infty := \mathbb{F}_q((\theta^{-1}))$ the ∞-adic completion of K, $\mathbb{C}_\infty$ the ∞-adic completion of an algebraic closure of K_∞, and $\overline{K}$ the algebraic closure of K in $\mathbb{C}_\infty$. Let $\underline{n} = (n_1, \ldots, n_d) \in (\mathbb{Z}_{\geq 1})^d$ be a d-tuple ($d \geq 1$) of positive integers. Such an $\underline{n}$ is called an *index* of *weight* $\mathrm{wt}(\underline{n}) := \sum n_i$ and *depth* $\mathrm{dep}(\underline{n}) := d$. For an index $\underline{n}$, Thakur ([9, Section 5.10]) defined the *multizeta value* in characteristic p by

$$\zeta(\underline{n}) := \sum_{\substack{a_1, \ldots, a_d \in A:\text{monic} \\ \deg(a_1) > \cdots > \deg(a_d) \geq 0}} \frac{1}{a_1^{n_1} \cdots a_d^{n_d}} \in K_\infty.$$

We are also interested in determining all relations over $\overline{K}$ among given MZVs. For an index $\underline{n} = (n_1, \ldots, n_d)$ and an integer $e \in \mathbb{Z}$, we set

$$p^e \underline{n} := (p^e n_1, \ldots, p^e n_d) \in \mathbb{Z}[1/p]^d.$$

If $p^e \underline{n} \in \mathbb{Z}^d$, the *p-th power relation*

$$\zeta(p^e \underline{n}) = \zeta(\underline{n})^{p^e}$$

follows immediately from the definition of $\zeta(\underline{n})$. The MZVs of depth one are defined by Carlitz ([4]) and called the *Carlitz zeta values*. Carlitz showed the relation

$$\zeta(n) = \widetilde{\pi}^n \cdot \frac{B_n}{\Gamma_{n+1}}$$

for each positive integer $n \geq 1$ which is divisible by $q - 1$, where

$$\widetilde{\pi} := (-\theta)^{\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1} \in (-\theta)^{\frac{1}{q-1}} \cdot K_\infty^\times$$

is the *Carlitz period*, $B_n \in A$ is the *Bernoulli-Carlitz number* and $\Gamma_{n+1} \in A$ is the *factorial* of Carlitz (see Section 2). These relations are called the *Euler-Carlitz relations*. These are analogues of Euler's relations of the special zeta values at positive even integers. We say that a positive integer $n \geq 1$ is "even" (resp. "odd") if n is divisible (resp. not divisible) by $q - 1$. After works of Wade ([10]) and Yu ([11], [12]), finally Chang and Yu ([5, Corollary 4.6]) proved that all relations over $\overline{K}$ among the Carlitz zeta values come from p-th power relations and Euler-Carlitz relations. This means that if $n_1, \ldots, n_d \geq 1$ are positive "odd" integers such that n_i/n_j is not an integral power of

p for each $i \neq j$, then $\widetilde{\pi}, \zeta(n_1), \ldots, \zeta(n_d)$ are algebraically independent over $\overline{K}$. This is generalized in [7, Theorem 1.1] as follows: if $n_1, \ldots, n_d$ satisfy the above assumptions, then the set

$$\{\widetilde{\pi}\} \cup \{\zeta(\underline{m}) | \underline{m} \in \mathrm{Sub}(\underline{n})\}$$

has $1 + d(d+1)/2$ elements and these elements are algebraically independent over $\overline{K}$, where for an index $\underline{n} = (n_1, \ldots, n_d)$, we set

$$\mathrm{Sub}(\underline{n}) := \{(n_j, n_{j+1}, \ldots, n_i) | 1 \leq j \leq i \leq d\}.$$

Our results in this paper contain this as a special case.

To explain the results, let us introduce some notations.

Definition 1.1. Let $\underline{n} = (n_1, \ldots, n_d)$ be an index.
(1) We set

$$\mathrm{Sub}'(\underline{n}) := \{(n_{i_1}, \ldots, n_{i_r}) | 1 \leq r \leq d,\ 1 \leq i_1 < \cdots < i_r \leq d\}.$$

Thus we have

$$\mathrm{Sub}'(\underline{n}) \supset \mathrm{Sub}(\underline{n}), \quad \#\,\mathrm{Sub}'(\underline{n}) \leq 2^d - 1 \ \text{ and } \ \#\,\mathrm{Sub}(\underline{n}) \leq \frac{d(d+1)}{2}.$$

(2) For each $1 \leq j < i \leq d+1$, we set

$$\underline{n}_{ij} := (n_j, n_{j+1}, \ldots, n_{i-1}).$$

Thus we have

$$\mathrm{Sub}(\underline{n}) = \{\underline{n}_{ij} | 1 \leq j < i \leq d+1\}.$$

(3) Let $\underline{n}'$ be another index. We say that $\underline{n}$ and $\underline{n}'$ are equivalent and denote by $\underline{n} \sim \underline{n}'$ if there exists an integer $e \in \mathbb{Z}$ such that $\underline{n} = p^e \underline{n}'$, or both $\underline{n}, \underline{n}'$ are of depth 1 and $\underline{n} = (m), \underline{n}' = (m')$ for some $m, m' \in (q-1)\mathbb{Z}$ (hence $\mathrm{dep}(\underline{n}) = \mathrm{dep}(\underline{n}')$ if $\underline{n} \sim \underline{n}'$). For positive integers m and m', we write $m \sim m'$ if the indices (m) and (m') of depth one are equivalent.
(4) Let S be a set of indices. We denote by $S/\sim$ the quotient set of S by the equivalence relation $\sim$.

The purpose of this paper is to generalize the result [7, Theorem 1.1] to the following three directions:

- n_i/n_j may be an integral power of p,
- n_i may be "even",
- treat elements $\zeta(\underline{m})$ for $\underline{m} \in \mathrm{Sub}'(\underline{n})$.

We expect that if the given indices satisfy some "good" conditions, then all algebraic relations over $\overline{K}$ among the multizeta values at such points come from p-th power relations and Euler-Carlitz relations. In fact, in this paper, we prove the following theorems:

Theorem 1.2. *Let $\underline{n} = (n_1, \dots, n_d)$ be an index such that the n_i's are "odd" and distinct from each other. Assume that there exists exactly one pair $j_1 < j_2$ such that $n_{j_1} \sim n_{j_2}$. We set*

$$S := \{\underline{m} | \underline{m} \in \mathrm{Sub}'(\underline{n}),\ (n_{j_1}, n_{j_2}) \notin \mathrm{Sub}(\underline{m})\}.$$

Then we have

$$\mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\widetilde{\pi}, \zeta(\underline{m}) | \underline{m} \in S) = \#((\{(q-1)\} \cup S)/\sim).$$

Note that the condition $(n_{j_1}, n_{j_2}) \notin \mathrm{Sub}(\underline{m})$ means that $\underline{m}$ is not an index of the form $\underline{m} = (\dots, n_{j_1}, n_{j_2}, \dots)$.

As a consequence of Theorem 1.2, we have the following corollary:

Corollary 1.3. *Let $\underline{n} = (n_1, \dots, n_d)$ be an index of positive "odd" integers such that n_i/n_j is not an integral power of p for each $i \neq j$. Then we have*

$$\mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\widetilde{\pi}, \zeta(\underline{m}) | \underline{m} \in \mathrm{Sub}'(\underline{n})) = 2^d.$$

We also have the following theorem:

Theorem 1.4. *Let $\underline{n} = (n_1, \dots, n_d)$ be an index such that the n_i's are "odd" and distinct from each other. Assume that there exists exactly one pair $j_1 \neq j_2$ such that $n_{j_1} \sim n_{j_2}$. Then we have*

$$\mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\widetilde{\pi}, \zeta(\underline{m}) | \underline{m} \in \mathrm{Sub}(\underline{n})) = \#((\{(q-1)\} \cup \mathrm{Sub}(\underline{n}))/\sim) = \frac{d(d+1)}{2}.$$

Remark. We do not know in general that when

$$\widetilde{\pi}, \zeta(n) \text{ and } \zeta((q-1)m, n) \quad (\text{or } \ \widetilde{\pi}, \zeta(n) \text{ and } \zeta(n, (q-1)m))$$

are algebraically independent over $\overline{K}$, where n is "odd". Thus we do not treat "even" integers in Theorems 1.2 and 1.4. When we treat "even" integers, we need to assume that the above elements are already algebraically independent over $\overline{K}$ as in Theorem 1.5.

For a set S of indices, we define a set $[S]$ by

$$[S] := \{\underline{m}\colon \text{index} | \underline{m} \sim \underline{n} \text{ for some } \underline{n} \in S\}.$$

Theorems 1.2 and 1.4 follow from the following theorem:

Theorem 1.5. *Let* $\underline{n}^{(m)} = (n_1^{(m)}, \dots, n_{d_m}^{(m)})$ $(1 \le m \le k,\ k \ge 2)$ *be indices. If the following conditions* (1) $\sim$ (5) *hold, then we have*

$$\mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\widetilde{\pi}, \zeta(\underline{n}) | \underline{n} \in \cup_{m=1}^{k} \mathrm{Sub}(\underline{n}^{(m)})) = \# \left(\left(\{(q-1)\} \cup \bigcup_{m=1}^{k} \mathrm{Sub}(\underline{n}^{(m)}) \right) / \sim \right).$$

(1) $\mathrm{dep}(\underline{n}^{(k)}) = d_k \ge 2$.

(2) $\underline{n}^{(k)} \ne (n_1^{(k)}, n_1^{(k)}, \dots, n_1^{(k)})$.

(3) $\mathrm{Sub}(\underline{n}^{(k)}) \smallsetminus [\cup_{m=1}^{k-1} \mathrm{Sub}(\underline{n}^{(m)})] = \{\underline{n}^{(k)}\}$.

(4) $\mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\widetilde{\pi}, \zeta(\underline{n}) | \underline{n} \in \cup_{m=1}^{k-1} \mathrm{Sub}(\underline{n}^{(m)})) = \#((\{(q-1)\} \cup \bigcup_{m=1}^{k-1} \mathrm{Sub}(\underline{n}^{(m)}))/\sim)$.

(5) *In the following, the* (m, i, j) *runs over all triples of integers such that*

$$1 \le m \le k-1, \quad 1 \le j < i \le d_m + 1, \quad i - j = d_k.$$

One of the following four conditions holds:

(5-1)

$$n_1^{(k)} \not\sim n_{i-1}^{(m)} \ \text{ or } \ \underline{n}_{d_k+1,2}^{(k)} = (n_2^{(k)}, \dots, n_{d_k}^{(k)}) \not\sim \underline{n}_{i-1,j}^{(m)} = (n_j^{(m)}, \dots, n_{i-2}^{(m)})$$

and

$$n_1^{(k)} \not\sim n_j^{(m)} \ \text{ or } \ \underline{n}_{d_k+1,2}^{(k)} = (n_2^{(k)}, \dots, n_{d_k}^{(k)}) \not\sim \underline{n}_{i,j+1}^{(m)} = (n_{j+1}^{(m)}, \dots, n_{i-1}^{(m)})$$

for each (m, i, j) *and* $n_1^{(k)}, \underline{n}_{d_k+1,2}^{(k)} \not\sim q - 1$.

(5-1)$'$

$$n_{d_k}^{(k)} \not\sim n_{i-1}^{(m)} \ \text{ or } \ \underline{n}_{d_k,1}^{(k)} = (n_1^{(k)}, \dots, n_{d_k-1}^{(k)}) \not\sim \underline{n}_{i-1,j}^{(m)} = (n_j^{(m)}, \dots, n_{i-2}^{(m)})$$

and

$$n_{d_k}^{(k)} \not\sim n_j^{(m)} \ \text{ or } \ \underline{n}_{d_k,1}^{(k)} = (n_1^{(k)}, \dots, n_{d_k-1}^{(k)}) \not\sim \underline{n}_{i,j+1}^{(m)} = (n_{j+1}^{(m)}, \dots, n_{i-1}^{(m)})$$

for each (m, i, j) *and* $n_{d_k}^{(k)}, \underline{n}_{d_k,1}^{(k)} \not\sim q - 1$.

(5-2) *There exists exactly one triple* (m_0, i_0, j_0) *such that*

$$n_1^{(k)} \sim n_{i_0-1}^{(m_0)}, \quad n_{d_k}^{(k)} \sim n_{j_0}^{(m_0)}, \quad \underline{n}_{d_k,1}^{(k)} \sim \underline{n}_{i_0,j_0+1}^{(m_0)}, \quad \underline{n}_{d_k+1,2}^{(k)} \sim \underline{n}_{i_0-1,j_0}^{(m_0)},$$

$n_\ell^{(k)} \ne n_{\ell+2}^{(k)}$ *for some* ℓ *(resp.* $\underline{n}^{(k)} \not\sim (n_2^{(m_0)}, n_1^{(m_0)})$*) if* $d_k \ge 3$ *(resp.* $d_k = 2$*), and for other* (m, i, j)*'s*

$$n_1^{(k)}, n_{d_k}^{(k)} \not\sim n_{i-1}^{(m)} \ \text{ or } \ \underline{n}_{d_k,1}^{(k)}, \underline{n}_{d_k+1,2}^{(k)} \not\sim \underline{n}_{i-1,j}^{(m)}$$

and

$$n_1^{(k)}, n_{d_k}^{(k)} \not\sim n_j^{(m)} \quad \text{or} \quad \underline{n}_{d_k,1}^{(k)}, \underline{n}_{d_k+1,2}^{(k)} \not\sim \underline{n}_{i,j+1}^{(m)},$$

and $n_1^{(k)}, n_{d_k}^{(k)} \not\sim q-1$, *and* $\underline{n}_{d_k,1}^{(k)} \not\sim \underline{n}_{d_k+1,2}^{(k)}$.

(5-2)′ *There exists exactly one triple* (m_0, i_0, j_0) *such that*

$$n_1^{(k)} \sim n_{j_0}^{(m_0)}, \quad n_{d_k}^{(k)} \sim n_{i_0-1}^{(m_0)}, \quad \underline{n}_{d_k,1}^{(k)} \sim \underline{n}_{i_0-1,j_0}^{(m_0)}, \quad \underline{n}_{d_k+1,2}^{(k)} \sim \underline{n}_{i_0,j_0+1}^{(m_0)},$$

and for other (m,i,j)*'s*

$$n_1^{(k)}, n_{d_k}^{(k)} \not\sim n_{i-1}^{(m)} \quad \text{or} \quad \underline{n}_{d_k,1}^{(k)}, \underline{n}_{d_k+1,2}^{(k)} \not\sim \underline{n}_{i-1,j}^{(m)}$$

and

$$n_1^{(k)}, n_{d_k}^{(k)} \not\sim n_j^{(m)} \quad \text{or} \quad \underline{n}_{d_k,1}^{(k)}, \underline{n}_{d_k+1,2}^{(k)} \not\sim \underline{n}_{i,j+1}^{(m)},$$

and $n_1^{(k)}, n_{d_k}^{(k)} \not\sim q-1$, *and* $\underline{n}_{d_k,1}^{(k)} \not\sim \underline{n}_{d_k+1,2}^{(k)}$.

Proof of Theorem 1.2. We fix an order of the set $S = \{\underline{n}^{(1)}, \underline{n}^{(2)}, \dots\}$ such that $\mathrm{dep}(\underline{n}^{(1)}) \leq \mathrm{dep}(\underline{n}^{(2)}) \leq \cdots$. For each $1 \leq k \leq \#S$, we show the equality

$$\text{(1.1)} \qquad \mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\tilde{\pi}, \zeta(\underline{n}^{(1)}), \dots, \zeta(\underline{n}^{(k)})) = \#(\{(q-1), \underline{n}^{(1)}, \dots, \underline{n}^{(k)}\}/\sim)$$

by induction on k. If $\mathrm{dep}(\underline{n}^{(k)}) = 1$, then the equality comes from the result of Chang and Yu ([5, Corollary 4.6]). Let $\mathrm{dep}(\underline{n}^{(k)}) \geq 2$, then it is clear that the conditions (1), (2) and (3) of Theorem 1.5 hold. By the induction hypothesis, the condition (4) also holds. When $n_1^{(k)} \notin \{n_{j_1}, n_{j_2}\}$, the condition (5-1) holds if $d_k \geq 3$, and the condition (5-1), (5-2) or (5-2)′ holds if $d_k = 2$. Similarly, when $n_{d_k}^{(k)} \notin \{n_{j_1}, n_{j_2}\}$, the condition (5-1)′ holds if $d_k \geq 3$, and the condition (5-1)′, (5-2) or (5-2)′ holds if $d_k = 2$. When $n_1^{(k)} \sim n_{d_k}^{(k)}$ (this means that $n_1^{(k)} = n_{j_1}$ and $n_{d_k}^{(k)} = n_{j_2}$), then we have $d_k \geq 3$ by the definition of S, and the conditions (5-1) and (5-1)′ hold. In any case, the condition (5) of Theorem 1.5 holds, and hence the equality (1.1) follows from Theorem 1.5. □

Proof of Theorem 1.4. By Theorem 1.2, we may assume that $j_2 = j_1 + 1$. The proof is similar to that of Theorem 1.2. We fix an order on S as before, and show the equality (1.1) by induction. Let $\mathrm{dep}(\underline{n}^{(k)}) \geq 2$. Then the conditions (1), (2) and (3) of Theorem 1.5 hold clearly, and the condition (4) follows from the induction hypothesis. In this case, the conditions (5-1) and (5-1)′ hold. □

The next proposition does not follow from Theorem 1.5, but we can show this by similar arguments of the proof of Theorem 1.5.

Proposition 1.6. *Let* $\underline{n} = (n_1, n_2, n_3)$ *be an index of depth three. If the* n_i*'s are "odd" and distinct from each other, then we have*

$$\mathrm{tr.deg}_{\overline{K}}\, \overline{K}(\tilde{\pi}, \zeta(\underline{m}) | \underline{m} \in \mathrm{Sub}(\underline{n})) = \#(\{(q-1)\} \cup \mathrm{Sub}(\underline{n}))/\sim.$$

In Section 2, we define notations which are used in this paper and briefly review Papanikolas' theory of pre-t-motives. In Section 3 (resp. 4), we study "lifts" of p-th power (resp. Euler-Carlitz) relations. To apply Papanikolas' theory to MZVs which have p-th power or Euler-Carlitz relations, we need their lifts. In Section 5, we prove Theorem 1.5 and Proposition 1.6. The proofs are refinements of the proofs in [7].

§ 2. Preliminaries

We continue to use the notations of the Introduction. Let t be a new variable independent from θ. We fix an ∞-adic valuation $|-|_\infty$ on $\mathbb{C}_\infty$. Let $\mathbb{T} := \{f \in \mathbb{C}_\infty[\![t]\!] | f \text{ converges on } |t|_\infty \leq 1\}$ be the Tate algebra over $\mathbb{C}_\infty$ and $\mathbb{L}$ the fraction field of $\mathbb{T}$. For a formal Laurent series $f = \sum_i a_i t^i \in \mathbb{C}_\infty((t))$ and an integer $n \in \mathbb{Z}$, we define the n-fold twisting of f by $f^{(n)} := \sum_i a_i^{q^n} t^i$. The fields $\overline{K}(t) \subset \mathbb{L}$ are stable under the action $f \mapsto f^{(n)}$ for each $n \in \mathbb{Z}$ and their fixed parts under the action $f \mapsto f^{(-1)}$ are $\mathbb{F}_q(t)$. Let $\|f\|_\infty := \max_i\{|a_i|_\infty\}$ denote the Gauss norm of f.

The formal power series

$$\Omega(t) := (-\theta)^{-\frac{q}{q-1}} \prod_{i=1}^{\infty} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \overline{K_\infty}[\![t]\!]$$

is an entire function and it is an element of $\mathbb{T}^\times$. Clearly, it satisfies

$$\Omega(\theta) = \frac{1}{\tilde{\pi}} \quad \text{and} \quad \Omega^{(-1)} = (t-\theta)\Omega.$$

Since $\Omega(t)$ has infinitely many zeros, it is transcendental over $\overline{K}(t)$.

Let $D_0 := 1$ and $D_i := \prod_{j=0}^{i-1}(\theta^{q^i} - \theta^{q^j})$ for $i \geq 1$. For an integer $n \geq 0$ with q-adic expansion $n = \sum_i n_i q^i$ $(0 \leq n_i < q)$, the factorial of Carlitz $\Gamma_{n+1} \in A$ is defined by

$$\Gamma_{n+1} := \prod_i D_i^{n_i}.$$

We set $D_n(t)$ (resp. $\Gamma_{n+1}(t)$) to be the inverse image of D_n (resp. Γ_{n+1}) by the $\mathbb{F}_q$-isomorphism $\mathbb{F}_q[t] \xrightarrow{\cong} A; t \mapsto \theta$. For an integer $n \geq 0$, the Bernoulli-Carlitz number $B_n \in A$ is defined by

$$\sum_{n=0}^{\infty} \frac{B_n}{\Gamma_{n+1}} z^n = z \left(\sum_{i=0}^{\infty} \frac{1}{D_i} z^{q^i} \right)^{-1}.$$

For each integer $n \geq 0$, Anderson and Thakur ([2, 3.7.1]) defined a polynomial $H_n \in A[t]$ by

$$\sum_{n=0}^{\infty} \frac{H_n}{\Gamma_{n+1}(t)} z^n = \left(1 - \sum_{i=0}^{\infty} \frac{\prod_{j=1}^{i}(t^{q^i} - \theta^{q^j})}{D_i(t)} z^{q^i} \right)^{-1}.$$

We also set

$$S_i(n) := \sum_{\substack{a\in A:\text{monic}\\ \deg(a)=i}} \frac{1}{a^n}$$

for each $n \geq 1$ and $i \geq 0$. These satisfy

$$\|H_{n-1}\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}} \quad \text{and} \quad (H_{n-1}\Omega^n)^{(i)}(\theta) = \frac{\Gamma_n S_i(n)}{\widetilde{\pi}^n}$$

for each $n \geq 1$ and $i \geq 0$ (see [2, 3.7.4], [3, 2.4.1]).

For an index $\underline{n} = (n_1, \ldots, n_d)$, the formal power series

$$L_{\underline{n}}(t) := \sum_{i_1 > \cdots > i_d \geq 0} \frac{H_{n_1-1}^{(i_1)} \cdots H_{n_d-1}^{(i_d)}}{((t-\theta^q)\cdots(t-\theta^{q^{i_1}}))^{n_1} \cdots ((t-\theta^q)\cdots(t-\theta^{q^{i_d}}))^{n_d}} \in \overline{K_\infty}[\![t]\!],$$

converges on $|t|_\infty < |\theta|_\infty^q$ and it is an element of $\mathbb{T}$. Clearly, it satisfies

$$L_{\underline{n}}^{(-1)} = \frac{H_{n_d-1}^{(-1)}}{(t-\theta)^{n_1+\cdots+n_{d-1}}} L_{\underline{n}_{d1}} + \frac{L_{\underline{n}}}{(t-\theta)^{n_1+\cdots+n_d}},$$

where we set $L_{\underline{n}_{11}} = L_\emptyset := 1$ when $d = 1$. Anderson and Thakur ([3, 2.5.6]) showed that

$$L_{\underline{n}}(\theta) = \Gamma_{n_1} \cdots \Gamma_{n_d} \zeta(\underline{n}).$$

Next, we recall Papanikolas' theory of pre-t-motives. We do not give the complete details, but see [8] for more on this theory. See also [6, Section 2], [7, Section 3].

A *pre-t-motive* M is a finite dimensional $\overline{K}(t)$-vector space equipped with a bijective additive map $\varphi\colon M \to M$ such that $\varphi(fm) = f^{(-1)}\varphi(m)$ for $f \in \overline{K}(t)$ and $m \in M$. We always assume that M is rigid analytically trivial. Thus such M is determined by the matrix $\Phi \in \mathrm{GL}_r(\overline{K}(t))$ ($r := \dim M$) representing the φ-action with respect to a fixed basis, such that

$$\Psi^{(-1)} = \Phi\Psi$$

for some matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$. The Betti realization $\omega(M)$ is defined and is functorial on M (see [8, 3.4 and 3.5]). The space $\omega(M)$ is an $\mathbb{F}_q(t)$-vector space and its dimension over $\mathbb{F}_q(t)$ is equal to the dimension of M over $\overline{K}(t)$. The category of (rigid analytically trivial) pre-t-motives forms a neutral Tannakian category over $\mathbb{F}_q(t)$ with fiber functor ω. We denote by G_M the fundamental group of the Tannakian subcategory generated by M. When we fix a basis of M and choose a matrix Ψ as above, we also define

$$G_\Psi := \mathrm{Spec}(\mathbb{F}_q(t)[X, 1/\det X]/\operatorname{Ker}\nu) \subset \mathrm{GL}_{r,\mathbb{F}_q(t)},$$

where $X = (X_{ij})$ is a matrix of $r \times r$ variables and ν is the $\mathbb{F}_q(t)$-morphism defined by

$$\nu\colon \mathbb{F}_q(t)[X, 1/\det X] \to \mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L};\ X_{ij} \mapsto \widetilde{\Psi}_{ij}.$$

Here we set $\widetilde{\Psi} := \Psi_1^{-1}\Psi_2 \in \mathrm{GL}_r(\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L})$, where Ψ_1 (resp. Ψ_2) $\in \mathrm{GL}_r(\mathbb{L} \otimes_{\overline{K}(t)} \mathbb{L})$ is the matrix defined by $(\Psi_1)_{ij} := \Psi_{ij} \otimes 1$ (resp. $(\Psi_2)_{ij} := 1 \otimes \Psi_{ij}$). Papanikolas ([8, Theorem 4.2.11]) showed that the scheme G_Ψ is a closed subgroup scheme of $\mathrm{GL}_{r,\mathbb{F}_q(t)}$. Moreover, he proved the following theorem:

Theorem 2.1 ([8, Theorems 4.3.1, 4.5.10, 5.2.2])**.** *There exists a natural isomorphism $G_\Psi \xrightarrow{\cong} G_M$ and the equality*

$$\dim G_\Psi = \mathrm{tr.deg}_{\overline{K}(t)} \overline{K}(t)(\Psi_{ij}|i,j)$$

holds. Moreover, this value is equal to

$$\mathrm{tr.deg}_{\overline{K}} \overline{K}(\Psi_{ij}(\theta)|i,j)$$

if $\Phi \in \mathrm{Mat}_r(\overline{K}[t])$, $\det \Phi = c(t-\theta)^n$ for some $n \in \mathbb{Z}_{\geq 0}$ and $c \in \overline{K}^\times$, each entry of Ψ is entire and $\Psi \in \mathrm{GL}_r(\mathbb{T})$.

Remark. The last part of Theorem 2.1 is proved by using a very deep result in [1, Theorem 3.1.1], which is called the *ABP-criterion.* We use the last part of Theorem 2.1 to prove our theorems.

Example 2.2. Let $\underline{n} = (n_1, \dots, n_d)$ be an index. Let $M[\underline{n}]$ be the pre-t-motive defined by the $(d+1) \times (d+1)$-matrix

$$\Phi[\underline{n}] := \begin{bmatrix} (t-\theta)^{n_1+\cdots+n_d} & 0 & 0 & \cdots & 0 \\ H_{n_1-1}^{(-1)}(t-\theta)^{n_1+\cdots+n_d} & (t-\theta)^{n_2+\cdots+n_d} & 0 & \cdots & 0 \\ 0 & H_{n_2-1}^{(-1)}(t-\theta)^{n_2+\cdots+n_d} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{n_d} & 0 \\ 0 & \cdots & 0 & H_{n_d-1}^{(-1)}(t-\theta)^{n_d} & 1 \end{bmatrix}.$$

We also set

$$\Psi[\underline{n}] := \begin{bmatrix} \Omega^{n_1+\cdots+n_d} & 0 & 0 & \cdots & 0 \\ \Omega^{n_1+\cdots+n_d} L_{\underline{n}_{21}} & \Omega^{n_2+\cdots+n_d} & 0 & \cdots & 0 \\ \Omega^{n_1+\cdots+n_d} L_{\underline{n}_{31}} & \Omega^{n_2+\cdots+n_d} L_{\underline{n}_{32}} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \Omega^{n_d} & 0 \\ \Omega^{n_1+\cdots+n_d} L_{\underline{n}_{d+1,1}} & \Omega^{n_2+\cdots+n_d} L_{\underline{n}_{d+1,2}} & \cdots & \Omega^{n_d} L_{\underline{n}_{d+1,d}} & 1 \end{bmatrix}.$$

Then we have $\Psi[\underline{n}]^{(-1)} = \Phi[\underline{n}]\Psi[\underline{n}]$. By Theorem 2.1, we have $G_{\Psi[\underline{n}]} \cong G_{M[\underline{n}]}$ and

$$\begin{aligned} \dim G_{\Psi[\underline{n}]} &= \mathrm{tr.deg}_{\overline{K}(t)} \overline{K}(t)(\Omega, L_{\underline{m}}|\underline{m} \in \mathrm{Sub}(\underline{n})) \\ &= \mathrm{tr.deg}_{\overline{K}} \overline{K}(\widetilde{\pi}, \zeta(\underline{m})|\underline{m} \in \mathrm{Sub}(\underline{n})). \end{aligned}$$

The matrix $\widetilde{\Psi[\underline{n}]} = (\widetilde{\Psi[\underline{n}]}_{ij})$ is calculated as

$$\widetilde{\Psi[\underline{n}]}_{ij} = (\Omega^{-1} \otimes \Omega)^{n_i+\cdots+n_d} \sum_{s=j}^{i} \sum_{r=0}^{i-s} (-1)^r \sum_{\substack{s=i_0<i_1<\cdots \\ <i_{r-1}<i_r=i}} L_{i_1 i_0} \cdots L_{i_r i_{r-1}} \otimes \Omega^{n_j+\cdots+n_{i-1}} L_{sj}$$

for each $j < i$, where we denote $L_{k\ell} := L_{\underline{n}_{k\ell}}$.

Example 2.3. For $k = 1, 2$, let $\Phi_k \in \mathrm{GL}_{r_k}(\overline{K}(t))$ and $\Psi_k \in \mathrm{GL}_{r_k}(\mathbb{L})$ be matrices such that $\Psi_k^{(-1)} = \Phi_k \Psi_k$, and let M_k be the pre-t-motive defined by Φ_k. Since M_k is a direct factor of $M_1 \oplus M_2$, there exists a surjective map

$$G_{\Psi_1 \oplus \Psi_2} \cong G_{M_1 \oplus M_2} \twoheadrightarrow G_{M_k} \cong G_{\Psi_k}$$

by Tannakian duality and Theorem 2.1. This coincides with the restriction of the k-th projection $\mathrm{GL}_{r_1} \times \mathrm{GL}_{r_2} \twoheadrightarrow \mathrm{GL}_{r_k}$.

§ 3. Lifts of p-th power relations

In this section, we study p-th power relations among $L_{\underline{n}}$'s, which are lifts of p-th power relations among MZVs. For an index $\underline{n} = (n_1, \ldots, n_d)$, we use the notations

$$\Omega^{\underline{n}} := \Omega^{n_1+\cdots+n_d} \quad \text{and} \quad \underline{n}' := (n_1, \ldots, n_{d-1}).$$

Lemma 3.1. *For each positive integer $n \geq 1$ and each non-negative integer $e \geq 0$, we have*

$$\frac{H_{p^e n-1}}{\Gamma_{p^e n}(t)} = \left(\frac{H_{n-1}}{\Gamma_n(t)}\right)^{p^e}.$$

Proof. We have

$$\left(\frac{H_{s-1}}{\Gamma_s(t)}(\theta^{q^{-i}})\right)^{q^i} = \left(\frac{H_{s-1}}{\Gamma_s(t)}\right)^{(i)}(\theta) = \frac{S_i(s)}{\widetilde{\pi}^s (\Omega^s)^{(i)}(\theta)}.$$

Therefore

$$\left(\frac{H_{p^e n-1}}{\Gamma_{p^e n}(t)}(\theta^{q^{-i}})\right)^{q^i} = \frac{S_i(p^e n)}{\widetilde{\pi}^{p^e n}(\Omega^{p^e n})^{(i)}(\theta)} = \left(\frac{S_i(n)}{\widetilde{\pi}^n (\Omega^n)^{(i)}(\theta)}\right)^{p^e}$$
$$= \left(\left(\frac{H_{n-1}}{\Gamma_n(t)}(\theta^{q^{-i}})\right)^{q^i}\right)^{p^e} = \left(\left(\frac{H_{n-1}}{\Gamma_n(t)}\right)^{p^e}(\theta^{q^{-i}})\right)^{q^i}$$

for each $i \geq 0$. Thus we have

$$\frac{H_{p^e n-1}}{\Gamma_{p^e n}(t)}(\theta^{q^{-i}}) = \left(\frac{H_{n-1}}{\Gamma_n(t)}\right)^{p^e}(\theta^{q^{-i}})$$

for each $i \geq 0$. $\square$

We set
$$\gamma_{e,n} := \frac{H_{p^e n-1}}{H_{n-1}^{p^e}} = \frac{\Gamma_{p^e n}(t)}{\Gamma_n(t)^{p^e}} \in \mathbb{F}_q(t)^\times$$
and
$$\gamma_{e,\underline{n}} := \prod_{i=1}^{d} \gamma_{e,n_i}$$
for any index $\underline{n} = (n_1, \ldots, n_d)$. When $p^{-e}\underline{n} \in \mathbb{Z}^d$, we also set
$$\gamma_{-e,\underline{n}} := \gamma_{e,p^{-e}\underline{n}}^{-p^{-e}}.$$

The next lemma gives a p-th power relation among $L_{\underline{n}}$ and $L_{p^e\underline{n}}$, which is a lift of the p-th power relation $\zeta(p^e\underline{n}) = \zeta(\underline{n})^{p^e}$.

Lemma 3.2. *For each index $\underline{n}$ of depth d and each integer $e \in \mathbb{Z}$ such that $p^e\underline{n} \in \mathbb{Z}^d$, we have*
$$L_{p^e\underline{n}} = \gamma_{e,\underline{n}} L_{\underline{n}}^{p^e}.$$

Proof. By the definition of $\gamma_{e,\underline{n}}$ for negative integers e, we may assume that $e \geq 0$. We prove this equality by induction on d. When $d = 0$, it is clear. We take $d \geq 1$ and assume that the above equality holds for any indices whose depths are lower than d. Then we have
$$\begin{aligned}
&(\Omega^{p^e\underline{n}} L_{p^e\underline{n}} - \gamma_{e,\underline{n}}(\Omega^{\underline{n}} L_{\underline{n}})^{p^e})^{(-1)} \\
&= H_{p^e n_d-1}^{(-1)}(t-\theta)^{p^e n_d}\Omega^{p^e\underline{n}} L_{p^e\underline{n}'} + \Omega^{p^e\underline{n}} L_{p^e\underline{n}} - \gamma_{e,\underline{n}}\left(H_{n_d-1}^{(-1)}(t-\theta)^{n_d}\Omega^{\underline{n}} L_{\underline{n}'} + \Omega^{\underline{n}} L_{\underline{n}}\right)^{p^e} \\
&= \Omega^{p^e\underline{n}} L_{p^e\underline{n}} - \gamma_{e,\underline{n}}(\Omega^{\underline{n}} L_{\underline{n}})^{p^e} + (t-\theta)^{p^e n_d}\Omega^{p^e\underline{n}}\Big(H_{p^e n_d-1}^{(-1)} L_{p^e\underline{n}'} - \gamma_{e,\underline{n}'}\gamma_{e,n_d}(H_{n_d-1}^{(-1)})^{p^e} L_{\underline{n}'}^{p^e}\Big) \\
&= \Omega^{p^e\underline{n}} L_{p^e\underline{n}} - \gamma_{e,\underline{n}}(\Omega^{\underline{n}} L_{\underline{n}})^{p^e} + (t-\theta)^{p^e n_d}\Omega^{p^e\underline{n}} L_{p^e\underline{n}'}\left(H_{p^e n_d-1}^{(-1)} - \gamma_{e,n_d}(H_{n_d-1}^{(-1)})^{p^e}\right) \\
&= \Omega^{p^e\underline{n}} L_{p^e\underline{n}} - \gamma_{e,\underline{n}}(\Omega^{\underline{n}} L_{\underline{n}})^{p^e} + (t-\theta)^{p^e n_d}\Omega^{p^e\underline{n}} L_{p^e\underline{n}'}\left(H_{p^e n_d-1}^{(-1)} - \gamma_{e,n_d}^{(-1)}(H_{n_d-1}^{(-1)})^{p^e}\right) \\
&= \Omega^{p^e\underline{n}} L_{p^e\underline{n}} - \gamma_{e,\underline{n}}(\Omega^{\underline{n}} L_{\underline{n}})^{p^e}.
\end{aligned}$$
Thus
$$\Omega^{p^e\underline{n}} L_{p^e\underline{n}} = \gamma_{e,\underline{n}}(\Omega^{\underline{n}} L_{\underline{n}})^{p^e} + c$$
for some $c \in \mathbb{F}_q(t)$. Since we have
$$\begin{aligned}
(\gamma_{e,\underline{n}}(\Omega^{\underline{n}} L_{\underline{n}})^{p^e})(\theta) &= \frac{\Gamma_{p^e n_1}}{\Gamma_{n_1}^{p^e}} \cdots \frac{\Gamma_{p^e n_d}}{\Gamma_{n_d}^{p^e}} \left(\frac{\Gamma_{n_1}\cdots\Gamma_{n_d}\zeta(\underline{n})}{\widetilde{\pi}^{n_1+\cdots+n_d}}\right)^{p^e} \\
&= \frac{\Gamma_{p^e n_1}\cdots\Gamma_{p^e n_d}\zeta(p^e\underline{n})}{\widetilde{\pi}^{p^e n_1+\cdots+p^e n_d}} \\
&= (\Omega^{p^e\underline{n}} L_{p^e\underline{n}})(\theta),
\end{aligned}$$

we conclude $c(\theta) = 0$, and hence $c = 0$. $\square$

Lemma 3.3. *Let* $\underline{n} = (n_1, \ldots, n_d)$ *be an index and* $\Psi := \Psi[\underline{n}] \in \mathrm{GL}_{d+1}(\mathbb{L})$ *the matrix defined in Example 2.2. Take* $1 \le j < i \le d+1$ *and* $1 \le \ell < k \le d+1$ *such that* $\underline{n}_{ij} = p^e \underline{n}_{k\ell}$ *for some integer* $e \in \mathbb{Z}$. *Then the equality*

$$\widetilde{\Psi}_{ij}/\widetilde{\Psi}_{ii} = \gamma_{e,\underline{n}_{k\ell}}(\widetilde{\Psi}_{k\ell}/\widetilde{\Psi}_{kk})^{p^e}$$

holds.

Proof. We may assume that $e \ge 0$. By Example 2.2, we have

$$\begin{aligned}
\widetilde{\Psi}_{ij}/\widetilde{\Psi}_{ii} &= \sum_{s=j}^{i}\sum_{r=0}^{i-s}(-1)^r \sum_{\substack{s=i_0<i_1<\cdots\\ <i_{r-1}<i_r=i}} L_{\underline{n}_{i_1 i_0}} \cdots L_{\underline{n}_{i_r i_{r-1}}} \otimes \Omega^{\underline{n}_{ij}} L_{\underline{n}_{sj}} \\
&= \sum_{s=\ell}^{k}\sum_{r=0}^{k-s}(-1)^r \sum_{\substack{s=i_0<i_1<\cdots\\ <i_{r-1}<i_r=k}} L_{p^e\underline{n}_{i_1 i_0}} \cdots L_{p^e\underline{n}_{i_r i_{r-1}}} \otimes \Omega^{p^e\underline{n}_{k\ell}} L_{p^e\underline{n}_{s\ell}} \\
&= \sum_{s=\ell}^{k}\sum_{r=0}^{k-s}(-1)^r \sum_{\substack{s=i_0<i_1<\cdots\\ <i_{r-1}<i_r=k}} \gamma_{e,\underline{n}_{i_1 i_0}} \cdots \gamma_{e,\underline{n}_{i_r i_{r-1}}} \gamma_{e,\underline{n}_{s\ell}} (L_{\underline{n}_{i_1 i_0}} \cdots L_{\underline{n}_{i_r i_{r-1}}} \otimes \Omega^{\underline{n}_{k\ell}} L_{\underline{n}_{s\ell}})^{p^e} \\
&= \gamma_{e,\underline{n}_{k\ell}} \left(\sum_{s=\ell}^{k}\sum_{r=0}^{k-s}(-1)^r \sum_{\substack{s=i_0<i_1<\cdots\\ <i_{r-1}<i_r=k}} L_{\underline{n}_{i_1 i_0}} \cdots L_{\underline{n}_{i_r i_{r-1}}} \otimes \Omega^{\underline{n}_{k\ell}} L_{\underline{n}_{s\ell}} \right)^{p^e} \\
&= \gamma_{e,\underline{n}_{k\ell}} (\widetilde{\Psi}_{k\ell}/\widetilde{\Psi}_{kk})^{p^e}.
\end{aligned}$$

$\square$

§ 4. Lifts of Euler-Carlitz relations

In this section, we study Euler-Carlitz relations among Ω and L_n's, which are lifts of Euler-Carlitz relations among $\widetilde{\pi}$ and Carlitz zeta values at positive "even" integers. Let $n \ge 1$ be a positive "even" integer. By [6, Remark 3.3], there exist $c_n \in \mathbb{F}_q(t)^\times$ and $f_n \in \overline{K}(t)$ such that

$$\Omega^n L_n - c_n = f_n \Omega^n.$$

This gives a lift of the Euler-Carlitz relation at n. The c_n is determined by

$$c_n(\theta) = \frac{\Gamma_n \zeta(n)}{\widetilde{\pi}^n} = \frac{\Gamma_n B_n}{\Gamma_{n+1}} \in K^\times.$$

Let $\underline{n}=(n_1,\dots,n_d)$ be an index and take j such that n_j is "even". Then we have

$$\widetilde{\Psi}_{j+1,j}/\widetilde{\Psi}_{j+1,j+1}=c_{n_j}(1-(\Omega^{-1}\otimes\Omega)^{n_j}),$$

where $\Psi:=\Psi[\underline{n}]$ is the matrix defined in Example 2.2.

§ 5. Proofs

In this section, we always assume that algebraic groups are defined over $\mathbb{F}_q(t)$. We need the following lemma. This can be proved easily and we omit the proof.

Lemma 5.1. *Let $V\subset{\mathbb{G}_a}^r$ be an algebraic subgroup of dimension zero. Let $m_1,\dots,m_r\in\mathbb{Z}$ be non-zero integers. Assume that V is stable under the $\mathbb{G}_m$-action on ${\mathbb{G}_a}^r$ defined by*

$$a.(x_1,\dots,x_r)=(a^{m_1}x_1,\dots,a^{m_r}x_r)\ \ (a\in\mathbb{G}_m,(x_i)\in{\mathbb{G}_a}^r).$$

Then $V(\overline{\mathbb{F}_q(t)})$ is trivial.

From now on, we identify group schemes over $\mathbb{F}_q(t)$ with the sets of $\overline{\mathbb{F}_q(t)}$-valued points of them. We use letters $a,x_{ij}^{(m)},\dots$ for coordinate variables of algebraic groups.

Proof of Theorem 1.5. For $1\le\ell\le k$, let $G^{\le\ell}$ be the algebraic group defined by the matrix $[\Omega]\oplus\bigoplus_{m=1}^{\ell}\Psi[\underline{n}_m]$. Then $G^{\le\ell}$ is an algebraic subgroup of

$$\left\{\begin{bmatrix}a\end{bmatrix}\oplus\bigoplus_{m=1}^{\ell}\begin{bmatrix}a^{n_1^{(m)}+\cdots n_{d_m}^{(m)}} & & & \\ x_{21}^{(m)} & \ddots & & \\ \vdots & \ddots & a^{n_{d_m}^{(m)}} & \\ x_{d_m+1,1}^{(m)} & \cdots & x_{d_m+1,d_m}^{(m)} & 1\end{bmatrix}\in\mathbb{G}_m\times\prod_{m=1}^{\ell}\mathrm{GL}_{d_m+1}\ \middle|\ \begin{array}{c} x_{i'j'}^{(m')}/a^{n_{i'}^{(m')}+\cdots+n_{d_{m'}}^{(m')}}=\gamma_{e,\underline{n}_{ij}^{(m)}}\left(x_{ij}^{(m)}/a^{n_i^{(m)}+\cdots+n_{d_m}^{(m)}}\right)^{p^e} \\ \text{if}\ \ \underline{n}_{i'j'}^{(m')}=p^e\underline{n}_{ij}^{(m)}, \\ x_{j+1,j}^{(m)}/a^{n_{j+1}^{(m)}+\cdots+n_{d_m}^{(m)}}=c_{n_j^{(m)}}(1-a^{n_j^{(m)}}) \\ \text{if}\ \ n_j^{(m)}\ \ \text{is "even"}\end{array}\right\}.$$

By Theorem 2.1 and the condition (4), this inclusion is actually an equality for $1\le\ell\le k-1$. It suffices to show that this inclusion is actually an equality for $\ell=k$. We already have

$$\dim G^{\le k-1}\le\dim G^{\le k}\le\dim G^{\le k-1}+1$$

by the condition (3) and it suffices to show that the second inequality is an equality. Let

$$\psi\colon G^{\le k}\twoheadrightarrow G^{\le k-1}\ \ \text{and}\ \ \pi^{\le\ell}\colon G^{\le\ell}\twoheadrightarrow G_\Omega\cong\mathbb{G}_m\ (1\le\ell\le k)$$

be the surjections obtained as in Example 2.3. We set $V^{\leq \ell} := \operatorname{Ker} \pi^{\leq \ell}$ to be the unipotent radical of $G^{\leq \ell}$. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & V^{\leq k} & \longrightarrow & G^{\leq k} & \xrightarrow{\pi^{\leq k}} & \mathbb{G}_m & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} & & \| & & \\ 1 & \longrightarrow & V^{\leq k-1} & \longrightarrow & G^{\leq k-1} & \xrightarrow{\pi^{\leq k-1}} & \mathbb{G}_m & \longrightarrow & 1 \end{array}$$

with exact rows, where ψ' is the restriction of ψ to $V^{\leq k}$. The morphism ψ' is surjective. It is clear that $\operatorname{Ker} \psi'$ is a normal subgroup of $G^{\leq k}$. The conjugate action $X \mapsto A^{-1}XA$ of $G^{\leq k}$ on $\operatorname{Ker} \psi'$ factors through the action of $\mathbb{G}_m$ on $\operatorname{Ker} \psi'$ by

$$a.(\dots, 0, x^{(k)}_{d_k+1,1}, 0, \dots) = (\dots, 0, a^{n^{(k)}_1 + \cdots + n^{(k)}_{d_k}} x^{(k)}_{d_k+1,1}, 0, \dots).$$

Now, we assume that $\dim G^{\leq k} = \dim G^{\leq k-1}$ and we shall induce a contradiction. Since $\dim \operatorname{Ker} \psi' = 0$, the group $\operatorname{Ker} \psi'$ is trivial by Lemma 5.1. Let $X = (x^{(m)}_{ij})$ (resp. $A = (a^{(m)}_{ij})$) be any element of $V^{\leq k}$ such that $x^{(m)}_{ij} = 0$ if $i - j \neq 1$ (resp. $a^{(m)}_{ij} = 0$ if $i - j \neq d_k - 1$) and $(m, i, j) \neq (k, d_k + 1, 1)$. For each $1 \leq m \leq k$, we set $x^{(m)}_j := x^{(m)}_{j+1,j}$ $(1 \leq j \leq d_m)$ and $a^{(m)}_j := a^{(m)}_{d_k-1+j,j}$ $(1 \leq j \leq d_m - d_k + 2)$. We also set $d := d_k - 1$ (which is ≥ 1 by the condition (1)). Then the m-th component of $A^{-1}XA$ is equal to

$$\begin{array}{r} \\ \\ \\ \\ \\ d+2 \\ \\ \\ \\ 2d+2 \\ \\ \\ \\ 3d+2 \\ \\ \\ \end{array}
\left[\begin{array}{ccc}
1 & & \\
x^{(m)}_1 & 1 & \\
0 & x^{(m)}_2 & \ddots \\
\vdots & 0 & \ddots \\
0 & \vdots & \\
a^{(m)}_1 x^{(m)}_{d+1} - a^{(m)}_2 x^{(m)}_1 & 0 & \\
0 & a^{(m)}_2 x^{(m)}_{d+2} - a^{(m)}_3 x^{(m)}_2 & \\
\vdots & 0 & \ddots \\
0 & \vdots & \\
-a^{(m)}_{d+2}(a^{(m)}_1 x^{(m)}_{d+1} - a^{(m)}_2 x^{(m)}_1) & 0 & \\
0 & -a^{(m)}_{d+3}(a^{(m)}_2 x^{(m)}_{d+2} - a^{(m)}_3 x^{(m)}_2) & \\
\vdots & 0 & \ddots \\
0 & \vdots & \\
a^{(m)}_{d+2} a^{(m)}_{2d+2}(a^{(m)}_1 x^{(m)}_{d+1} - a^{(m)}_2 x^{(m)}_1) & 0 & \\
0 & a^{(m)}_{d+3} a^{(m)}_{2d+3}(a^{(m)}_2 x^{(m)}_{d+2} - a^{(m)}_3 x^{(m)}_2) & \\
\vdots & \vdots & \ddots
\end{array}\right]$$

for each $1 \le m < k$. Thus the (i,j)-th entry of the m-th (resp. k-th) component of $A^{-1}XA$ is

$$\begin{cases} 1 & (i-j=0) \\ x_j^{(m)} & (i-j=1) \\ (a_j^{(m)} x_{d+j}^{(m)} - a_{j+1}^{(m)} x_j^{(m)})(-1)^{r-1} \prod_{s=1}^{r-1} a_{sd+j+1}^{(m)} & (i-j=rd+1,\ r\ge 1) \\ \left(\text{resp. } a_1^{(k)} x_{d+1}^{(k)} - a_2^{(k)} x_1^{(k)} + x_{d_k+1,1}^{(k)}\right) & (\text{resp. } i-j=d+1=d_k)) \\ 0 & (\text{otherwise}) \end{cases}$$

Therefore, if the equalities

$$a_{i-1,j}^{(m)} x_{i,i-1}^{(m)} - a_{i,j+1}^{(m)} x_{j+1,j}^{(m)} = 0 \quad (1 \le m < k,\quad 1 \le j < i \le d_m+1,\quad i-j=d_k) \tag{5.1}$$

hold, then $X^{-1}A^{-1}XA \in \operatorname{Ker}\psi' = \{1\}$ and the equality

$$a_{d_k,1}^{(k)} x_{d_k+1,d_k}^{(k)} - a_{d_k+1,2}^{(k)} x_{21}^{(k)} = 0 \tag{5.2}$$

must hold. We show that this implication induces a contradiction and hence we have $\dim G^{\le k} = \dim G^{\le k-1} + 1$.

First, we assume that the condition (5-1) holds. If $\underline{n}_{d_k,1}^{(k)} \not\sim \underline{n}_{d_k+1,2}^{(k)}$, we can take

$$a_{d_k+1,2}^{(k)} x_{21}^{(k)} \ne 0,\quad a_{d_k,1}^{(k)} x_{d_k+1,d_k}^{(k)} = a_{i-1,j}^{(m)} x_{i,i-1}^{(m)} = a_{i,j+1}^{(m)} x_{i,i-1}^{(m)} = 0$$

for each (m,i,j). Then the equalities (5.1) hold and hence the equality (5.2) also holds. However, it becomes $a_{d_k+1,2}^{(k)} x_{21}^{(k)} = 0$. This is a contradiction. If $\underline{n}_{d_k,1}^{(k)} \sim \underline{n}_{d_k+1,2}^{(k)}$ (and hence $n_1^{(k)} \sim n_{d_k}^{(k)}$), there exists an integer e such that $\underline{n}_{d_k,1}^{(k)} = p^e \underline{n}_{d_k+1,2}^{(k)}$. Then we can take

$$a_{i-1,j}^{(m)} x_{i,i-1}^{(m)} = a_{i,j+1}^{(m)} x_{i,i-1}^{(m)} = 0$$

for each (m,i,j) and the equality (5.2) becomes

$$\gamma_{e,\underline{n}_{d_k+1,2}^{(k)}} (a_{d_k+1,2}^{(k)})^{p^e} x_{d_k+1,d_k}^{(k)} - a_{d_k+1,2}^{(k)} x_{21}^{(k)} = 0$$

for each $a_{d_k+1,2}^{(k)}$ and $x_{21}^{(k)}$. Then e must be zero. However, since $\underline{n}^{(k)} \ne (n_1^{(k)}, \dots, n_1^{(k)})$ by the condition (2), e is non-zero. This is a contradiction.

Similarly, when the condition (5-1)$'$ holds, we obtain a contradiction.

Next, we assume that the condition (5-2) holds. Then we can take

$$x_{21}^{(k)}, x_{d_k+1,d_k}^{(k)} \ne 0,\quad a_{i-1,j}^{(m)} x_{i,i-1}^{(m)} = a_{i,j+1}^{(m)} x_{i,i-1}^{(m)} = 0$$

for each $(m,i,j) \ne (m_0,i_0,j_0)$ and $a_{d_k,1}^{(k)}, a_{d_k+1,2}^{(k)}$ as any elements. There exist integers $e_1, e_2, e_3, e_4 \in \mathbb{Z}$ such that

$$\underline{n}_{i_0-1,j_0}^{(m_0)} = p^{e_1} \underline{n}_{d_k+1,2}^{(k)},\quad \underline{n}_{i_0,j_0+1}^{(m_0)} = p^{e_2} \underline{n}_{d_k,1}^{(k)},\quad n_{j_0}^{(m_0)} = p^{e_3} n_{d_k}^{(k)},\quad n_{i_0-1}^{m_0} = p^{e_4} n_1^{(k)}.$$

Then the (m_0, i_0, j_0)-th equality of the equalities (5.1) becomes

$$\gamma_{e_1,\underline{n}^{(k)}_{d_k+1,2}}(a^{(k)}_{d_k+1,2})^{p^{e_1}}\gamma_{e_4,n^{(k)}_1}(x^{(k)}_{21})^{p^{e_4}} - \gamma_{e_2,\underline{n}^{(k)}_{d_k,1}}(a^{(k)}_{d_k,1})^{p^{e_2}}\gamma_{e_3,n^{(k)}_{d_k}}(x^{(k)}_{d_k+1,d_k})^{p^{e_3}} = 0.$$

We take any $a^{(k)}_{d_k+1,2}$ and set

$$a^{(k)}_{d_k,1} := \left(\frac{\gamma_{e_1,\underline{n}^{(k)}_{d_k+1,2}}\gamma_{e_4,n^{(k)}_1}(x^{(k)}_{21})^{p^{e_4}}}{\gamma_{e_2,\underline{n}^{(k)}_{d_k,1}}\gamma_{e_3,n^{(k)}_{d_k}}(x^{(k)}_{d_k+1,d_k})^{p^{e_3}}}\right)^{p^{-e_2}}(a^{(k)}_{d_k+1,2})^{p^{e_1-e_2}}.$$

Then the equalities (5.1) hold and the equality (5.2) becomes

$$\left(\frac{\gamma_{e_1,\underline{n}^{(k)}_{d_k+1,2}}\gamma_{e_4,n^{(k)}_1}(x^{(k)}_{21})^{p^{e_4}}}{\gamma_{e_2,\underline{n}^{(k)}_{d_k,1}}\gamma_{e_3,n^{(k)}_{d_k}}(x^{(k)}_{d_k+1,d_k})^{p^{e_3}}}\right)^{p^{-e_2}}(a^{(k)}_{d_k+1,2})^{p^{e_1-e_2}}x^{(k)}_{d_k+1,d_k} - a^{(k)}_{d_k+1,2}x^{(k)}_{21} = 0$$

and holds for each $a^{(k)}_{d_k+1,2}$. This implies $e_1 = e_2$. Then we have $n^{(k)}_\ell = n^{(k)}_{\ell+2}$ for each ℓ (resp. $\underline{n}^{(k)} \sim (n^{(m_0)}_2, n^{(m_0)}_1)$) if $d_k \geq 3$ (resp. $d_k = 2$). In any case, we obtain a contradiction.

Similarly, when the condition (5-2)$'$ holds, we have $\underline{n}^{(k)} = (n^{(k)}_1, \ldots, n^{(k)}_1)$ (resp. $\underline{n}^{(k)} \sim \underline{n}^{(m_0)}$) if $d_k \geq 3$ (resp. $d_k = 2$). In any case, we obtain a contradiction. □

Proof of Proposition 1.6. We use the notations in the proof of Theorems 1.5. We set

$$\underline{n}^{(1)} := n_1, \quad \underline{n}^{(2)} := n_2, \quad \underline{n}^{(3)} := n_3,$$
$$\underline{n}^{(4)} := (n_1, n_2), \quad \underline{n}^{(5)} := (n_2, n_3), \quad \underline{n}^{(6)} := (n_1, n_2, n_3).$$

By Theorem 1.5 and the result of Chang and Yu ([5, Corollary 4.6]), we have

$$\text{tr.deg}_{\overline{K}}\,\overline{K}(\widetilde{\pi}, \zeta(\underline{n}^{(m)})|1 \leq m \leq 4) = \#((\{(q-1)\} \cup \{\underline{n}^{(m)}|1 \leq m \leq 4\})/\sim).$$

If we prove

$$\text{tr.deg}_{\overline{K}}\,\overline{K}(\widetilde{\pi}, \zeta(\underline{n}^{(m)})|1 \leq m \leq 5) = \#((\{(q-1)\} \cup \{\underline{n}^{(m)}|1 \leq m \leq 5\})/\sim),$$

then Proposition 1.6 follows from Theorem 1.5.

Assume that $\underline{n}^{(5)} \not\sim \underline{n}^{(4)}$ and $\dim G^{\leq 5} = \dim G^{\leq 4}$. In this case, the equality $a^{(4)}_{21}x^{(4)}_{32} - a^{(4)}_{32}x^{(4)}_{21} = 0$ implies the equality $a^{(5)}_{21}x^{(5)}_{32} - a^{(5)}_{32}x^{(5)}_{21} = 0$. We may assume that $n_1 \sim n_2 \sim n_3$, otherwise we obtain a contradiction from [7, Theorem 1.1] and Theorem 1.4. We set

$$a_1 := a^{(4)}_{21},\ a_2 := a^{(4)}_{32} = a^{(5)}_{21},\ a_3 := a^{(5)}_{32},$$
$$x_1 := x^{(4)}_{21},\ x_2 := x^{(4)}_{32} = x^{(5)}_{21},\ x_3 := x^{(5)}_{32}.$$

For each j, we have $n_j = p^{e_j}n$ for some $n \geq 1$ and $e_j \geq 0$ with $\min\{e_j\} = 0$. We set $a := a_{j_0}$ and $x := x_{j_0}$ for some j_0 such that $e_{j_0} = 0$. Thus we have $a_j = \gamma_{e_j,n}a^{p^{e_j}}$ and $x_j = \gamma_{e_j,n}x^{p^{e_j}}$ for each j. Then

$$a^{p^{e_1}}x^{p^{e_2}} - a^{p^{e_2}}x^{p^{e_1}} = 0 \text{ implies } a^{p^{e_2}}x^{p^{e_3}} - a^{p^{e_3}}x^{p^{e_2}} = 0$$

for any $a, x \in \overline{\mathbb{F}_q(t)}$. Since $e_1 \neq e_2$, we conclude that $e_1 - e_2$ divides $e_2 - e_3$. By symmetric arguments, since $e_2 \neq e_3$, we conclude that $e_3 - e_2$ divides $e_2 - e_1$ This means that $e_1 - e_2 = \pm(e_2 - e_3)$. However this is a contradiction because we assume that $\underline{n}^{(5)} \not\sim \underline{n}^{(4)}$ and $e_1 \neq e_3$. □

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References

[1] G. W. Anderson, W. D. Brownawell, M. A. Papanikolas, *Determination of the algebraic relations among special* Γ*-values in positive characteristic*, Ann. of Math. **160** (2004) 237–313.

[2] G. W. Anderson, D. S. Thakur, *Tensor powers of the Carlitz module and zeta values*, Ann. of Math. **132** (1990) 159–191.

[3] G. W. Anderson, D. S. Thakur, *Multizeta values for* $\mathbb{F}_q[t]$*, their period interpretation, and relations between them*, Int. Math. Res. Not. IMRN (2009), no. 11, 2038–2055.

[4] L. Carlitz, *On certain functions connected with polynomials in a Galois field*, Duke. Math. J. **1** (1935) 137-168.

[5] C.-Y. Chang, J. Yu, *Determination of algebraic relations among special zeta values in positive characteristic*, Adv. Math. **216** (2007) 321–345.

[6] Y. Mishiba, *Algebraic independence of the Carlitz period and the positive characteristic multizeta values at* n *and* (n, n), Proc. Amer. Math. Soc. **143** (2015), no. 9, 3753–3763.

[7] Y. Mishiba, *On algebraic independence of certain multizeta values in characteristic* p, preprint (2014),
`http://arxiv.org/abs/1401.3628`

[8] M. A. Papanikolas, *Tannakian duality for Anderson-Drinfeld motives and algebraic independence of Carlitz logarithms*, Invent. Math. **171** (2008) 123–174.

[9] D. S. Thakur, *Function Field Arithmetic*, World Scientific Pub., 2004.

[10] L. I. Wade, *Certain quantities transcendental over over* $GF(p^n, x)$, Duke Math. J. **8**, (1941), 701–720.

[11] J. Yu, *Transcendence and special zeta values in characteristic* p, Ann. of Math. **134** (1991) 1–23.

[12] J. Yu, *Analytic homomorphisms into Drinfeld modules*, Ann. of Math. (2) **145** (1997), no. 2, 215–233.