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Ramification theory and perfectoid spaces — a survey

By

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Abstract

In 1980s, Deligne proved that, if two complete discrete valuation fields with perfect residue fields are "close" enough, then their absolute Galois groups are isomorphic to each other modulo certain upper ramification subgroups. In this article, we give a brief survey on the author's generalization of this result to the case where the residue fields are imperfect.

§ 1. Classical ramification theory

This article is a survey of the author's result on a comparison between ramification of complete discrete valuation fields of mixed and equal characteristics [10], generalizing Deligne's theorem for the case of perfect residue fields [5]. After giving some background in the first four sections, the main theorem (Theorem 5.1) and its corollaries are stated in Section 5. We also give a sketch of the proof of the main theorem in Section 6.

Let p be a rational prime and K a complete discrete valuation field whose residue field k is of characteristic p. We denote the ring of integers of K by $\mathcal{O}_K$. We fix an algebraic closure $\bar{K}$ of K, and let K^{sep} be the separable closure of K inside $\bar{K}$. We put $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$. Let v_K be the additive valuation of K normalized as $v_K(K^\times) = \mathbb{Z}$, and we extend it naturally to $\bar{K}$. We denote a uniformizer of K by π_K and the completion of $\bar{K}$ by $\mathbb{C}$.

Ramification theory measures an extent of how far a finite separable extension L of K is from unramified extensions: we can define a normal subgroup G_K^j of G_K for any positive rational number j, which is called the j-th (non-log) upper ramification subgroup of G_K. For any finite separable extension L/K, the Galois group G_K acts

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continuously on the finite set $\mathcal{F}_K(L) = \mathrm{Hom}_{K\text{-alg.}}(L, \bar{K})$. Then the extension L/K is unramified if and only if the group G_K^j acts trivially on the set $\mathcal{F}_K(L)$ for any positive rational number j. The number

$$c(L/K) = \inf\{j \in \mathbb{Q}_{>0} \mid G_K^j \text{ acts trivially on } \mathcal{F}_K(L)\}$$

can be considered as a measure of ramification of the extension L/K, and it is called the conductor of the extension L/K. We say that the ramification of L/K is bounded by j if $c(L/K) < j$.

For the case where the residue field k is perfect, the definition of the group G_K^j is classical (see [17]): Let L/K be a finite Galois extension with Galois group $G = \mathrm{Gal}(L/K)$. For any real number $i \geq -1$, we define the i-th lower ramification subgroup G_i of G by

$$G_i = \{g \in G \mid v_L(g(\pi_L) - \pi_L) \geq i + 1\}.$$

This group is not compatible with quotients. Namely, for a Galois extension M/K inside L with Galois group $H = \mathrm{Gal}(M/K)$, the image of G_i by the surjection $G \to H$ is not necessarily equal to H_i. Thus we renumber them to define G_K^j by using the Hasse-Herbrand function

$$\varphi_{L/K}(s) = \int_0^s \frac{dt}{[G_0 : G_t]}.$$

Set $\psi_{L/K}(j)$ to be the inverse function of $\varphi_{L/K}$ and put $G^j = G_{\psi_{L/K}(j-1)}$ (Note that our definition is shifted from that of [17] by one). Then it is compatible with quotients. Define

$$G_K^j = \varprojlim_{L/K} \mathrm{Gal}(L/K)^j,$$

where the limit is taken over the filtered ordered set of finite Galois extensions L/K inside $\bar{K}$. By the compatibility with quotients, the image of G_K^j by the natural surjection $G_K \to \mathrm{Gal}(L/K)$ is equal to $\mathrm{Gal}(L/K)^j$. Note that, since the residue field k is perfect, the extension L/K can be written as an Eisenstein extension of an unramified extension of K. In other words, in the case of perfect residue fields, we can reduce the study of ramification to extensions L/K such that the $\mathcal{O}_K$-algebra $\mathcal{O}_L$ is generated by a single element (i.e. *monogenic*). This property is a key point both to define G^j and to prove its compatibility with quotients in the classical ramification theory.

§ 2. Deligne's equivalence

Consider two complete discrete valuation fields K and F, and suppose that their residue fields are isomorphic to each other. Since the category $\mathrm{FE}_K^{<0+}$ (*resp.* $\mathrm{FE}_F^{<0+}$) of finite unramified extensions of K (*resp.* F) is equivalent to that of finite separable

extensions of its residue field, we obtain an equivalence of categories $\mathrm{FE}_K^{<0+} \simeq \mathrm{FE}_F^{<0+}$. Let $\mathrm{FE}_K^{<j}$ be the category of finite separable extensions of K whose ramification is bounded by j. In this article, we mainly focus on the following:

Question 2.1. Suppose that we have an isomorphism of rings $\mathcal{O}_K/(\pi_K^m) \simeq \mathcal{O}_F/(\pi_F^m)$ for some positive integer m. Then, can we show an equivalence of categories $\mathrm{FE}_K^{<j} \simeq \mathrm{FE}_F^{<j}$ for some j?

For the case where the residue field of K (and F) is perfect, this was studied by Deligne and he proved the following theorem, which gives a striking isomorphism between the absolute Galois groups of complete discrete valuation fields of possibly different characteristics modulo ramification subgroups:

Theorem 2.2 ([5], 1.3). *Let K and F be complete discrete valuation fields with perfect residue fields and m a positive integer. Suppose that we have an isomorphism of rings $\mathcal{O}_K/(\pi_K^m) \simeq \mathcal{O}_F/(\pi_F^m)$. Then there exists an equivalence of categories $\mathrm{FE}_K^{<j} \simeq \mathrm{FE}_F^{<j}$ for any j satisfying $0 < j \leq m$. In particular, there exists an isomorphism of topological groups $G_K/G_K^j \simeq G_F/G_F^j$.*

Suppose moreover that F is of characteristic p. Then for any m, we can find a p-adic field K_m such that there exists an isomorphism of rings $\mathcal{O}_F/(\pi_F^m) \simeq \mathcal{O}_{K_m}/(\pi_{K_m}^m)$ (Indeed, we can take $\mathcal{O}_{K_m} = C(k)[[u]]/(u^m - p)$, where $C(k)$ is a Cohen ring of the residue field k of F). In other words, any complete discrete valuation field of equal characteristic can be written as a limit of a family of complete discrete valuation fields of mixed characteristic. Combining this fact with Theorem 2.2, we see the following: If the residue field is perfect, then we can reduce the study of the absolute Galois group of a complete discrete valuation field of equal characteristic to that of mixed characteristic! This trick is known as (the Galois side of) the theory of close local fields due to Deligne and Kazhdan, and it provides a powerful tool for proving the local Langlands correspondence for reductive groups over local fields of equal characteristic where harmonic analysis is much harder than the mixed characteristic case (see [4] for the inner forms of SL_n, and [6] for GSp_4).

The key point of the proof of Theorem 2.2 is the fact that ramification of a finite Galois extension L/K with trivial residue extension can be read off from the Newton polygon of a translation of the minimal polynomial $f(X)$ of a uniformizer π_L over $\mathcal{O}_K$. Write as

$$f(X + \pi_L) = a_0 X^n + a_1 X^{n-1} + \cdots + a_{n-1} X.$$

Consider the subset $\{(i, v_K(a_i))\} \subseteq \mathbb{R}^2$. The Newton polygon $\mathrm{NP}_K(L, \pi_L)$ associated to the extension L/K and the uniformizer π_L is by definition its lower convex hull. Then the y-intercept of the leftmost slope is equal to $c(L/K)$ [5, Proposition 1.5.1].

More precisely, the Hasse-Herbrand function $\varphi_{L/K}$ can be recovered completely from $P = \mathrm{NP}_K(L, \pi_L)$ by the formula

$$\varphi_{L/K}(s) = \varphi_P(\frac{s+1}{e(L/K)}) - 1,$$

where $e(L/K)$ is the relative ramification index of L/K and φ_P is the dual polygon of P:

$$\varphi_P(s) = \inf\{t \in \mathbb{R} \mid y = -sx + t \text{ intersects } P\}$$

(for example, see [7, Section 3]. Note that the corresponding formula there contains an error). From this we see that if $c(L/K) < m$, then the Newton polygon $\mathrm{NP}_K(L, \pi_L)$ depends only on $\mathcal{O}_L/(\pi_K^m)$. This allows us to prove that the category $\mathrm{FE}_K^{<j}$ depends only on $\mathcal{O}_K/(\pi_K^m)$, and hence Theorem 2.2 follows.

§ 3. Residually imperfect case: ramification theory of Abbes-Saito

Next we consider the case where the residue field k of K is imperfect. Let L/K be a finite Galois extension. The problem is that, in this case, the integer ring $\mathcal{O}_L$ is not necessarily monogenic over the integer ring of the maximal unramified extension inside L/K, and ramification appears not only in the uniformizer π_L but also in the inseparable residue extension. Therefore the classical definition of upper ramification subgroups does not work for this case. A ramification theory generalizing the classical one to such non-monogenic extensions is due to Abbes-Saito [2, 3], for which we briefly explain their idea.

First we assume again that k is perfect, to illustrate the idea. Let L/K be a finite Galois extension with trivial residue extension and Galois group G. We fix a uniformizer π_L of L and let $f(X)$ be its minimal polynomial over $\mathcal{O}_K$, as above. Let $z_1 = \pi_L, z_2, \ldots, z_n$ be the roots of $f(X)$ in $\bar{K}$. Put

$$X_{\mathbb{C}}^j(\mathcal{O}_L, \pi_L) = \{x \in \mathcal{O}_{\mathbb{C}} \mid v_K(f(x)) \geq j\}.$$

Note that it also depends on the base field K, and that G_K acts naturally on it. We write it as the disjoint union of discs, as follows. For any $x \in \mathcal{O}_{\mathbb{C}}$, we have

$$v_K(f(x)) = \sum_{l=1}^{n} v_K(x - z_l).$$

Take i such that $v_K(x - z_i) = \max_{l=1,\ldots,n} v_K(x - z_l)$. Since $v_K(x - z_l) = v_K((x - z_i) + (z_i - z_l))$, we have

$$v_K(x - z_i) \leq v_K(z_i - z_l) \Rightarrow v_K(x - z_l) = v_K(x - z_i),$$
$$v_K(x - z_i) > v_K(z_i - z_l) \Rightarrow v_K(x - z_l) = v_K(z_i - z_l)$$

and

$$v_K(f(x)) = \sum_{l\text{: first case}} v_K(x - z_i) + \sum_{l\text{: second case}} v_K(z_i - z_l).$$

Since G acts transitively on the roots of $f(X)$, the set $\{v_K(z_i - z_l) \mid l \neq i\}$ is independent of i and thus the valuation $v_K(f(x))$ depends only on $u = \max_{l=1,\dots,n} v_K(x - z_l)$. Put $v_K(f(x)) = \tilde{\varphi}(u)$. This is a piecewise linear function passing the origin. On a smooth point $(u, \tilde{\varphi}(u))$, the slope of this function is equal to

$$\sharp\{l \mid u \leq v_K(z_i - z_l)\} = \sharp\{g \in G \mid v_K(g(\pi_L) - \pi_L) \geq u\} = \sharp G_{e(L/K)u-1}.$$

Hence we obtain the equality

$$\tilde{\varphi}(u) = \varphi_{L/K}(e(L/K)u - 1) + 1,$$

and its inverse function $\tilde{\psi}(j)$ is

$$\tilde{\psi}(j) = \frac{1}{e(L/K)}(\psi_{L/K}(j-1) + 1).$$

Put $\theta = 1/p$ and write the disc with radius θ^j centered at z as

$$D(z, \theta^j) = \{x \in \mathcal{O}_{\mathbb{C}} \mid v_K(x - z) \geq j\}.$$

Then the set $X^j_{\mathbb{C}}(\mathcal{O}_L, \pi_L)$ can be written as

$$X^j_{\mathbb{C}}(\mathcal{O}_L, \pi_L) = \bigcup_{l=1,\dots,n} D(z_l, \theta^{\tilde{\psi}(j)}).$$

We define an equivalence relation $\sim_s$ on $\mathcal{O}_{\mathbb{C}}$ by

$$z \sim_s w \Leftrightarrow v_K(z - w) \geq s.$$

It satisfies

$$z \sim_s z' \Leftrightarrow D(z, \theta^s) = D(z', \theta^s),$$
$$z \nsim_s z' \Leftrightarrow D(z, \theta^s) \cap D(z', \theta^s) = \emptyset.$$

Note that, for any $g, h \in G$, we have

$$g(\pi_L) \sim_{\tilde{\psi}(j)} h(\pi_L) \Leftrightarrow g^{-1}h \in G^j.$$

Thus we obtain the decomposition

$$X^j_{\mathbb{C}}(\mathcal{O}_L, \pi_L) = \coprod_{g \in G/G^j} D(g(\pi_L), \theta^{\tilde{\psi}(j)}).$$

This decomposition can be interpreted as the decomposition into connected components if we consider both sides as analytic varieties over $\mathbb{C}$. Namely, we can recover the ramification subgroup G^j as the stabilizer in G of any single element of the set $\pi_0(X^j_{\mathbb{C}}(\mathcal{O}_L, \pi_L))$. This observation can be generalized to the non-monogenic case, which is the idea of Abbes-Saito to define ramification subgroups G^j_K for K with imperfect residue field.

Now we recall their definition in a slightly different, but equivalent, manner from the original one [2]. Here we describe their ramification theory via adic spaces, in order to pass to perfectoid spaces (see [10, Section 2]). For this, we briefly explain about the definition of adic spaces (see [12, Section 3] and [16, Section 2]).

Let R be a ring. A valuation on R is a multiplicative map $|\cdot| : R \to \Gamma \cup \{0\}$, where Γ is a totally ordered abelian group with its group structure written multiplicatively, such that $|0| = 0$, $|1| = 1$ and $|f+g| \leq \max\{|f|, |g|\}$ for any $f, g \in R$. We denote by $\Gamma_{|\cdot|}$ the subgroup of Γ generated by $\{|f| \mid f \neq 0 \in R\}$. For a topological ring R, a valuation $|\cdot| : R \to \Gamma \cup \{0\}$ on R is said to be continuous if the subset $\{f \in R \mid |f| < \gamma\}$ is open in R for any $\gamma \in \Gamma_{|\cdot|}$. Two valuations $|\cdot|$ and $|\cdot|'$ are said to be equivalent if the condition

$$|f| \leq |g| \Leftrightarrow |f|' \leq |g|'$$

is satisfied for any $f, g \in R$. We denote this equivalence relation by $\sim$.

Let F be a topological field whose topology is given by a non-trivial valuation of rank one. A topological F-algebra R is called a Tate F-algebra if there exists a subring R_0 of R such that $\{aR_0 \mid a \in F^\times\}$ forms a basis of open neighborhoods of 0. An element f of a Tate F-algebra R is said to be power-bounded if there exists $a \in F^\times$ such that $\{f^n \mid n \in \mathbb{Z}_{\geq 0}\} \subseteq aR_0$. We denote the subring of power-bounded elements in R by R°. Let R^+ be an open subring of R which is integrally closed in R and contained in R°. We define

$$\mathrm{Spa}(R, R^+) = \{|\cdot| : R \to \Gamma \cup \{0\} \mid |\cdot| \text{ is continuous and } |f| \leq 1 \text{ for any } f \in R^+\}/\sim .$$

Hence any point $x \in \mathrm{Spa}(R, R^+)$ defines an equivalence class of valuations on R. We write a valuation representing x as $f \mapsto |f(x)|$. Note that the group Γ may vary for each point $x \in \mathrm{Spa}(R, R^+)$.

Let $f_1, \ldots, f_r, g$ be elements of R satisfying $(f_1, \ldots, f_r) = R$. Then the subset

$$\{x \in \mathrm{Spa}(R, R^+) \mid |f_i(x)| \leq |g(x)| \neq 0 \text{ for any } i\}$$

is called a rational subset of $\mathrm{Spa}(R, R^+)$. We give $\mathrm{Spa}(R, R^+)$ the topology generated by the rational subsets.

Put $\pi = \pi_K$. Let $\tilde{B}$ be a finite flat $\mathcal{O}_K$-algebra. Fix a finite system of generators $Z = (z_1, \ldots, z_n)$ of the $\mathcal{O}_K$-algebra $\tilde{B}$. Consider the surjection $\mathcal{O}_K[X_1, \ldots, X_n] \to \tilde{B}$

defined by $X_i \mapsto z_i$ and write its kernel as $(f_1, \dots, f_r)$. Let us write as $X = (X_1, \dots, X_n)$ and

$$\mathcal{O}_{\mathbb{C}}\langle X\rangle = \mathcal{O}_{\mathbb{C}}[X_1, \dots, X_n]^{\wedge}, \quad \mathbb{C}\langle X\rangle = \mathcal{O}_{\mathbb{C}}\langle X\rangle[1/\pi],$$

where $\wedge$ means the π-adic completion. Put

$$X_{\mathbb{C}}^{\mathrm{ad}} = \mathrm{Spa}(\mathbb{C}\langle X\rangle, \mathcal{O}_{\mathbb{C}}\langle X\rangle),$$

which also depends on n. Define a rational subset $X_{\mathbb{C}}^{j,\mathrm{ad}}(\tilde{B}, Z)$ of $X_{\mathbb{C}}^{\mathrm{ad}}$ by

$$X_{\mathbb{C}}^{j,\mathrm{ad}}(\tilde{B}, Z) = \{x \in X_{\mathbb{C}}^{\mathrm{ad}} \mid |f_i(x)| \leq |\pi(x)|^j \text{ for any } i\}$$

and put

$$\mathcal{F}_K^j(\tilde{B}) = \pi_0(X_{\mathbb{C}}^{j,\mathrm{ad}}(\tilde{B}, Z)).$$

Then we can show that this is a finite G_K-set which is independent of the choice of Z up to an isomorphism and functorial on $\tilde{B}$.

Definition 3.1. We say that the ramification of $\tilde{B}/\mathcal{O}_K$ is bounded by j if $\sharp\mathcal{F}_K^j(\tilde{B}) = \mathrm{rank}_{\mathcal{O}_K}(\tilde{B})$. For a finite extension L/K, we say that the ramification of L/K is bounded by j if the ramification of $\mathcal{O}_L/\mathcal{O}_K$ is bounded by j.

This definition is equivalent to [2, Definition 6.3] if $\tilde{B}$ is of relative complete intersection over $\mathcal{O}_K$ and $\tilde{B} \otimes_{\mathcal{O}_K} K$ is etale over K. Then we have the following compatibility with base change.

Lemma 3.2 ([1], Proof of Lemme 2.1.5)**.** *Let K'/K be an extension of complete discrete valuation fields which is not necessarily finite. Let $\tilde{B}$ be a finite flat $\mathcal{O}_K$-algebra. Then we have a natural bijection*

$$\mathcal{F}_K^j(\tilde{B}) \simeq \mathcal{F}_{K'}^{je(K'/K)}(\tilde{B} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}).$$

For any finite Galois extension L/K with Galois group G, we define the j-th upper ramification subgroup G^j as

$$G^j = \mathrm{Ker}(G \to \mathrm{Aut}(\mathcal{F}_K^j(\mathcal{O}_L)))$$

and G_K^j as their projective limit. In this case, the above definition of the boundedness of ramification is equivalent to what is mentioned earlier. Moreover, for the residually perfect case, this subgroup G_K^j coincides with the one defined before.

As we can see from the above definition, one of the advantages of Abbes-Saito's ramification theory is that we can measure ramification of finite flat algebras. This advantage had been used for a study of canonical subgroups of abelian varieties [1, 8,

9, 18, 19]. To pass to finite flat algebras to measure ramification is also a key step in what follows.

§ 4. Ramification of truncated discrete valuation rings

Since we have the higher dimensional ramification theory of Abbes-Saito, it is natural to expect that Theorem 2.2 can be generalized to the residually imperfect case. The point is to define an analogue of the category $\mathrm{FE}_K^{<j}$ over $\mathcal{O}_K/(\pi_K^m) \simeq \mathcal{O}_F/(\pi_F^m)$ to bridge between the sides of K and F. In other words, what we need is an intrinsic definition of a ramification theory of local rings similar to the ring $\mathcal{O}_K/(\pi_K^m)$ — truncated discrete valuation rings.

Definition 4.1 ([5], Subsection 1.1, [11], Section 2)**.** We say that a ring A is a truncated discrete valuation ring if A is a local ring with nilpotent principal maximal ideal. We refer to a generator of its principal maximal ideal as a uniformizer of A. A truncated discrete valuation ring A is said to be of length m if the length of A as an A-module is m.

A typical example of truncated discrete valuation rings of length m is the ring $\mathcal{O}_K/(\pi_K^m)$. Conversely, for any truncated discrete valuation ring A of length m, we can always find a complete discrete valuation field K and a local surjection $\iota : \mathcal{O}_K \to A$ inducing an isomorphism $\mathcal{O}_K/(\pi_K^m) \simeq A$ [5, Subsection 1.1]. We refer to such a pair (K, ι) as a lift of A.

A notion of finite extension of A is defined as follows:

Definition 4.2 ([11], Section 2)**.** Let A be a truncated discrete valuation ring of length m. A finite flat A-algebra B is said to be a finite extension of truncated discrete valuation rings over A if B is a truncated discrete valuation ring for $m \geq 2$, and B is a field if $m = 1$.

For any finite extension L/K of complete discrete valuation fields and any integer $m \geq 2$, the $\mathcal{O}_K/(\pi_K^m)$-algebra $\mathcal{O}_L/(\pi_K^m)$ is a finite extension of truncated discrete valuation rings in this sense. We have the following converse:

Theorem 4.3 ([11], Proposition 2.2)**.** *Let A be a truncated discrete valuation ring of length m and B/A a finite extension of truncated discrete valuation rings. Let (K, ι) be a lift of A. Then there exists a finite separable extension L/K with a cocartesian diagram*

$$\begin{tikzcd} \mathcal{O}_K \arrow[r] \arrow[d] & \mathcal{O}_L \arrow[d] \\ A \arrow[r] & B. \end{tikzcd}$$

Remark. Though the proof of [11, Proposition 2.2] for the claim that we can take a separable L/K has a gap, it can be easily fixed. Note that in [10] we do not use the claim, since [10, Lemma 4.10 (ii)] is enough for applications.

The idea of Hiranouchi-Taguchi [11] to attack a generalization of Theorem 2.2 is to define ramification of any finite extension B/A of truncated discrete valuation rings as ramification of L/K as in Theorem 4.3. Namely, for any positive rational number $j \leq m$, they defined that the ramification of B/A is bounded by j if the ramification of L/K is bounded by j. This notion is independent of the choice of L once we fix a lift (K, ι) of A, by [7, Lemma 1]. Then they also defined a category $\mathrm{FFP}^{<j}_{A,(K,\iota)}$ of finite extensions of A whose ramification is bounded by j along the lift (K, ι) (in fact, they denote it by $\mathrm{FFP}^{\leq j}_{A}$) and showed that it is naturally equivalent to the category $\mathrm{FE}^{<j}_{K}$. If the category $\mathrm{FFP}^{<j}_{A,(K,\iota)}$ is independent of the choice of a lift (K, ι), then we can obtain an equivalence of categories generalizing Theorem 2.2 immediately. However, this independence had remained open.

§ 5. Main theorems

For any complete discrete valuation field K of residue characteristic p, we set $e(K)$ to be the absolute ramification index of K if K is of characteristic zero and an arbitrary positive integer if K is of characteristic p. Then the main result of the author's paper [10] is the following, which settles this problem for the case where $pA = 0$.

Theorem 5.1 ([10], Theorem 1.1 (i)). *Let L_1/K_1 and L_2/K_2 be finite extensions of complete discrete valuation fields of residue characteristic p. Let m be a positive integer satisfying $m \leq \min_i e(K_i)$. Suppose that we have compatible isomorphisms of rings $\mathcal{O}_{K_1}/(\pi^m_{K_1}) \simeq \mathcal{O}_{K_2}/(\pi^m_{K_2})$ and $\mathcal{O}_{L_1}/(\pi^m_{K_1}) \simeq \mathcal{O}_{L_2}/(\pi^m_{K_2})$. Then, for any positive rational number $j \leq m$, the ramification of L_1/K_1 is bounded by j if and only if the ramification of L_2/K_2 is bounded by j.*

The assumption $m \leq \min_i e(K_i)$, which means that what we are considering are only the truncated discrete valuation rings killed by p, is crucial. This is because, to prove the theorem, we first lift $A = \mathcal{O}_{K_i}/(\pi^m_{K_i})$ to a complete discrete valuation field F of equal characteristic using $pA = 0$, and then compare ramification over F and K_i by passing to perfectoid spaces. The author has no idea of how to drop this assumption, while he thinks that the p-torsion case is the main case of interest, since it enables us to switch mixed and equal characteristics.

We also have the following corollaries of Theorem 5.1.

Corollary 5.2.

1. *([10], Theorem 4.16) If $pA = 0$, then the category $\mathrm{FFP}^{<j}_{A,(K,\iota)}$ is independent of the choice of a lift (K, ι) of A.*

2. *([10], Corollary 4.18) Let K_1 and K_2 be complete discrete valuation fields, with residue fields k_1 and k_2 of characteristic p, respectively. Let j be a positive rational number satisfying $j \leq \min_i e(K_i)$. Suppose that the fields k_1 and k_2 are isomorphic to each other. Then there exists an equivalence of categories*

$$\mathrm{FE}^{<j}_{K_1} \simeq \mathrm{FE}^{<j}_{K_2}.$$

In particular, there exists an isomorphism of topological groups

$$G_{K_1}/G^j_{K_1} \simeq G_{K_2}/G^j_{K_2}.$$

3. *([10], Theorem 6.2) The functor of higher fields of norms [15] is compatible with ramification in the sense of Abbes-Saito.*

4. *([10], Theorem 7.2) Suppose $\mathrm{char}(K) = 0$. Let V be a p-adic representation of G_K with finite local monodromy. Define the Artin conductor of V as*

$$\mathrm{Art}(V) = \sum_{j \in \mathbb{Q}_{>0}} j \dim_{\mathbb{Q}_p}(V^{G^{j+}_K}/V^{G^j_K}).$$

If $\mathrm{Art}(V) < e(K)$, then $\mathrm{Art}(V)$ is an integer.

Remark.

1. Corollary 5.2 (3) gives a totally different proof of a theorem of Ohkubo [13, Theorem 3.42].

2. In the ramification theory of Abbes-Saito, we also have another ramification subgroup which generalizes the classical ramification subgroup G^{j-1}_K in our notation—the j-th log ramification subgroup $G^j_{K,\log}$. Theorem 5.1 and Corollary 5.2 can be generalized to the case of log ramification under the slightly stronger assumption of $j \leq \min\{e(K_i) - 2\}$, except Corollary 5.2 (1) (see [10]).

   The reason why we need the stronger assumption is as follows: For the log case, we have to compare the elements $\pi_{L_i}^{e(L_i/K_i)}/\pi_{K_i} \bmod \pi^m_{K_i}$ for $i = 1, 2$. For these two elements to be equal, we need to have an isomorphism $\mathcal{O}_{K_1}/(\pi^{m+1}_{K_1}) \simeq \mathcal{O}_{K_2}/(\pi^{m+1}_{K_2})$ and this forces us to have $m+1 \leq e(K_i)$. Moreover, the log ramification is defined by counting the number of connected components of an analytic variety whose defining equations include

$$|(X_n^{e(L_i/K_i)} - \pi_{K_i} g)(x)| \leq |\pi_{K_i}(x)|^{j+1}$$

with some polynomial $g \in \mathcal{O}_{K_i}[X]$, and this forces $j+1 \leq m$.

On the other hand, the reason why Corollary 5.2 (1) remains open for the log case is that, if we try to prove a functoriality for any extension L_i'/L_i of a comparison result of ramification, we encounter an analytic variety whose defining equations include

$$|(X_n^{e(L_i'/K_i)} - \pi_{K_i}^{e(L_i'/L_i)} g')(x)| \leq |\pi_{K_i}(x)|^{j+e(L_i'/L_i)}.$$

The power on the right-hand side can be arbitrarily large and we cannot compare ramification functorially through modulo $\pi_{K_i}^m$ with a fixed m.

3. Corollary 5.2 (4) is a part of a theorem of Xiao [20, Subsection 1.1, Theorem]. However, the log version of this corollary proves the integrality of the Swan conductor of V for a case which had not been known previously.

§ 6. Sketch of the proof

In this section, we explain main ideas for the proof of Theorem 5.1. For the residually perfect case, the key point of the proof of Theorem 2.2 is to bridge between ramification theories of complete discrete valuation fields with possibly different characteristics, by extracting the combinatorial objects of Newton polygons which recover ramification and then comparing these objects inside $\mathbb{R}^2$. As we mentioned before, if the residue field k of a complete discrete valuation field K is imperfect, then the integer ring $\mathcal{O}_L$ of a finite Galois extension L/K is not necessarily monogenic and we cannot define the Newton polygon.

Tropical analytic geometry (see [14]) produces higher dimensional combinatorial objects generalizing the Newton polygon. Using them, it may be possible to study lower ramification subgroups of $\mathrm{Gal}(L/K)$ whose definition involves the valuations of differences of common zeros of the defining equations of $\mathcal{O}_L$ over $\mathcal{O}_K$ as in the classical case. However, in the non-monogenic case, upper ramification subgroups do not necessarily coincide with lower ramification subgroups even after renumbering [3, Subsection 2.1, Example]. Thus it is unclear how to define a combinatorial object which recovers the Abbes-Saito ramification of L/K. The idea to bypass this difficulty is a use of perfectoid spaces [16] to bridge between ramification theories of mixed and equal characteristics.

Let m be a positive integer and j a positive rational number satisfying $j \leq m$. Let A be a truncated discrete valuation ring of length m which is killed by p and B a finite flat A-algebra. Let (K, ι) and $(F, \boldsymbol{\iota})$ be two lifts of A. Suppose that we have a

cocartesian diagram

$$\begin{tikzcd} \tilde{B}_K \arrow[r] & B & \tilde{B}_F \arrow[l] \\ \mathcal{O}_K \arrow[u] \arrow[r, "\iota"'] & A \arrow[u] & \mathcal{O}_F \arrow[u] \arrow[l, "\boldsymbol{\iota}"] \end{tikzcd} \tag{6.1}$$

with $\tilde{B}_K = \mathcal{O}_L$ and $\tilde{B}_F = \mathcal{O}_E$ for some finite separable extensions L/K and E/F. What we have to show is that the ramification of L/K is bounded by j if and only if the ramification of E/F is bounded by j. For this, it is enough to show the equality

$$\sharp\mathcal{F}_K^j(\tilde{B}_K) = \sharp\mathcal{F}_F^j(\tilde{B}_F).$$

We identify the residue field of A with those of K and F and denote it also by k. Since $pA = 0$, we can choose a section $k \to A$ of the reduction map $A \to k$. We fix once and for all such a k-algebra structure of A. Let $\bar{\pi}$ be a uniformizer of A. Then the map $k[[u]] \to A$ sending u to $\bar{\pi}$ defines a lift of A, and by Theorem 4.3, we can find another similar cocartesian diagram over $k[[u]]$. Thus, by comparing ramification over $k((u))$ with those over K and F, we may assume $F = k((u))$ and $\boldsymbol{\iota}(u) = \bar{\pi}$. Let π be a uniformizer of K satisfying $\iota(\pi) = \bar{\pi}$. Let $\bar{K}$ and $\mathbb{C}$ be as before.

For simplicity, we assume that K is of characteristic zero. Fix a Cohen ring $C(k)$ of k. Then we can find a local homomorphism $C(k) \to \mathcal{O}_K$ which makes the following diagram commutative.

$$\begin{tikzcd} C(k) \arrow[r] \arrow[d] & \mathcal{O}_K \arrow[d, "\iota"] \\ k \arrow[r] & A \end{tikzcd}$$

By fixing such a local homomorphism, we consider $K_0 = \mathrm{Frac}(C(k))$ as a subfield of K. We also fix a p-basis $\{\bar{b}_\lambda\}_{\lambda\in\Lambda}$ of k and its lift $\{b_\lambda\}_{\lambda\in\Lambda}$ in $C(k)$, and a system of p-power roots $(b_{\lambda,l})_{l\geq 0}$ of b_λ in $\bar{K}$ satisfying $b_{\lambda,0} = b_\lambda$ and $b_{\lambda,l+1}^p = b_{\lambda,l}$. Let K_0' be the completion of the discrete valuation field

$$\bigcup_{\lambda,l} K_0(b_{\lambda,l}),$$

which is naturally considered as a subfield of $\mathbb{C}$. Put $K' = K_0'K$, the composite field inside $\mathbb{C}$. Then we have $e(K'/K) = 1$, $\mathcal{O}_{K'} = \mathcal{O}_K \otimes_{C(k)} \mathcal{O}_{K_0'}$ and the residue field k' of K_0' is the perfect closure of k in the residue field $\bar{k}$ of $\mathbb{C}$.

Consider the k-algebra A as a $C(k)$-algebra by the map $C(k) \to k \to A$. Put $A' = A \otimes_k k' = A \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ and similarly for B'. Then the lift $\boldsymbol{\iota} : k[[u]] \to A$ induces a lift $\boldsymbol{\iota}' : k'[[u]] \to A'$. Put $F' = k'((u))$. Then we obtain the following cocartesian

diagram

$$\begin{tikzcd}
\tilde{B}_K \otimes_{\mathcal{O}_K} \mathcal{O}_{K'} \arrow[r] & B' & \tilde{B}_F \otimes_{\mathcal{O}_F} \mathcal{O}_{F'} \arrow[l] \\
\mathcal{O}_{K'} \arrow[u] \arrow[r] & A' \arrow[u] & \mathcal{O}_{F'}. \arrow[u] \arrow[l]
\end{tikzcd}$$

By Lemma 3.2 and the equalities $e(K'/K) = e(F'/F) = 1$, it is enough to show the equality

$$\sharp\mathcal{F}^j_{K'}(\tilde{B}_K \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}) = \sharp\mathcal{F}^j_{F'}(\tilde{B}_F \otimes_{\mathcal{O}_F} \mathcal{O}_{F'}).$$

Note that the residue field k' of both K' and F' is perfect, while the rings $\tilde{B}_K \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ and $\tilde{B}_F \otimes_{\mathcal{O}_F} \mathcal{O}_{F'}$ are not necessarily normal anymore. Namely, by this base changing argument, we may assume that the residue field k is **perfect**, at the cost of assuming that $\tilde{B}_K$ and $\tilde{B}_F$ are just **finite flat algebras** over $\mathcal{O}_K$ and $\mathcal{O}_F$.

Next we choose the completion $\mathbb{C}^\flat$ of an algebraic closure of F as follows. Define the ring $\mathcal{O}_{\mathbb{C}^\flat}$ as the inverse limit ring

$$\mathcal{O}_{\mathbb{C}^\flat} = \varprojlim(\mathcal{O}_{\mathbb{C}}/(\pi^m) \leftarrow \mathcal{O}_{\mathbb{C}}/(\pi^m) \leftarrow \cdots),$$

where all the transition maps are given by $x \mapsto x^p$. Here we consider the leftmost entry in the limit as the zeroth entry, and write any element x of $\mathcal{O}_{\mathbb{C}^\flat}$ as $x = (x_0, x_1, \ldots)$. We can see that the reduction modulo π^m induces a bijection

$$\varprojlim(\mathcal{O}_{\mathbb{C}} \leftarrow \mathcal{O}_{\mathbb{C}} \leftarrow \cdots) \to \varprojlim(\mathcal{O}_{\mathbb{C}}/(\pi^m) \leftarrow \mathcal{O}_{\mathbb{C}}/(\pi^m) \leftarrow \cdots),$$

where all the transition maps in the former limit are also given by $x \mapsto x^p$. Composing this bijection with the zeroth projection, we obtain a natural multiplicative map $(\cdot)^\sharp : \mathcal{O}_{\mathbb{C}^\flat} \to \mathcal{O}_{\mathbb{C}}$, which can be written explicitly as follows: for $x = (x_0, x_1, \ldots) \in \mathcal{O}_{\mathbb{C}^\flat}$, take any lift $\hat{x}_l$ of x_l in $\mathcal{O}_{\mathbb{C}}$ and put $x^\sharp = \lim_{l\to\infty} \hat{x}_l^{p^l}$. This map extends to a multiplicative map $\mathbb{C}^\flat = \mathrm{Frac}(\mathcal{O}_{\mathbb{C}^\flat}) \to \mathbb{C}$. The field $\mathbb{C}^\flat$ is an algebraically closed complete valuation field whose integer ring is $\mathcal{O}_{\mathbb{C}^\flat}$ and additive valuation is $v_{\mathbb{C}^\flat}(x) = v_K(x^\sharp)$.

We fix a system $(\pi_l)_{l\geq 0}$ of p-power roots of π satisfying $\pi_0 = \pi$ and $\pi_{l+1}^p = \pi_l$. This gives the element $\underline{\pi} = (\pi_0, \pi_1, \ldots) \in \mathcal{O}_{\mathbb{C}^\flat}$ with $\underline{\pi}^\sharp = \pi$. Since we are assuming that k is perfect, the residue field of the maximal unramified extension K^{nr} of K in $\bar{K}$ is $\bar{k}$, and we have a unique multiplicative section $[\cdot] : \bar{k} \to \mathcal{O}_{K^{\mathrm{nr}}}$ of the reduction map. Using this, we consider $\mathcal{O}_{\mathbb{C}^\flat}$ as a $\bar{k}$-algebra by the map

$$x \mapsto ([x], [x^{1/p}], [x^{1/p^2}], \ldots).$$

We also consider $\mathcal{O}_{\mathbb{C}^\flat}$ as an $\mathcal{O}_F$-algebra by this map and $u \mapsto \underline{\pi}$. Then we see that the field $\mathbb{C}^\flat$ is the completion of an algebraic closure of F. Moreover, we have a commutative

diagram of k-algebras

$$\begin{tikzcd} \mathcal{O}_F \arrow[r] \arrow[d, "\boldsymbol{\iota}"'] & \mathcal{O}_{\mathbb{C}^\flat} \arrow[dd, "\mathrm{pr}_0"] \\ A \arrow[d, "\iota^{-1}"', "\wr"] & \\ \mathcal{O}_K/(\pi^m) \arrow[r] & \mathcal{O}_{\bar{K}}/(\pi^m). \end{tikzcd} \tag{6.2}$$

Let us return to the diagram (6.1) with finite flat algebras $\tilde{B}_K$ and $\tilde{B}_F$, as we are assuming. Let $\bar{Z} = (\bar{z}_1, \dots, \bar{z}_n)$ be a finite system of generators of the A-algebra B. Consider the surjection $A[X] \to B$ defined by $X_i \mapsto \bar{z}_i$, and write its kernel as $\bar{I} = (\bar{f}_1, \dots, \bar{f}_r)$. Let f_i be a lift of $\bar{f}_i$ to the ring $\mathcal{O}_K[X]$ by the map $\iota : \mathcal{O}_K \to A$, and let $\mathbf{f}_i$ be its lift to the ring $\mathcal{O}_F[X]$ by the map $\boldsymbol{\iota} : \mathcal{O}_F \to A$. Let z_i be a lift of $\bar{z}_i$ to $\tilde{B}_K$. Then the set $Z = (z_1, \dots, z_n)$ is a finite system of generators of the $\mathcal{O}_K$-algebra $\tilde{B}_K$. Let I be the kernel of the surjection $\mathcal{O}_K[X] \to \tilde{B}_K$ defined by Z. Then we have

$$I + \pi^m \mathcal{O}_K[X] = (f_1, \dots, f_r) + \pi^m \mathcal{O}_K[X].$$

Since $j \leq m$, the rational subset $X_{\mathbb{C}}^{j,\mathrm{ad}}(\tilde{B}_K, Z)$ is equal to

$$X_{\mathbb{C}}^{j,\mathrm{ad}}(B, \bar{Z}) = \{x \in X_{\mathbb{C}}^{\mathrm{ad}} \mid |f_i(x)| \leq |\pi(x)|^j \text{ for any } i\}.$$

Similarly, the rational subset we have to consider on the side of F is equal to

$$X_{\mathbb{C}^\flat}^{j,\mathrm{ad}}(B, \bar{Z}) = \{x \in X_{\mathbb{C}^\flat}^{\mathrm{ad}} \mid |\mathbf{f}_i(x)| \leq |\underline{\pi}(x)|^j \text{ for any } i\}.$$

To compare the sets of connected components of these two adic spaces over $\mathbb{C}$ and $\mathbb{C}^\flat$, we pass to perfectoid spaces [16]. Put

$$\mathcal{O}_{\mathbb{C}}[X^{1/p^l}] = \mathcal{O}_{\mathbb{C}}[X_1^{1/p^l}, \dots, X_n^{1/p^l}], \quad \mathcal{O}_{\mathbb{C}}[X^{1/p^\infty}] = \varinjlim_{l \in \mathbb{Z}_{\geq 0}} \mathcal{O}_{\mathbb{C}}[X_1^{1/p^l}, \dots, X_n^{1/p^l}]$$

and denote their π-adic completions by $\mathcal{O}_{\mathbb{C}}\langle X^{1/p^l}\rangle$ and $\mathcal{O}_{\mathbb{C}}\langle X^{1/p^\infty}\rangle$. We also put

$$\mathbb{C}\langle X^{1/p^l}\rangle = \mathcal{O}_{\mathbb{C}}\langle X^{1/p^l}\rangle[1/\pi], \quad \mathbb{C}\langle X^{1/p^\infty}\rangle = \mathcal{O}_{\mathbb{C}}\langle X^{1/p^\infty}\rangle[1/\pi].$$

We define the rings $\mathcal{O}_{\mathbb{C}^\flat}\langle X^{1/p^l}\rangle$, $\mathcal{O}_{\mathbb{C}^\flat}\langle X^{1/p^\infty}\rangle$, $\mathbb{C}^\flat\langle X^{1/p^l}\rangle$ and $\mathbb{C}^\flat\langle X^{1/p^\infty}\rangle$ on the side of F similarly, using $\underline{\pi}$ instead of π. Then the ring $\mathbb{C}\langle X^{1/p^\infty}\rangle$ is a perfectoid $\mathbb{C}$-algebra with tilt $\mathbb{C}^\flat\langle X^{1/p^\infty}\rangle$ [16, Definition 5.1, Proposition 5.20]. In particular, we have a canonical ring isomorphism

$$\mathcal{O}_{\mathbb{C}^\flat}\langle X^{1/p^\infty}\rangle \simeq \varprojlim(\mathcal{O}_{\mathbb{C}}\langle X^{1/p^\infty}\rangle/(\pi^m) \leftarrow \mathcal{O}_{\mathbb{C}}\langle X^{1/p^\infty}\rangle/(\pi^m) \leftarrow \cdots),$$

where all the transition maps are $x \mapsto x^p$, such that the composite with the zeroth projection is equal to

$$\mathcal{O}_{\mathbb{C}^\flat}\langle X^{1/p^\infty}\rangle \to \mathcal{O}_{\mathbb{C}^\flat}\langle X^{1/p^\infty}\rangle/(\underline{\pi}^m) \simeq \mathcal{O}_{\mathbb{C}}\langle X^{1/p^\infty}\rangle/(\pi^m),$$

where the right isomorphism is defined by $\mathrm{pr}_0 : \mathcal{O}_{\mathbb{C}^\flat}/(\underline{\pi}^m) \simeq \mathcal{O}_{\mathbb{C}}/(\pi^m)$ and $X_i^{1/p^l} \mapsto X_i^{1/p^l}$ [10, Lemma 3.2]. Moreover, the reduction modulo π^m induces a bijection

$$\varprojlim(\mathcal{O}_{\mathbb{C}}\langle X^{1/p^\infty}\rangle \leftarrow \mathcal{O}_{\mathbb{C}}\langle X^{1/p^\infty}\rangle \leftarrow \cdots) \\ \to \varprojlim(\mathcal{O}_{\mathbb{C}}\langle X^{1/p^\infty}\rangle/(\pi^m) \leftarrow \mathcal{O}_{\mathbb{C}}\langle X^{1/p^\infty}\rangle/(\pi^m) \leftarrow \cdots),$$

as in the case of $\mathcal{O}_{\mathbb{C}}$ and $\mathcal{O}_{\mathbb{C}^\flat}$. Composing these bijections with the zeroth projection, we obtain a continuous multiplicative map

$$(\cdot)^\sharp : \mathbb{C}^\flat\langle X^{1/p^\infty}\rangle \to \mathbb{C}\langle X^{1/p^\infty}\rangle$$

[16, Proposition 5.17]. From the commutative diagram (6.2), we also obtain the congruence

$$\mathbf{f}_i^\sharp \equiv f_i \bmod \pi^m. \tag{6.3}$$

Put

$$X^{\mathrm{ad}}_{\mathbb{C},l} = \mathrm{Spa}(\mathbb{C}\langle X^{1/p^l}\rangle, \mathcal{O}_{\mathbb{C}}\langle X^{1/p^l}\rangle), \quad X^{\mathrm{ad}}_{\mathbb{C},\infty} = \mathrm{Spa}(\mathbb{C}\langle X^{1/p^\infty}\rangle, \mathcal{O}_{\mathbb{C}}\langle X^{1/p^\infty}\rangle)$$

and similarly for $X^{\mathrm{ad}}_{\mathbb{C}^\flat,l}$, $X^{\mathrm{ad}}_{\mathbb{C}^\flat,\infty}$. By [16, Theorem 6.3], we have a homeomorphism

$$\tau : X^{\mathrm{ad}}_{\mathbb{C},\infty} \to X^{\mathrm{ad}}_{\mathbb{C}^\flat,\infty}$$

defined by $|\mathbf{f}(\tau(x))| = |\mathbf{f}^\sharp(x)|$ for any $x \in X^{\mathrm{ad}}_{\mathbb{C},\infty}$ and any $\mathbf{f} \in \mathbb{C}^\flat\langle X^{1/p^\infty}\rangle$.

Now we denote by $X^{j,\mathrm{ad}}_{\mathbb{C},\infty}(B,\bar{Z})$ the inverse image of the rational subset $X^{j,\mathrm{ad}}_{\mathbb{C}}(B,\bar{Z}) \subseteq X^{\mathrm{ad}}_{\mathbb{C}}$ by the natural projection $p_{\infty,0} : X^{\mathrm{ad}}_{\mathbb{C},\infty} \to X^{\mathrm{ad}}_{\mathbb{C}}$. This is the rational subset

$$\{x \in X^{\mathrm{ad}}_{\mathbb{C},\infty} \mid |f_i(x)| \le |\pi(x)|^j \text{ for any } i\}$$

of the adic space $X^{\mathrm{ad}}_{\mathbb{C},\infty}$. Similarly, we define $X^{j,\mathrm{ad}}_{\mathbb{C}^\flat,\infty}(B,\bar{Z})$ to be the inverse image of $X^{j,\mathrm{ad}}_{\mathbb{C}^\flat}(B,\bar{Z})$ by the projection $p^\flat_{\infty,0} : X^{\mathrm{ad}}_{\mathbb{C}^\flat,\infty} \to X^{\mathrm{ad}}_{\mathbb{C}^\flat}$. This is equal to

$$\{x \in X^{\mathrm{ad}}_{\mathbb{C}^\flat,\infty} \mid |\mathbf{f}_i(x)| \le |\underline{\pi}(x)|^j \text{ for any } i\}.$$

Thus we have a commutative diagram

$$\begin{tikzcd}
X^{j,\mathrm{ad}}_{\mathbb{C},\infty}(B,\bar{Z}) \arrow[r, hook] \arrow[d] & X^{\mathrm{ad}}_{\mathbb{C},\infty} \arrow[r, "\tau", "\sim"'] \arrow[d, "p_{\infty,0}"'] & X^{\mathrm{ad}}_{\mathbb{C}^\flat,\infty} \arrow[d, "p^\flat_{\infty,0}"] & X^{j,\mathrm{ad}}_{\mathbb{C}^\flat,\infty}(B,\bar{Z}) \arrow[l, hook'] \arrow[d] \\
X^{j,\mathrm{ad}}_{\mathbb{C}}(B,\bar{Z}) \arrow[r, hook] & X^{\mathrm{ad}}_{\mathbb{C}} & X^{\mathrm{ad}}_{\mathbb{C}^\flat} & X^{j,\mathrm{ad}}_{\mathbb{C}^\flat}(B,\bar{Z}) \arrow[l, hook']
\end{tikzcd}$$

whose two squares are cartesian. Then Theorem 5.1 follows from the proposition below:

Proposition 6.1.

1. *([10], Lemma 3.5)* $\tau^{-1}(X^{j,\mathrm{ad}}_{\mathbb{C}^\flat,\infty}(B,\bar{Z})) = X^{j,\mathrm{ad}}_{\mathbb{C},\infty}(B,\bar{Z})$.
2. *([10], Lemma 3.6) The projections* $p_{\infty,0}$ *and* $p^\flat_{\infty,0}$ *induce bijections*

$$\pi_0(X^{j,\mathrm{ad}}_{\mathbb{C},\infty}(B,\bar{Z})) \to \pi_0(X^{j,\mathrm{ad}}_{\mathbb{C}}(B,\bar{Z})), \quad \pi_0(X^{j,\mathrm{ad}}_{\mathbb{C}^\flat,\infty}(B,\bar{Z})) \to \pi_0(X^{j,\mathrm{ad}}_{\mathbb{C}^\flat}(B,\bar{Z})).$$

Outline of proof. The first part of the proposition follows from the congruence (6.3) and the assumption $j \leq m$, since

$$|f_i(x)| \leq |\pi(x)|^j \Leftrightarrow |\mathbf{f}_i^\sharp(x)| \leq |\underline{\pi}^\sharp(x)|^j = |\pi(x)|^j.$$

For the second part, the assertion on $p^\flat_{\infty,0}$ follows from the fact that this projection is a homeomorphism. Consider the assertion on $p_{\infty,0}$. By a limit argument, it is enough to show a similar assertion for the projection $p_{l,0} : X^{\mathrm{ad}}_{\mathbb{C},l} \to X^{\mathrm{ad}}_{\mathbb{C}}$ for any $l \geq 0$.

Put $X^{j,\mathrm{ad}}_{\mathbb{C},l}(B,\bar{Z}) = p_{l,0}^{-1}(X^{j,\mathrm{ad}}_{\mathbb{C}}(B,\bar{Z}))$. Note that the map $p_{l,0}$ is surjective, and also open since it is flat and finitely presented. It is enough to show that for any connected component C of $X^{j,\mathrm{ad}}_{\mathbb{C}}(B,\bar{Z})$, its inverse image $p_{l,0}^{-1}(C)$ is connected. Since the adic spaces $X^{j,\mathrm{ad}}_{\mathbb{C}}(B,\bar{Z})$ and $X^{j,\mathrm{ad}}_{\mathbb{C},l}(B,\bar{Z})$ are locally of finite type, the numbers of their connected components are finite and the components are open. The inverse image $p_{l,0}^{-1}(C)$ is the disjoint union of some connected components of $X^{j,\mathrm{ad}}_{\mathbb{C},l}(B,\bar{Z})$. If it had more than one connected components, then the intersection of their images in C would be a nonempty open subset and thus contain a classical point (namely, a point of $X^{\mathrm{ad}}_{\mathbb{C}}$ defined by the valuation $f \mapsto p^{-v_K(f(x))}$ for some closed point $x \in \mathrm{Spec}(\mathbb{C}\langle X\rangle)$). Hence it suffices to show that, for any classical point x in $X^{j,\mathrm{ad}}_{\mathbb{C}}(B,\bar{Z})$ defined by the map $X_i \mapsto x_i$ with $x_i \in \mathcal{O}_{\mathbb{C}}$, any two points $y, y' \in p_{l,0}^{-1}(x)$ are contained in the same connected component of $X^{j,\mathrm{ad}}_{\mathbb{C},l}(B,\bar{Z})$.

Consider the polydisc

$$U = \{z \in X^{\mathrm{ad}}_{\mathbb{C}} \mid |(X_i - x_i)(z)| \leq |\pi(z)|^m \text{ for any } i\}.$$

Since $j \leq m$, this polydisc is an open neighborhood of x contained in $X^{j,\mathrm{ad}}_{\mathbb{C}}(B,\bar{Z})$. Its inverse image $p_{l,0}^{-1}(U)$ is equal to

$$\{z \in X^{\mathrm{ad}}_{\mathbb{C},l} \mid |(X_i - x_i)(z)| \leq |\pi(z)|^m \text{ for any } i\}.$$

Since $m \leq e(K)$, we have

$$|(X_i - x_i)(z)| \leq |\pi(z)|^m \Leftrightarrow |(X_i^{1/p^l} - x_i^{1/p^l})(z)| \leq |\pi(z)|^{m/p^l}.$$

Hence, the inverse image $p_{l,0}^{-1}(U)$ is also a polydisc, thus connected, which is contained in $X^{j,\mathrm{ad}}_{\mathbb{C},l}(B,\bar{Z})$ and contains both y and y'. This concludes the proof. $\square$

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