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**B53** (2015), 117–131

# On algebraicity of special values of $L$-functions for $\mathrm{SO}(\mathbf{V}) \times \mathrm{GL}_2$

By

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## Abstract

In this paper, we announce the result of the joint work [8] with Masaaki Furusawa, in which we state a conjecture (see Conjecture 2) on the algebraicity of special values of tensor product $L$-functions for automorphic representations of $\mathrm{SO}(\mathbf{V})$ and $\mathrm{GL}_2$ explicating Deligne periods and we show this conjecture at various critical points under certain assumption on the weight (see Theorem 3.1). This algebraicity result is a generalization of our previous result [7, Theorem 1] with respect to critical points and infinity types of automorphic representations of $\mathrm{SO}(\mathbf{V})$.

## § 1. Notation

Let $\mathbf{V}$ be a quadratic space over $\mathbb{Q}$ such that $\dim \mathbf{V} = n \geq 2$ and $\mathbf{V} \otimes_{\mathbb{Q}} \mathbb{R}$ is positive definite. When $n$ is even, let $\chi_{\mathbf{V}}$ denote the quadratic character of $\mathbb{A}_{\mathbb{Q}}^{\times}$ given by

$$\chi_{\mathbf{V}}(x) = \Big(x, (-1)^{n/2}\, d(\mathbf{V})\Big)_{\mathbb{Q}}$$

where $(\ ,\ )_{\mathbb{Q}}$ is the Hilbert symbol of $\mathbb{Q}$ and $d(\mathbf{V})$ is the discriminant of $\mathbf{V}$. When $n$ is odd, let $\chi_{\mathbf{V}}$ be the trivial character of $\mathbb{A}_{\mathbb{Q}}^{\times}$.

Received March 25, 2014. Revised October 20, 2014.
2010 Mathematics Subject Classification(s): 11F67, 11F70.
*Key Words*: *Deligne's conjecture, critical values, automorphic L-functions.*
The research of the author was supported in part by Grant-in-Aid for JSPS Fellows (23-6883) and JSPS Institutional Program for Young Researcher Overseas Visits project: Promoting international young researchers in mathematics and mathematical sciences led by OCAMI
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We recall that by the highest weight theory, isomorphism classes of finite dimensional irreducible complex representations of $\mathrm{SO}(\mathbf{V},\mathbb{R})$ are completely parametrized by

$$\Lambda_n = \begin{cases} \left\{(m_1,\dots,m_{n'}) \in \mathbb{Z}^{n'} \mid m_1 \geq \cdots \geq m_{n'} \geq 0\right\} & \text{when } n \text{ is odd,} \\ \left\{(m_1,\dots,m_{n'}) \in \mathbb{Z}^{n'} \mid m_1 \geq \cdots \geq m_{n'-1} \geq |m_{n'}|\right\} & \text{when } n \text{ is even,} \end{cases}$$

with $n' = \left[\frac{n}{2}\right]$ (for example, see [12, Section 3]). Here $[x]$ denotes the integer such that $[x] \leq x < [x]+1$ for $x \in \mathbb{R}$.

For a Hecke character $\eta$ of $\mathbb{A}_{\mathbb{Q}}^{\times}$ of finite order, let $\eta_0$ be its associated Dirichlet character. Then we denote by $\mathfrak{g}(\eta)$ the Gauss sum $\mathfrak{g}(\eta_0)$ for $\eta_0$.

For an integer $k \geq 2$, let $S_k^{\mathrm{new}}(\Gamma_0(N),\varepsilon)$ denote the set of normalized newforms for $\Gamma_0(N)$ of weight $k$ with Nebentypus $\varepsilon$. For $f \in S_k^{\mathrm{new}}(\Gamma_0(N),\varepsilon)$, let

$$f(z) = \sum_{n=1}^{\infty} c_n(f) e^{2\pi i n z}$$

be the Fourier expansion of $f$ at the infinity. Then for $\sigma \in \mathrm{Aut}(\mathbb{C})$, the group of field automorphisms of $\mathbb{C}$, we define $f^{\sigma} \in S_k^{\mathrm{new}}(\Gamma_0(N),\varepsilon^{\sigma})$ by

$$f^{\sigma}(z) = \sum_{n=1}^{\infty} c_n(f)^{\sigma} \cdot e^{2\pi i n z}.$$

For $\sigma \in \mathrm{Aut}(\mathbb{C})$, let us define the $\sigma$-twist of a complex representation $(\pi, V)$ as in Waldspurger [22, 1.1]. Let $V'$ be a $\mathbb{C}$-vector space with $\sigma$-linear (i.e. $t'(av) = \sigma(a)\,t'(v)$) isomorphism $t' : V \to V'$. Then we define the $\sigma$-twist of $\pi$ by

$$\pi^{\sigma} := t' \circ \pi \circ t'^{-1}.$$

This definition is independent of $t'$ and $V'$ up to equivalence of representations.

Finally, for an automorphic representation $\pi$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$, we define the $\sigma$-twist of $\pi$ by

$$\pi^{\sigma} := \pi_{\mathrm{fin}}^{\sigma} \otimes \pi_{\infty}$$

where $\pi_{\mathrm{fin}}$ and $\pi_{\infty}$ are the finite and infinite part of $\pi$, respectively, and $\pi_{\mathrm{fin}}^{\sigma}$ is the $\sigma$-twist of $\pi_{\mathrm{fin}}$ as a complex representation.

## § 2. Deligne's conjecture and its explication for $\mathrm{SO}(\mathbf{V}) \times \mathrm{GL}_2$

In this section, we shall explicate Deligne's conjecture on the algebraicity of special values of motivic $L$-functions at critical points when a motive is given by a tensor product of motives corresponding to automorphic representations of $\mathrm{SO}(\mathbf{V})$ and $\mathrm{GL}_2$.

Let us briefly recall Deligne's conjecture. Readers are referred to Deligne [6, Section 2] and Yoshida [23, Section 2]. In fact, we follow the latter rather closely.

Recall that there are various categories of pure motives. However it is not very important to consider a particular category since our purpose is not to discuss Deligne's conjecture itself but to give a conjecture on special values of $L$-functions for automorphic representations explicating Deligne periods. That is why we do not specify a category of motives, and we call an object of any category of motives merely motive.

Let $\mathcal{M}$ be a motive over $\mathbb{Q}$ with coefficients in a number field $E$. Let $J_E$ denote the set of all embeddings of $E$ into $\mathbb{C}$. For a finite place $\lambda$ of $E$, let $H_\lambda(\mathcal{M})$ be the $\lambda$-adic realization of $\mathcal{M}$, which determines a $\lambda$-adic representation $\sigma_\lambda : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to \mathrm{GL}\left(H_\lambda\left(\mathcal{M}\right)\right)$. For a prime number $p$ such that $(\lambda, p) = 1$, put

$$Z_p(\mathcal{M}, X) = \det\left(1 - X\,\sigma_\lambda\left(\Phi_p\right)|_{H_\lambda(\mathcal{M})^{I_p}}\right)^{-1},$$

where $\Phi_p$ denotes a geometric Frobenius at $p$ and $I_p$ denotes the inertia group at $p$. Then there exists a finite set $S$ of primes, so that $\sigma_\lambda$ is unramifed at a prime number $p \notin S$ and $Z_p\left(\mathcal{M}, X\right)^{-1} \in E_\lambda[X]$ is independent of $\lambda$. In general, it is conjectured that this local factor is independent of $\lambda$ for any $p$ and that we have $Z_p\left(\mathcal{M}, X\right)^{-1} \in E[X]$ (see [6, 1.1, (1.2.1)]). From now on, we assume these conjectures. Then for $\sigma \in J_E$, put

$$L_p(\sigma, \mathcal{M}, s) = \sigma Z_p\left(\mathcal{M}, p^{-s}\right),$$

and

$$L(\sigma, \mathcal{M}, s) = \prod_p L_p(\sigma, \mathcal{M}, s).$$

Let

$$H_B(\mathcal{M}) = \bigoplus_{p,q\in\mathbb{Z}} H^{p\,q}(\mathcal{M})$$

be the Hodge decomposition of the Betti realization $H_B(\mathcal{M})$ of $\mathcal{M}$. Suppose that $\mathcal{M}$ is of pure weight $w$, i.e., $H^{p\,q}(\mathcal{M}) = 0$ whenever $p + q \neq w$. Then it is conjectured that $L(\sigma, \mathcal{M}, s)$ converges absolutely for $\mathrm{Re}(s) > \frac{w}{2} + 1$ and has a meromorphic continuation to the whole plane $\mathbb{C}$. Further, associated to the Hodge structure of the Betti realization $H_B(\mathcal{M})$, the archimedean local $L$-factor $L_\infty(\mathcal{M}, s)$ is defined independently from $\sigma \in J_E$ (see Serre [19, Section 3]). Then it is conjectured that $L(\sigma, \mathcal{M}, s)L_\infty(\mathcal{M}, s)$ has a functional equation with respect to $s \mapsto w + 1 - s$. Hereafter, we assume these conjectures.

We say that an integer $m \in \mathbb{Z}$ is critical for $\mathcal{M}$ if neither $L_\infty(s, \mathcal{M})$ nor $L_\infty(w + 1 - s, \mathcal{M})$ has a pole at $s = m$. If $m$ is critical for $\mathcal{M}$, then we should have

$$p < m \leq q \quad \text{whenever } H^{p\,q}(\mathcal{M}) \neq \{0\},\ p < q. \tag{2.1}$$

The condition (2.1) is sufficient for $m$ to be critical if $w$ is odd. On the other hand, if $w$ is even and we put $w = 2p$, the complex conjugation $c_\infty$ must act on $H^{p\,p}(\mathcal{M})$ by the scalar for which $\mathcal{M}$ has a critical integer. Put

$$c_\infty = (-1)^{p+\varepsilon}, \quad \varepsilon = 0 \text{ or } 1 \text{ on } H^{p\,p}(\mathcal{M}).$$

Then we should have

$$\begin{cases} m > p - \varepsilon & \text{if } p + \varepsilon + m \text{ is even,} \\ m < p + \varepsilon + 1 & \text{if } p + \varepsilon + m \text{ is odd.} \end{cases} \tag{2.2}$$

Moreover (2.1) and (2.2) are sufficient for $m \in \mathbb{Z}$ to be critical for $\mathcal{M}$. Assume that $\mathcal{M}$ has a critical integer. Then we define Deligne periods $c^{\pm}(\mathcal{M})$ (see [6, p.323]) as follows.

Let $\{F^i\left(H_{DR}\left(\mathcal{M}\right)\right)\}$ be the Hodge filtration of $H_{DR}(\mathcal{M})$. When $w$ is odd, let $F^+\left(\mathcal{M}\right) = F^-\left(\mathcal{M}\right) = F^{(w-1)/2}\left(\mathcal{M}\right)$. When $w = 2p$ is even and $p + \varepsilon$ is even, let

$$F^+\left(\mathcal{M}\right) = F^p\left(H_{DR}\left(\mathcal{M}\right)\right) \quad \text{and} \quad F^-\left(\mathcal{M}\right) = F^{p+1}\left(H_{DR}\left(\mathcal{M}\right)\right).$$

When $w = 2p$ is even and $p + \varepsilon$ is odd, let

$$F^+\left(\mathcal{M}\right) = F^{p+1}\left(H_{DR}\left(\mathcal{M}\right)\right) \quad \text{and} \quad F^-\left(\mathcal{M}\right) = F^p\left(H_{DR}\left(\mathcal{M}\right)\right).$$

Let us put

$$H_{DR}^+\left(\mathcal{M}\right) = H_{DR}\left(\mathcal{M}\right)/F^-\left(\mathcal{M}\right) \quad \text{and} \quad H_{DR}^-\left(\mathcal{M}\right) = H_{DR}\left(\mathcal{M}\right)/F^+\left(\mathcal{M}\right).$$

Then we have the canonical isomorphisms

$$I^{\pm} : H_B^{\pm}(\mathcal{M}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq H_{DR}^{\pm}(\mathcal{M}) \otimes_{\mathbb{Q}} \mathbb{C}$$

as $E \otimes_{\mathbb{Q}} \mathbb{C}$-modules. Let

$$c^{\pm}(\mathcal{M}) = \det(I^{\pm}) \in (E \otimes_{\mathbb{Q}} \mathbb{C})^{\times}$$

be the determinants calculated by an $E$-rational basis. Then Deligne periods $c^{\pm}(\mathcal{M})$ are determined up to a multiplication by elements of $E^{\times}$. Further, we remark that Deligne periods are written as periods integrals (see [6, 1.7]).

Now, let us state Deligne's conjecture. We define a function $L^*(\mathcal{M}, s)$ taking values in $E \otimes_{\mathbb{Q}} \mathbb{C}$ by assigning $\{L(\sigma, \mathcal{M}, s)\}_{\sigma \in J_E}$ through the identification $E \otimes_{\mathbb{Q}} \mathbb{C} \simeq \mathbb{C}^{J_E}$. Then Deligne conjectured that (see [6, Conjecture 2.8])

$$\frac{L^*(\mathcal{M}, m)}{(1 \otimes (2\pi\sqrt{-1})^{d^{\pm}(\mathcal{M})m})c^{\pm}(\mathcal{M})} \in E \tag{2.3}$$

where double-sign corresponds. Here $d^{\pm}(\mathcal{M})$ is the dimension of $\pm$-eigen space of the Betti realization of $\mathcal{M}$, and $E$ is embedded into $\mathbb{C}^{J_E}$ by $e \mapsto (\sigma(e))_{\sigma}$.

Our aim is to explicate Deligne periods for a tensor product motive $M \otimes N$ when motives $M$ and $N$ correspond to automorphic representations of $\mathrm{SO}(\mathbf{V}, \mathbb{A}_{\mathbb{Q}})$ and $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$, respectively. Recall that in various situations, Deligne periods for tensor product motives are considered (for example, see Bhagwat–Raghuram [2], Blasius [3] and Yoshida [23], [24]). We shall consider Deligne periods in the following situation.

Suppose that $f \in S_k^{\mathrm{new}}(\Gamma_0(N), \varepsilon)$ is a Hecke eigenform. We denote by $\pi = \pi(f)$ the irreducible unitary cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ associated to $f$. Let $\tau$ be an irreducible automorphic representation of $\mathrm{SO}(\mathbf{V}, \mathbb{A}_{\mathbb{Q}})$ whose infinite component $\tau_\infty$ is an irreducible representation of $\mathrm{SO}(\mathbf{V}, \mathbb{R})$ with the highest weight $(m_1, \cdots, m_{n'}) \in \Lambda_n$. For our computation, we impose three assumptions. First, we assume that a special case of the conjecture by Clozel [5, Conjecture 4.5] holds, which states an existence of motives corresponding to algebraic automorphic representations of general linear groups. Indeed, we assume the following conjecture.

**Conjecture 1** (Conjecture 4.5 in Clozel [5]). *Let $\Pi$ be an irreducible algebraic automorphic representation of $\mathrm{GL}_{2r}(\mathbb{A}_{\mathbb{Q}})$. Then there exists a motive $M_\Pi$ over $\mathbb{Q}$ with coefficients in a number field $E_\Pi$ such that $M_\Pi$ is of pure weight and we have*

$$L_v(\sigma, M_\Pi, s) = L\left(s - \frac{1}{2}, \Pi_v^\sigma\right)$$

*for any place $v$ of $\mathbb{Q}$ and $\sigma \in J_{E_\Pi}$.*

Second of all, we impose the following two assumptions on $\tau$.

*Assumptions.*

1. There exists an irreducible automorphic representation $\Pi_\tau$ of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ such that $(\Pi_\tau)_v$ is the local transfer of $\tau_v$ for almost all finite places and the infinite place $v$ of $\mathbb{Q}$ where
$$m = \begin{cases} n-1 & \text{when } n \text{ is odd,} \\ n & \text{when } n \text{ is even,} \end{cases}$$
with $n = \dim \mathbf{V}$.

2. $\tau$ is tempered.

We give some remarks on these assumptions. In Arthur [1], he showed an existence of a functorial lift from quasi-split inner forms of $\mathrm{SO}(\mathbf{V})$ to $\mathrm{GL}_m$, conditional on the stabilization of twisted trace formulas. Further, when $\tau$ is tempered (i.e. the second assumption holds), he states an existence of the functorial lift in the first assumption without a proof (see Arthur [1, Theorem 9.5.3]). One can be optimistic that the first assumption for our $\tau$ will be verified in a near future.

Further, we would like to give a remark on an algebraicity of automorphic representations of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ given in the first assumption. In order to simplify our argument in this remark, we suppose that $\dim \mathbf{V}$ is even. However, as we remark later, a similar result holds even when $\dim \mathbf{V}$ is odd.

Let us recall the Langlands parameter of $\tau_\infty$. Let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$, which contains $\mathbb{C}^\times$ as a normal subgroup of index two. For each $a \in \frac{1}{2}\mathbb{Z}$, we define a unitary character $\chi_a : \mathbb{C}^\times \to \mathbb{C}^1$ by

$$\chi_a(z) = \left(\frac{z}{\bar{z}}\right)^a$$

and we define a two-dimensional representation of $W_{\mathbb{R}}$ by

$$V(a) = \mathrm{Ind}_{\mathbb{C}^\times}^{W_{\mathbb{R}}}(\chi_a).$$

We know that the Langlands parameter of $\tau_\infty$ is given by

$$\bigoplus_{i=1}^{n/2} V(m_i + n - i)$$

where $(m_1, \ldots, m_{n/2})$ is the highest weight of $\tau_\infty$. Then we can easily show that $(\Pi_\tau)_\infty$ is algebraic in the sense of Clozel [5]. Since $\Pi_\tau$ should be an isobaric automorphic representation from the second assumption, $\Pi_\tau$ is an algebraic automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. We also note that $\Pi_\tau$ is uniquely determined by the strong multiplicity one theorem for isobaric automorphic representations. On the other hand, when $\dim \mathbf{V}$ is odd, in a similar argument, we see that $\Pi_\tau \otimes |-|^{1/2}$ is an algebraic automorphic representation.

Let us go back to a situation with no assumption on $\dim \mathbf{V}$.

*Then the above remark and Conjecture 1 imply that there exists a motive $M_\tau$ over $\mathbb{Q}$ with coefficients in a number field $E_\tau$ such that $M_\tau$ is of pure weight and we have*

$$L_v(\sigma, M_\tau, s) = L\left(s - \frac{i}{2}, \tau_v^\sigma\right) \quad \text{where } i = \begin{cases} 1 & \text{when } n \text{ is odd,} \\ 0 & \text{when } n \text{ is even,} \end{cases}$$

*for any place $v$ of $\mathbb{Q}$ and $\sigma \in J_{E_\tau}$. Here we define $L(s, \tau_v) = L(s, (\Pi_\tau)_v)$ for any place $v$ of $\mathbb{Q}$.*

On the other hand, Scholl [18, Theorem 1.2.4] showed that there exists a rank two (Grothendieck) motive $N_\pi$ over $\mathbb{Q}$ with coefficients in a number field $\mathbb{Q}(f)$ such that $N_\pi$ is of pure weight and we have

$$L_v(\sigma, N_\pi, s) = L\left(s - \frac{j}{2}, \pi_v^\sigma\right) \quad \text{where } j = \begin{cases} 1 & \text{when } k \text{ is even,} \\ 0 & \text{when } k \text{ is odd,} \end{cases}$$

for any place $v$ of $\mathbb{Q}$ and $\sigma \in J_{\mathbb{Q}(f)}$. Here, $\mathbb{Q}(f)$ is the algebraic number field generated by Fourier coefficients of $f$ over $\mathbb{Q}$. Then we computed $c^{\pm}(M_\tau \otimes N_\pi)$ in Appendix to [8], and we obtained the following formulas.

**Lemma 2.1** (Appendix to [8]). *Suppose all assumptions above, and assume that*

$$k - 2m_1 \geq n.$$

*Let $E$ be the composite field of $\mathbb{Q}(f)$ and $E_\tau$. Then we have the following formulas as elements of $(E \otimes \mathbb{C})^\times / E^\times$ (Recall that Deligne periods $c^{\pm}(M_\tau \otimes N_\pi)$ are determined up to a multiplication by elements of $E^\times$). When $n$ is odd, we have*

$$c^{\pm}(M_\tau \otimes N_\pi) = \begin{cases} \left\{ \left( \left(2\pi\sqrt{-1}\right)^{-2} \cdot J(f^\sigma) \right)^{\left[\frac{n}{2}\right]} \right\}_{\sigma \in J_E} & \textit{if } k \textit{ is even,} \\ \left\{ \left( \left(2\pi\sqrt{-1}\right)^{-1} \cdot J(f^\sigma) \right)^{\left[\frac{n}{2}\right]} \right\}_{\sigma \in J_E} & \textit{if } k \textit{ is odd.} \end{cases}$$

*When $n$ is even, we have*

$$c^{\pm}(M_\tau \otimes N_\pi) = \begin{cases} \mathfrak{g}(\chi_V) \left\{ \left( \left(2\pi\sqrt{-1}\right)^{-1} \cdot J(f^\sigma) \right)^{\left[\frac{n}{2}\right]} \right\}_{\sigma \in J_E} & \textit{if } k \textit{ is even,} \\ \mathfrak{g}(\chi_V) \left\{ \left( J(f^\sigma) \right)^{\left[\frac{n}{2}\right]} \right\}_{\sigma \in J_E} & \textit{if } k \textit{ is odd.} \end{cases}$$

*Here $J(f)$ is defined by*

$$J(f) = \pi^k \, \mathfrak{g}(\varepsilon) \, \langle f, f \rangle$$

*where*

$$\langle f, f \rangle = \mathrm{vol}\left(\Gamma_0(N) \backslash \mathfrak{h}\right)^{-1} \int_{\Gamma_0(N) \backslash \mathfrak{h}} |f(z)|^2 y^{k-2} \, dx \, dy$$

*with $\mathfrak{h} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.*

Because of this lemma, we may explicate Deligne's conjecture (2.3) and we may state the following conjecture on the algebraicity of special values of $L$-functions for $\mathrm{SO}(\mathbf{V}) \times \mathrm{GL}_2$.

**Conjecture 2** (Appendix to [8]). *Let $\pi(f)$ (resp. $\tau$) be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ (resp. $\mathrm{SO}(\mathbf{V}, \mathbb{A}_\mathbb{Q})$) as above. Let us define the field of rationality of $\tau$ by*

$$\mathbb{Q}(\tau) = \{a \in \mathbb{C} \mid \sigma(a) = a \textit{ for any } \sigma \in \mathrm{Aut}(\mathbb{C}) \textit{ such that } \tau_{\mathrm{fin}}^\sigma \simeq \tau_{\mathrm{fin}}\},$$

*and we write the composite field of* $\mathbb{Q}(\tau)$ *and* $\mathbb{Q}(f)$ *by* $\mathbb{Q}(f,\tau)$*. Suppose that we have*

$$k - 2m_1 \geq n.$$

*Then for* $m$ *such that*

$$m = \frac{k-2m_1-n+1}{2} - l \quad \text{where } l \in \mathbb{Z} \text{ and } 0 \leq l \leq \frac{k-2m_1-n}{2},$$

*we have*

$$\frac{L\left(m, \pi(f) \otimes \tau\right)}{\left(2\pi\sqrt{-1}\right)^{2m\cdot\left[\frac{n}{2}\right]} \mathfrak{g}\left(\chi_{\mathbf{V}}\right) J\left(f\right)^{\left[\frac{n}{2}\right]}} \in \mathbb{Q}(f,\tau)$$

*where the* $L$*-function* $L\left(s, \pi\left(f\right) \otimes \tau\right)$ *is normalized so that it has the functional equation with respect to* $s \mapsto 1-s$*.*

*Remark* 1. When $n=2$, we have

$$\mathrm{SO}(\mathbf{V}) \simeq E^1 = \left\{a \in E \mid N_{E/F}(a) = 1\right\}$$

for some imaginary quadratic extension $E$ of $\mathbb{Q}$. Then our $L$-function is a Rankin–Selberg $L$-function for $\mathrm{GL}_2 \times \mathrm{GL}_2$. Moreover, when $n=3$, we have

$$\mathrm{SO}(\mathbf{V}) \simeq D^\times / \mathbb{Q}^\times$$

for some definite quaternion division algebra over $\mathbb{Q}$. As in the previous case, by the Jacquet–Langlands correspondence, our $L$-function is regarded as a Rankin–Selberg $L$-function for $\mathrm{GL}_2 \times \mathrm{GL}_2$. Hence, for these cases, our conjecture is indeed a theorem of Shimura [20, Theorem 3].

*Remark* 2. Suppose $n=4$. Since we have

$$\mathrm{SO}\left(D\right) \simeq \left\{(d_1, d_2) \in D^\times \times D^\times \mid n_D\left(d_1\right) = n_D\left(d_2\right)\right\} / \left\{(a,a) \mid a \in \mathbb{Q}^\times\right\}$$

where $D$ is a definite quaternion division algebra over $\mathbb{Q}$ and $n_D$ denotes its reduced norm, our $L$-function may be regarded as a Rankin triple product $L$-function for $\mathrm{GL}_2$ because of the Jacquet-Langlands correspondence. Note that Deligne's conjecture for the Rankin triple $L$-function was explicated by Blasius in [3]. Then our conjecture corresponds to the *unbalanced* case in the following conjecture.

**Conjecture 3** (Blasius [3]). *For* $i = 1,2,3$*, let* $f_i \in S_{k_i}^{\mathrm{new}}\left(\Gamma_0\left(N_i\right), \varepsilon_i\right)$ *such that* $k_1 \geq k_2 \geq k_3$*. Let* $\mathbb{Q}\left(f_1, f_2, f_3\right)$ *be the number field generated by the Fourier coefficients of* $f_1$*,* $f_2$ *and* $f_3$ *over* $\mathbb{Q}$*. We denote by* $L\left(s, f_1 \otimes f_2 \otimes f_3\right)$ *the Rankin triple product* $L$*-function. Here we normalize the* $L$*-function so that it has a functional equation with respect to* $s \mapsto k_1 + k_2 + k_3 - 2 - s$*.*

1. *(Balanced case) Suppose that* $k_1 < k_2 + k_3$. *Let* $A = 3 - k_1 - k_2 - k_3$.

   *Then for* $n \in \mathbb{Z}$ *such that* $k_1 \le n \le k_2 + k_3 - 2$, *we have*

$$\frac{L\left(n, f_1 \otimes f_2 \otimes f_3\right)}{\pi^{4n+A}\, \mathfrak{g}(\varepsilon_1\varepsilon_2\varepsilon_3)^2 \langle f_1, f_1\rangle \langle f_2, f_2\rangle \langle f_3, f_3\rangle} \in \mathbb{Q}\left(f_1, f_2, f_3\right).$$

2. *(Unbalanced case) Suppose that* $k_1 \ge k_2 + k_3$. *Let* $B = 4 - 2k_2 - 2k_3$.

   *Then for* $n \in \mathbb{Z}$ *such that* $k_2 + k_3 - 1 \le n \le k_1 - 1$, *we have*

$$\frac{L\left(n, f_1 \otimes f_2 \otimes f_3\right)}{\pi^{4n+B}\, \mathfrak{g}(\varepsilon_1\varepsilon_2\varepsilon_3)^2 \langle f_1, f_1\rangle^2} \in \mathbb{Q}\left(f_1, f_2, f_3\right).$$

In the balanced case, using a remarkable integral representation by Garrett [9], algebraicity of the special values of Rankin triple product $L$-functions have been studied by Garrett [9], Orloff [16], Satoh [17], Harris–Kudla [14], Garrett–Harris [10] and Böcherer–Schulze-Pillot [4]. On the other hand, Harris–Kudla [14] showed Jacquet's conjecture, and they proved Blasius' conjecture at the central critical point not only in the balanced case but also in the unbalanced case under the assumption $\varepsilon_1\varepsilon_2\varepsilon_3 = 1$.

## § 3. Main Result

We now state our main result in [8].

**Theorem 3.1.** *Let* $\pi(f)$ *and* $\tau$ *be as in Conjecture 2. Suppose that*

$$k - 2m_1 > 2n. \tag{3.1}$$

*Then there exists a finite set* $S^\circ$ *of places of* $\mathbb{Q}$ *containing the infinite place such that*

$$P_S\left(m, f, \tau\right) := \frac{L_S\left(m, \pi(f) \otimes \tau\right)}{\left(2\pi\sqrt{-1}\right)^{2m\cdot\left[\frac{n}{2}\right]} \mathfrak{g}\left(\chi_{\mathbf{V}}\right) J\left(f\right)^{\left[\frac{n}{2}\right]}} \in \overline{\mathbb{Q}} \tag{3.2}$$

*and*

$$P_S\left(m, f, \tau\right)^\sigma = P_S\left(m, f^\sigma, \tau^\sigma\right) \quad \text{for any } \sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$$

*holds for any finite set* $S$ *of places of* $F$ *containing* $S^\circ$ *and for* $m$ *such that*

$$m = \frac{k - 2m_1 - n + 1}{2} - l \quad \text{where } l \in \mathbb{Z} \text{ and } 0 \le l \le \frac{k - 2m_1 - 2n - 1}{2}. \tag{3.3}$$

*In particular,*

$$P_S\left(m, f, \tau\right) \in \mathbb{Q}(f, \tau).$$

*Here* $L_S\left(s, \pi(f) \otimes \tau\right)$ *denotes the partial* $L$*-function defined by*

$$L_S\left(s, \pi(f) \otimes \tau\right) = \prod_{v \notin S} L\left(s, \pi(f)_v \otimes \tau_v\right).$$

*Remark* 3. As we shall discuss below, our proof relies on the algebraicity of Fourier coefficients of Eisenstein series on Shimura varieties in the domain of absolute convergence proved by Harris [13, Theorem 8.5]. Thus the upper bound for $l$ in (3.3) is forced upon us and the condition (3.1) becomes necessary so that the set of critical points $m$ satisfying (3.3) to be non-empty.

*Remark* 4. In our paper [8], Conjecture 2 and Theorem 3.1 are, indeed, stated and proved, respectively, over arbitrary totally real number fields. The setting for this note is over a rational number field $\mathbb{Q}$ just to simplify the exposition.

*Remark* 5. In our previous paper [7], we showed only the algebraicity (3.2) in the simplest case, namely when $\tau_\infty$ is trivial and $m$ is the rightmost critical point. Thus Theorem 3.1 is a generalization of [7, Theorem 1].

*Remark* 6. If we have the local Langlands correspondence for $\mathrm{SO}\,(\mathbf{V}, \mathbb{Q}_p)$ at each prime $p$, then by an argument similar to the one given in [7, Section 6], the algberaicity for complete $L$-functions holds (see the proof of Corollary 3.3 when $\dim \mathbf{V} = 4$).

Let us give an outline of our proof of Theorem 3.1. For simplicity, we merely prove algebraicity of special values of our $L$-function.

*Outline of the proof of Theorem 3.1.* In order to study our $L$-function, we use an integral representation by Ginzburg–Piatetski-Shapiro–Rallis [11], which is given as follows.

We choose a co-dimension one subspace $\mathbf{W}$ of $\mathbf{V}$, and let $\mathbb{V} = \mathbf{W} \oplus \mathbb{H} \oplus \mathbb{H}$ where $\mathbb{H}$ is the hyperbolic plane. We note that the symmetric space associated to $\mathrm{SO}(\mathbb{V})$ gives a type IV tube domain.

We know that $\mathrm{SO}(\mathbb{V})$ has a maximal parabolic subgroup $P$ whose Levi component is $\mathrm{GL}_2 \times \mathrm{SO}(W)$. Moreover, $\mathrm{SO}(\mathbb{V})$ has another maximal parabolic subgroup $Q$ whose unipotent radical $U_Q$ is abelian.

Let us take an irreducible automorphic representation $(\rho, V_\rho)$ of $\mathrm{SO}(\mathbf{W}, \mathbb{A}_\mathbb{Q})$ such that a $\mathrm{SO}(\mathbf{W}, \mathbb{A}_\mathbb{Q})$-invariant bilinear form on $V_\tau \times V_\rho$ given by

$$V_\tau \times V_\rho \ni (\phi_\tau, \phi_\rho) \mapsto \int_{\mathrm{SO}(\mathbf{W},\mathbb{Q})\backslash \mathrm{SO}(\mathbf{W},\mathbb{A}_\mathbb{Q})} \phi_\tau(g)\phi_\rho(g)\, dg$$

is not identically zero. For $f_s \in \mathrm{Ind}_{P(\mathbb{A}_\mathbb{Q})}^{\mathrm{SO}(\mathbb{V},\mathbb{A}_\mathbb{Q})}(\pi \otimes \rho \otimes \delta_P^s)$, we define an Eisenstein series by

$$E(g,s) := \sum_{\gamma \in P(\mathbb{Q})\backslash \mathrm{SO}(\mathbb{V},\mathbb{Q})} f_s(\gamma g).$$

Let us take a character $\theta$ of $U_Q(\mathbb{A}_\mathbb{Q})/U_Q(\mathbb{Q})$ such that the identity component of the stabilizer of $\theta$ in the Levi component $M_Q(\mathbb{A}_\mathbb{Q})$ of $Q(\mathbb{A}_\mathbb{Q})$ is $\mathrm{SO}(\mathbf{V}, \mathbb{A}_\mathbb{Q})$. Then the global zeta integral is defined by

$$Z(s) = \int_{(\mathrm{SO}(\mathbf{V})\times U_Q)(\mathbb{Q})\backslash(\mathrm{SO}(\mathbf{V})\times U_Q)(\mathbb{A}_\mathbb{Q})} E(h,s)\Phi(g)\theta(u)\,dh\,du$$

for $\Phi \in V_\tau$ and $f_s \in \mathrm{Ind}_{P(\mathbb{A}_\mathbb{Q})}^{\mathrm{SO}(\mathbb{V},\mathbb{A}_\mathbb{Q})}(\pi \otimes \rho \otimes \delta_P^s)$. When $\Phi$ and $f_s$ are decomposable, this global integral becomes an infinite product of local integrals

$$Z(s) = \prod_v Z_v(s),$$

and Ginzburg, Piatetski-Shapiro and Rallis computed local zeta integrals explicitly when all the data involved are unramified.

**Theorem 3.2** (Ginzburg–Piatetski-Shapiro–Rallis [11]). *Let $v$ be a finite place of $\mathbb{Q}$ satisfying the following conditions.*

1. $\mathrm{SO}(\mathbf{V},\mathbb{Q}_v)$ *and* $\mathrm{SO}(\mathbf{W},\mathbb{Q}_v)$ *are quasi-split.*
2. $\pi_v$, $\rho_v$, $\tau_v$ *and* $\theta_v$ *are unramified,*
3. $v$ *does not lie over 2,*

*Suppose that local components of $\Phi$ and $f_s$ at the place $v$ are unramified and suitably normalized. For* $\mathrm{Re}(s) >> 0$, *we have*

$$Z_v(s) = \frac{L\left(ns+\frac{1}{2},\pi_v\otimes\tau_v\right)}{L\left(ns+1,\pi_v\otimes\rho_v\right)L\left(2ns,\pi_v,r\right)}$$

*where* $n = \dim\ \mathbf{V}$ *and*

$$r = \begin{cases} \wedge^2 & \text{when } n \text{ is odd,} \\ \mathrm{Sym}^2 & \text{when } n \text{ is even.} \end{cases}$$

For an integer or a half of integer $m$ given in (3.3), we let $s_m = \frac{1}{n}\left(m - \frac{1}{2}\right)$. Then we may choose $\Phi$ and $f_s$ satisfying following three conditions.

1. At $s = s_m$, the local zeta integral $Z_v(s)$ converges and becomes a non-zero algebraic number.
2. At $s = s_m$, the Eisenstein series absolutely converges and becomes holomorphic Eisenstein series on the type IV tube domain. Further, it has algebraic Fourier coefficients by Harris [13, Theorem 8.5].
3. The global zeta integral $Z(s_m)$ becomes algebraic number (since $Z(s_m)$ can be written as a finite sum of Fourier coefficients of the holomorphic Eisenstein series).

With the above choice, we may compute $Z_\infty(s_m)$ explicitly, and we get the following identity

$$Z(s_m) = \frac{P_S(m, f, \tau)}{P_S\left(m + \frac{1}{2}, f, \rho\right) Q_S(2m, \pi, r)} \cdot \prod_{v \in S \setminus \{\infty\}} Z_v(s_m)$$

for some finite set of places $S$ of $\mathbb{Q}$ containing the archimedean place $\infty$. Here, we put

$$Q_S(2m, \pi, r) = \begin{cases} \dfrac{L_S\left(2m, \pi, \wedge^2\right)}{\left(2\pi\sqrt{-1}\right)^{2m} \mathfrak{g}(\varepsilon)} & \text{when } n \text{ is odd,} \\[2ex] \dfrac{L_S\left(2m, \pi, \mathrm{Sym}^2\right)}{\left(2\pi\sqrt{-1}\right)^{4m} \mathfrak{g}(\varepsilon) J(f)} & \text{when } n \text{ is even.} \end{cases}$$

We know that $Q_S(2m, \pi, r) \in \overline{\mathbb{Q}}^\times$ by Klingen [15] when $n$ is odd and by Sturm [21, Theorem 1] when $n$ is even. Thus, our choice of $f_s$ and $\Phi$ implies that

$$\frac{P_S(m, f, \tau)}{P_S\left(m + \frac{1}{2}, f, \rho\right)} \in \overline{\mathbb{Q}}. \tag{3.4}$$

As we remarked, our algebraicity result in the case of $n = 2, 3$ is a theorem by Shimura [20, Theorem 3]. Hence, by the induction on $n$, our required algebraicity result follows from (3.4). □

As we remarked in Remark 2, when $n = 4$, Theorem 3.1 gives an algebraicity result for Rankin triple $L$-functions for $\mathrm{GL}_2$. Since the local Langlands conjecture holds for $\mathrm{GL}_2$, our assertion is for the complete $L$-function.

**Corollary 3.3.** *For $i = 1, 2, 3$, let $f_i \in S_{k_i}^{\mathrm{new}}\left(\Gamma_0(N_i), \varepsilon_i\right)$ be a Hecke eigenform. Let $\pi_i$ denote the irreducible unitary cuspidal representation of $\mathrm{GL}_2\left(\mathbb{A}_\mathbb{Q}\right)$ associated to $f_i$. We suppose that:*

1. *$k_2 + k_3$ is even,*
2. *$k_1 > k_2 + k_3 + 4$,*
3. *$\varepsilon_2 \varepsilon_3 = 1$,*
4. *there exists a definite quaternion division algebra $D$ over $\mathbb{Q}$ such that both $\pi_2$ and $\pi_3$ have the Jacquet-Langlands transfer to $D^\times\left(\mathbb{A}_\mathbb{Q}\right)$.*

*Then for an integer $m$ satisfying*

$$\frac{k_1 + k_2 + k_3 + 2}{2} < m \leq k_1 - 1,$$

*we have*

$$P(m, f_1, f_2, f_3) := \frac{L(m, f_1 \otimes f_2 \otimes f_3)}{\pi^{2(2m-k_1-k_2-k_3+3)} J(f_1)^2} \in \overline{\mathbb{Q}}$$

*and*

$$P(m, f_1, f_2, f_3)^\sigma = P(m, f_1^\sigma, f_2^\sigma, f_3^\sigma) \quad \textit{for any } \sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right).$$

*In particular,*

$$P(m, f_1, f_2, f_3) \in \mathbb{Q}(f_1, f_2, f_3).$$

*Here we normalize $L(s, f_1 \otimes f_2 \otimes f_3)$ so that it has a functional equation with respect to $s \mapsto k_1 + k_2 + k_3 - 2 - s$, and $\mathbb{Q}(f_1, f_2, f_3)$ is the number field generated by the Fourier coefficients of $f_1$, $f_2$ and $f_3$ over $\mathbb{Q}$.*

Before proceeding with a proof of this corollary, we give a remark on above assumptions. The first assumption follows from the third one, but this assumption is important for clarifying our situation, and thus we put it separately.

*Proof of Corollary 3.3.* From the fourth assumption, there is a Jacquet-Langlands transfer $\pi_i^D$ of $\pi_i$ (i=2, 3) to $D^\times(\mathbb{A}_\mathbb{Q})$. Then the third assumption implies that $\pi_2^D \otimes \pi_3^D$ gives an automorphic representation of $\mathrm{SO}(D, \mathbb{A}_\mathbb{Q})$, which is possibly reducible. Let $\tau_D$ be one of irreducible constituents of this automorphic representation of $\mathrm{SO}(D, \mathbb{A}_\mathbb{Q})$. Then we know that

$$L(s, \pi_{1,v} \otimes \tau_{D,v}) = L(s, \pi_{1,v} \otimes \pi_{2,v} \otimes \pi_{3,v})$$

for every place $v$ of $\mathbb{Q}$. Further, we note that the archimedean component of $\tau_D$ is the representation of $\mathrm{SO}(D, \mathbb{R})$ with the highest weight $\left(\frac{k_2+k_3}{2} - 2, \frac{|k_2-k_3|}{2}\right)$. Thus, because of the second assumption, we may apply Theorem 3.1 to $\pi_1$ and $\tau_D$. Indeed, taking the normalization of $L$-functions into account, we have for some finite set of places $S$ of $\mathbb{Q}$,

$$\frac{L_S(m, f_1 \otimes f_2 \otimes f_3)}{\pi^{2(2m-k_1-k_2-k_3+3)} J(f_1)^2} \in \mathbb{Q}(f_1, f_2, f_3)$$

with the integer $m$ such that $\frac{k_1+k_2+k_3+2}{2} < m \leq k_1 - 1$. Furthermore, for every finite place $v \in S$, we have

$$L_v(m, f_1 \otimes f_2 \otimes f_2) \in \overline{\mathbb{Q}}$$

and

$$L_v(m, f_1 \otimes f_2 \otimes f_2)^\sigma = L_v(m, f_1^\sigma \otimes f_2^\sigma \otimes f_2^\sigma)$$

for every $\sigma \in \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ (see [10, (4.9.14.1)] and a proof of [7, Therem 2]). Here, this local $L$-factor is the local $L$-factor attached to the local Langlands parameter of $\pi_{i,v}$. Then augmenting $L_v(m, f_1 \otimes f_2 \otimes f_2)$ to $L_S(m, f_1 \otimes f_2 \otimes f_3)$ formally, we obtain our required result. □

## References


[1] Arthur, J., *The endoscopic classification of representations*, Colloquium Publications **61**, American Mathematical Society, 2013.

[2] Bhagwat, C. and Raghuram, A., Ratios of periods for tensor product motives, *Mathematics Reseach Letters.*, to appear.

[3] Blasius, D., *Critical values of certain tensor product L-functions*, Appendix to T. Orloff, *Special values and mixed weight triple products*, Invent. Math. **90** (1987), 181–188.

[4] Böcherer, S. and Schulze-Pillot, R., On the central critical value of the triple product $L$-function. *London Math. Soc. Lecture Note Ser.* **235**, Number theory (Paris, 1993–1994), 1–46. Cambridge Univ. Press, Cambridge (1996).

[5] Clozel, L., Motifs et formes automorphes: applications du principe de fonctorialité, Automorphic forms, Shimura varieties, and $L$-functions, Vol. I (Ann Arbor, MI, 1988), *Perspect. Math.*, **10**, Academic Press, Boston, MA, 1990, pp. 77–159.

[6] Deligne, P., Valeurs de fonctions $L$ et périodes d'intégrales, With an appendix by N. Koblitz and A. Ogus, *Proc. Sympos. Pure Math.*, XXXIII, Automorphic forms, representations and $L$-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, 313–346. Amer. Math. Soc., Providence, R. I. (1979).

[7] Furusawa, M. and Morimoto, K., On special values of certain $L$-functions, *Amer. J. Math.*, **136** (2014), 1385–1407.

[8] Furusawa, M. and Morimoto, K., On special values of certain $L$-functions II, Submitted.

[9] Garrett, P., Decomposition of Eisenstein series: Rankin triple products. *Ann. of Math.* (2) **125**, 209–235 (1987).

[10] Garrett, P. and Harris, M., Special values of triple product $L$-functions. *Amer. J. Math.* **115**, 161–240 (1993).

[11] Ginzburg, D., Piatetski-Shapiro, I. and Rallis, S., *L functions for the orthogonal group*, Mem. Amer. Math. Soc. **128**, no. 611, viii+218 pp (1997).

[12] Goodman, R. and Wallach, N. R., *Symmetry, representations, and invariants*, Graduate Texts in Mathematics **255**, xx+716 pp. Springer, Dordrecht (2009).

[13] Harris, M., Arithmetic vector bundles and automorphic forms on Shimura varieties. II, *Compositio Math.*, **60** (1986), 323–378.

[14] Harris, M. and Kudla, S., The central critical value of a triple product $L$-function. *Ann. of Math.* (2) **133**, 605–672 (1991).

[15] Klingen, H., Über den arithmetischen Charakter der Fourierkoeffizienten von Modulformen *Math. Ann.* **147**, 176–188 (1962).

[16] Orloff, T., Special values and mixed weight triple products (with an appendix by Don Blasius). *Invent. Math.* **90**, 169–188 (1987).

[17] Satoh, T., Some remarks on triple $L$-functions. *Math. Ann.* **276**, 687–698 (1987).

[18] Scholl, A. J., Motives for modular forms, *Invent. Math.*, **100** (1990), 419–430.

[19] Serre, J. P., *Facteurs locaux des fonctions zêta des variétés algébriques (définitions et conjectures)*, Séminaire Delange-Pisot-Poitou. Théorie des nombres, 11 no. 2 (1969–1970), Exp. No. 19, 15 p.

[20] Shimura, G., The special values of the zeta functions associated with cusp forms. *Comm. Pure Appl. Math.* **29**, 783–804 (1976).

[21] Sturm, J., Special values of zeta functions, and Eisenstein series of half integral weight. *Amer. J. Math.* **102**, 219–240 (1980).

[22] Waldspurger, J. L., Quelques propriétés arithmétiques de certaines formes automorphes sur GL(2), *Compositio Math.*, **54** (1985), 121–171.

[23] Yoshida, H., On the zeta functions of Shimura varieties and periods of Hilbert modular forms, *Duke Math. J.*, **75** (1994), 121–191.
[24] Yoshida, H., Motives and Siegel modular forms, *Amer. J. Math.*, **123** (2001), 1171–1197.