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On Galois equivariance of homomorphisms between torsion potentially crystalline representations : A resume

By

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Abstract

In this article, we announce some results on Galois equivariance properties for torsion crystalline representations. These give torsion analogues of Kisin's theorem on the full faithfulness of a certain restriction functor on crystalline p-adic representations.

§ 1. Introduction

The main purpose of this article is to give a survey on several results concerning certain properties of homomorphisms of torsion crystalline representations, proved in [Oz2].

Let K be a complete discrete valuation field of mixed characteristics $(0,p)$ with perfect residue field. To understand various algebraic objects over K (e.g., abelian varieties (elliptic curves), p-divisible groups, modular forms), it is often useful to study certain associated l-adic and p-adic representations (for a prime $l \neq p$). In this article, we focus on p-adic representations. For this study, Fontaine's p-adic Hodge theory turns out to be useful. Fontaine defined some important classes of $\mathbb{Q}_p$-representations such as Hodge-Tate, de Rham, semi-stable and crystalline representations. In this article, we mainly focus on crystalline representations. Crystalline $\mathbb{Q}_p$-representations arise as

- subquotients of p-adic étale cohomology groups of algebraic varieties over K with good reduction,

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- Tate modules of p-divisible groups over the integer ring of K and
- p-adic representations associated with modular forms of prime to p level, by regarding them as representations of the absolute Galois group of $\mathbb{Q}_p$.

A natural approach for studying representations of G_K is to study their restriction to a certain subgroup "G_∞" of G_K. We are interested in "G_∞" defined as follows. Let π be a uniformizer of K and $(\pi_n)_{n\geq 0}$ a system of p-power roots of π such that $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$. Put $K_s = K(\pi_s)$ and $K_\infty = \cup_{n\geq 0} K_n$. Denote by G_s and G_∞ the absolute Galois groups of K_s and K_∞, respectively. By definition we have the following decreasing sequence of Galois groups:

$$G_K = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_\infty.$$

By the theory of fields of norms, G_∞ is isomorphic to the absolute Galois group of a field of characteristic p, and therefore representations of G_∞ have an "easy" interpretation via Fontaine's étale φ-modules. Thus it seems natural to pose the following question.

Question 1.1 (rough version)**.** Let T be a $\mathbb{Z}_p$- or $\mathbb{Q}_p$-representation of G_K. How small can we choose $s \geq 0$ to recover "enough" information about $T|_{G_s}$ from $T|_{G_\infty}$?

For crystalline $\mathbb{Q}_p$-representations of G_K, the following answer to Question 1.1 is proved by Kisin, which was a conjecture of Breuil.

Theorem 1.2 ([Kis1, Theorem (0.2)])**.** *The restriction functor from the category of crystalline $\mathbb{Q}_p$-representations of G_K into the category of $\mathbb{Q}_p$-representations of G_∞ is fully faithful.*

The (spirit of the) above question is also interesting for torsion representations. For example, Caruso-Liu ([CL2]) obtained ramification bounds for torsion semi-stable representations T of G_K from calculations of certain linear data corresponding to $T|_{G_\infty}$. Furthermore, Kisin constructed a moduli space of "finite flat models" of representations arising from finite flat group schemes by studying associated representations restricted to G_∞ ([Kis2]).

Main results in [Oz2] are (generalizations of) torsion analogues of Theorem 1.2. From now on, we fix an integer $s \geq 0$. We mainly consider the three categories $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{ht},\mathrm{pcris}(s)}(G_K)$, $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ and $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{st}}(G_K)$ defined as follows:

- $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{ht},\mathrm{pcris}(s)}(G_K)$ is the category of torsion $\mathbb{Z}_p$-representations T of G_K, which satisfy the following: there exist free $\mathbb{Z}_p$-representations L and L' of G_K, of height $\leq r$ (cf. Definition 2.2), such that
  - $L|_{G_s}$ is a subrepresentation of $L'|_{G_s}$. Furthermore, $L|_{G_s}$ and $L'|_{G_s}$ are lattices in some crystalline $\mathbb{Q}_p$-representation of G_s with Hodge-Tate weights in $[0, r]$;

– $T|_{G_s} \simeq (L'|_{G_s})/(L|_{G_s})$.

- $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ (resp. $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{st}}(G_K)$) is the category of torsion crystalline (resp. semi-stable) representations of G_K with Hodge-Tate weights in $[0,r]$.

Here, a torsion $\mathbb{Z}_p$-representation of G_K is *torsion crystalline* (resp. *torsion semi-stable*) *with Hodge-Tate weights in* $[0,r]$ if it can be written as the quotient of lattices in some crystalline (resp. semi-stable) $\mathbb{Q}_p$-representation of G_K with Hodge-Tate weights in $[0,r]$. It is known that a torsion $\mathbb{Z}_p$-representation T of G_K is an object of $\mathrm{Rep}_{\mathrm{tor}}^{1,\mathrm{cris}}(G_K)$ if and only if T is isomorphic to $G(\overline{K})$ for some finite flat group scheme G of p-power order over the integer ring of K. Here $\overline{K}$ is an algebraic closure of K.

We denote by e the absolute ramification index of K. Now we describe the main results of [Oz2].

Theorem 1.3. *Suppose that p is odd. Let T and T' be torsion $\mathbb{Z}_p$-representations of G_K. Then any G_∞-equivariant homomorphism $T \to T'$ is G_s-equivariant if one of the following conditions is fulfilled:*
(1) ([Oz2, Theorem 1.2]) *$e(r-1) < p-1$, T and T' are objects of* $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{ht},\mathrm{pcris}(s)}(G_K)$.
(2) ([Oz2, Theorem 1.4]) *$s > n-1+\log_p(r-(p-1)/e)$, T and T' are objects of* $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$ *which are killed by p^n.*
(3) ([Oz2, Theorem 4.16]) *$s > n-1+\log_p r$, T and T' are objects of* $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{st}}(G_K)$ *which are killed by p^n.*

We note that $\mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{ht},\mathrm{pcris}(0)}(G_K) = \mathrm{Rep}_{\mathrm{tor}}^{r,\mathrm{cris}}(G_K)$. Therefore, we obtain the following torsion analogue of Theorem 1.2 as an immediate consequence of Theorem 1.3 (1).

Corollary 1.4 ([Oz2, Corollary 1.3])**.** *Suppose that p is odd and $e(r-1) < p-1$. Then the restriction functor from the category of torsion crystalline $\mathbb{Z}_p$-representations of G_K with Hodge-Tate weights in $[0,r]$ into the category of torsion $\mathbb{Z}_p$-representations of G_∞ is fully faithful.*

The case $p = 2$ of the corollary has been shown independently in [Kim], [La] and [Li3]. For $p > 2$ the corollary was already known in the following cases:

- $e = 1$ and $r < p-1$ ([Br1], the proof of Théorèm 5.2).
- $r \leq 1$ ([Br2, Theorem 3.4.3]).
- $r < p$ and K is a finite unramified extension of $\mathbb{Q}_p$ ([Ab, Section 8.3.3]).

Furthermore, we can check that the bound "$e(r-1) < p-1$" is optimal for the above full faithfulness property for many finite extensions K over $\mathbb{Q}_p$. In fact, here is a counter example (in the case where $e=1$ and $r=p$): Suppose $K=\mathbb{Q}_p$ and let E be the Tate curve associated with π. We can show that $E[p]$ and $\mathbb{F}_p\oplus\mathbb{F}_p(1)$ are torsion crystalline with Hodge-Tate weights in $[0,p]$, and they are isomorphic as representations of G_∞. However, they are not isomorphic as representations of G_K. For more examples, we refer to Section 5 of [Oz2].

Convention: In the rest of this article, *we always assume that p is an odd prime number.*

For any topological group H, a *$\mathbb{Q}_p$-representation* (resp. *free $\mathbb{Z}_p$-representation*, resp. *torsion $\mathbb{Z}_p$-representation*) of H is a finite dimensional $\mathbb{Q}_p$-vector space (resp. free $\mathbb{Z}_p$-module of finite type, resp. torsion $\mathbb{Z}_p$-module of finite type) equipped with a continuous H-action. We denote by $\mathrm{Rep}_{\mathrm{tor}}(H)$ the category of torsion $\mathbb{Z}_p$-representations of H. All actions of Galois groups appearing in this paper are assumed to be continuous (with respect to the appropriate topologies).

We should remark that some notations in this article have been simplified and differ from [Oz2]. Another reason for this simplification is that the "base change argument" (as in Lemma 4.6 and 4.7 of *loc. cit.*) is not explained in this article.

§ 2. $(\varphi,\hat{G})$-modules

In this section, we recall the definition of $(\varphi,\hat{G})$-modules (cf. [Li1]) and describe some important properties.

§ 2.1. Notation

We continue to use the same notation $K, s, K_s, G_K, \ldots$ as in the Introduction.

We denote by $\mathcal{O}_{\overline{K}}$ the integer ring of $\overline{K}$. We define $R=\varprojlim \mathcal{O}_{\overline{K}}/p$, where the transition maps are given by the p-th power map. This is a complete valuation ring, of characteristic p, with residue field $\bar{k}$. Here $\bar{k}$ is an algebraic closure of k. The ring R is equipped with a natural G_K-action. We define $\underline{\pi_s}:=(\pi_{s+n} \bmod p)_{n\geq 0}\in R$.

Let $\mathfrak{S}_s=W(k)[\![u_s]\!]$ be the formal power series ring with indeterminate u_s. We define a Frobenius endomorphism φ of $\mathfrak{S}_s$ by $u_s\mapsto u_s^p$, extending the Frobenius of $W(k)$. The $W(k)$-algebra embedding $W(k)[u_s]\hookrightarrow W(R)$ defined by $u_s\mapsto[\underline{\pi_s}]$ extends to $\mathfrak{S}_s\hookrightarrow W(R)$. Here, for an element $*$ of R, $[*]$ stands for its Teichmüller representative. Note that G_∞ acts trivially on $\mathfrak{S}_s$ as a subring of $W(R)$.

We denote by $E_s(u_s)$ the minimal polynomial of π_s over $W(k)[1/p]$. We note that $E_s(u_s)$ does not depend on the choice of $s\geq 0$ as an element of $W(R)$. Let S_s be the p-adic completion of $W(k)[u_s,\frac{E_s(u_s)^i}{i!}]_{i\geq 0}$ and endow S_s with the following structures:

- a continuous $\varphi_{W(k)}$-semilinear Frobenius $\varphi\colon S_s\to S_s$ defined by $u_s\mapsto u_s^p$,

- a continuous $W(k)$-linear derivation $N\colon S_s \to S_s$ defined by $N(u_s) = -u_s$ and
- a decreasing filtration $(\mathrm{Fil}^i S_s)_{i\geq 0}$ on S_s. Here $\mathrm{Fil}^i S_s$ is the p-adic closure of the ideal generated by the divided powers $\gamma_j(E_s(u_s)) = \frac{E_s(u_s)^j}{j!}$ for all $j \geq i$.

Then φ, N and the filtration on S_s extends to $S_s[1/p]$. The inclusion $W(k)[u_s] \hookrightarrow W(R)$ defined above induces φ-compatible inclusions $\mathfrak{S}_s \hookrightarrow S_s \hookrightarrow A_{\mathrm{cris}}$ and $S_s[1/p] \hookrightarrow B^+_{\mathrm{cris}}$. Note that G_∞ acts trivially on S_s and $S_s[1/p]$ as subrings of B^+_{cris}. We also note that all extensions in Figure 1 are compatible with φ.

Remark 1. We often omit the subscript s in the case where $s = 0$ (e.g., $u_0 = u,\ \mathfrak{S}_0 = \mathfrak{S},\ S_0 = S$).

$$\begin{tikzcd}
S[1/p] = S_0[1/p] \arrow[r, dash] \arrow[d, dash] & S_s[1/p] \arrow[r, dash] \arrow[d, dash] & S_{s+1}[1/p] \arrow[r, dash] \arrow[d, dash] & B^+_{\mathrm{cris}} \arrow[d, dash] \\
S = S_0 \arrow[r, dash] \arrow[d, dash] & S_s \arrow[r, dash] \arrow[d, dash] & S_{s+1} \arrow[r, dash] \arrow[d, dash] & A_{\mathrm{cris}} \arrow[d, dash] \\
\mathfrak{S} = \mathfrak{S}_0 \arrow[r, dash] & \mathfrak{S}_s \arrow[r, dash] & \mathfrak{S}_{s+1} \arrow[r, dash] & W(R)
\end{tikzcd}$$

Figure 1. Ring extensions

§ 2.2. Kisin modules

We recall the notion of Kisin modules. A *φ-module over* $\mathfrak{S}_s$ is an $\mathfrak{S}_s$-module $\mathfrak{M}$ equipped with a φ-semilinear map $\varphi\colon \mathfrak{M} \to \mathfrak{M}$.

Definition 2.1. (1) We denote by $\mathrm{Mod}^r_{/\mathfrak{S}_s}$ the category of φ-modules $\mathfrak{M}$ over $\mathfrak{S}_s$ which satisfy the following:

- $\mathfrak{M}$ is free of finite type over $\mathfrak{S}_s$ and
- $\mathfrak{M}$ is of height $\leq r$ in the sense that $\mathrm{coker}(1 \otimes \varphi\colon \mathfrak{S}_s \otimes_{\varphi, \mathfrak{S}_s} \mathfrak{M} \to \mathfrak{M})$ is killed by $E_s(u_s)^r$.

We call objects in this category *free Kisin modules of height* $\leq r$.
(2) We denote by $\mathrm{Mod}^r_{/\mathfrak{S}_{s,\infty}}$ the category of φ-modules $\mathfrak{M}$ over $\mathfrak{S}_s$ which satisfy the following:

- $\mathfrak{M}$ is of finite type over $\mathfrak{S}_s$, killed by a power of p and u_s-torsion free and

- $\mathfrak{M}$ is of height $\leq r$.

We call objects in this category *torsion Kisin modules of height* $\leq r$.

We should remark (cf. [Kis1], [Kim], [La] and [Li3]) that the category $\mathrm{Mod}^1_{/\mathfrak{S}_s}$ (resp. $\mathrm{Mod}^1_{/\mathfrak{S}_{s,\infty}}$) is equivalent to the category of p-divisible groups over the integer ring of K_s (resp. the category of finite flat group schemes over the integer ring of K_s of p-power order).

For a free or torsion Kisin module $\mathfrak{M}$, we define a $\mathbb{Z}_p[G_\infty]$-module $T_{\mathfrak{S}_s}(\mathfrak{M})$ by

$$T_{\mathfrak{S}_s}(\mathfrak{M}) = \begin{cases} \mathrm{Hom}_{\mathfrak{S}_s,\varphi}(\mathfrak{M}, W(R)) & \text{if } \mathfrak{M} \text{ is free,} \\ \mathrm{Hom}_{\mathfrak{S}_s,\varphi}(\mathfrak{M}, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} W(R)) & \text{if } \mathfrak{M} \text{ is torsion.} \end{cases}$$

Here the G_∞-action on $T_{\mathfrak{S}_s}(\mathfrak{M})$ is given by $(\sigma.g)(x) = \sigma(g(x))$ for $\sigma \in G_\infty, g \in T_{\mathfrak{S}_s}(\mathfrak{M}), x \in \mathfrak{M}$.

Definition 2.2. A free $\mathbb{Z}_p$-representation T of G_K is *of height* $\leq r$ if $T|_{G_\infty}$ is isomorphic to $T_{\mathfrak{S}}(\mathfrak{M})(= T_{\mathfrak{S}_0}(\mathfrak{M}))$ for some $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}}(= \mathrm{Mod}^r_{/\mathfrak{S}_0})$. By definition, this notion depends on K, r and the system $(\pi_n)_{n\geq 0}$.

§ 2.3. $(\varphi, \hat{G})$-modules

Roughly speaking, $(\varphi, \hat{G})$-modules are Kisin modules equipped with a certain Galois action. To describe their definition precisely, we need some additional notation.

Let K_{s,p^∞} be the field obtained by adjoining to K_s all p-power roots of unity. We denote by $\hat{K}$ the composite field of K_{s,p^∞} and K_∞ (which is clearly independent of s). We define various Galois groups as in Figure 2 below.

Since p is odd, $G_{p^\infty}(= G_{0,p^\infty})$ is topologically isomorphic to $\mathbb{Z}_p$. We fix a topological generator τ of G_{p^∞}. Then $\tau_s := \tau^{p^s}$ generates G_{s,p^∞} topologically. Note also that we have an equality $\hat{G}_s = G_{s,p^\infty} \rtimes H_K$ with the relation $g\tau_s = \tau_s^{\chi(g)} g$ for $g \in H_K$. Here, χ is the p-adic cyclotomic character.

We define $\underline{\epsilon} := \tau(\underline{\pi})/\underline{\pi} = \tau_s(\underline{\pi_s})/\underline{\pi_s} \in R$ and put $t = -\log([\underline{\epsilon}]) \in A_{\mathrm{cris}}$. For any integer $n \geq 0$, let $t^{\{n\}} := t^{r(n)}\gamma_{\tilde{q}(n)}(\frac{t^{p-1}}{p})$ where $n = (p-1)\tilde{q}(n) + r(n)$ with $\tilde{q}(n) \geq 0,\ 0 \leq r(n) < p-1$ and $\gamma_i(x) = \frac{x^i}{i!}$ is the standard divided power. We define a subring $\mathcal{R}_s$ of B^+_{cris} as below:

$$\mathcal{R}_s := \left\{ \sum_{i=0}^{\infty} f_i t^{\{i\}} \mid f_i \in S_s[1/p] \text{ and } f_i \to 0 \text{ as } i \to \infty \right\}.$$

Furthermore, we define $\widehat{\mathcal{R}}_s := \mathcal{R}_s \cap W(R)$. By [Li1, Lemma 2.2.1] we know that $\mathcal{R}_s$ (resp. $\widehat{\mathcal{R}}_s$) is an $\mathfrak{S}_s$-subalgebra of B^+_{cris} (resp. $W(R)$) which is φ-stable and G_s-stable. By definition the G_s-action on it factors through $\hat{G}_s$.

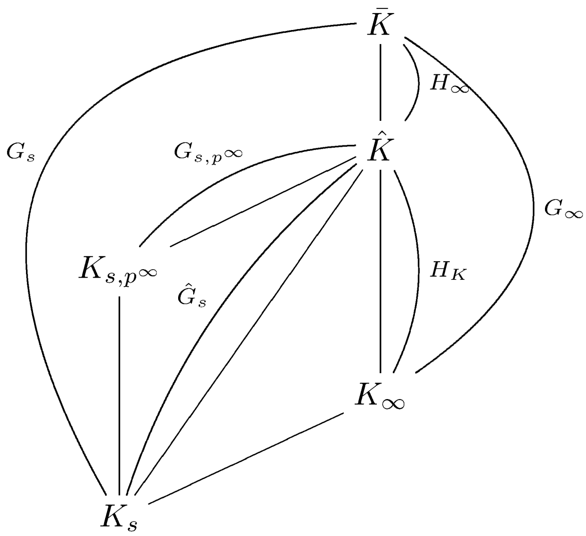

Figure 2. Galois groups of field extensions

Denote by $\nu\colon W(R) \to W(\bar{k})$ a unique lift of the projection $R \to \bar{k}$, which extends to $\nu\colon B^+_{\mathrm{cris}} \to W(\bar{k})[1/p]$. For any subring A in B^+_{cris}, we define $I_+A := \ker(\nu \text{ on } B^+_{\mathrm{cris}}) \cap A$. Since ν is G_K-equivariant, we see that $I_+\mathcal{R}_s$ and $I_+\widehat{\mathcal{R}}_s$ are $\hat{G}_s$-stable.

For any free or torsion Kisin module $\mathfrak{M}$ over $\mathfrak{S}_s$ (resp. $\mathfrak{S}$), the map $\mathfrak{M} \to \widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}_s} \mathfrak{M}$ (resp. $\mathfrak{M} \to \widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$), defined by $x \mapsto 1 \otimes x$, is injective. By this injection, we often regard $\mathfrak{M}$ as a $\varphi(\mathfrak{S}_s)$-stable submodule of $\widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}_s} \mathfrak{M}$ (resp. $\widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}} \mathfrak{M}$).

Definition 2.3. *A free (resp. torsion) $(\varphi, \hat{G}_s)$-module of height $\leq r$ over $\mathfrak{S}_s$* is a triple $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G}_s)$ where

(1) $(\mathfrak{M}, \varphi)$ is a free (resp. torsion) Kisin module of height $\leq r$ over $\mathfrak{S}_s$,

(2) $\hat{G}_s$ is an $\widehat{\mathcal{R}}_s$-semi-linear $\hat{G}_s$-action on $\widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}_s} \mathfrak{M}$,

(3) the $\hat{G}_s$-action commutes with $\varphi_{\widehat{\mathcal{R}}_s} \otimes \varphi_{\mathfrak{M}}$,

(4) $\mathfrak{M} \subset (\widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}_s} \mathfrak{M})^{H_K}$ and

(5) $\hat{G}_s$ acts on $\widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}_s} \mathfrak{M}/I_+\widehat{\mathcal{R}}_s(\widehat{\mathcal{R}}_s \otimes_{\varphi,\mathfrak{S}_s} \mathfrak{M})$ trivially.

We denote the category of free (resp. torsion) $(\varphi, \hat{G}_s)$-module of height $\leq r$ over $\mathfrak{S}_s$ by $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_s}$ (resp. $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_{s,\infty}}$).

By replacing $\mathfrak{S}_s$ with $\mathfrak{S}$, we define the notion of *free (resp. torsion) $(\varphi, \hat{G}_s)$-modules of height $\leq r$ over $\mathfrak{S}$*. Again, we denote by $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}}$ (resp. $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty}$) the corresponding category.

Remark 2. In [Oz2], the category $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}}$ (resp. $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty}$) is written as $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}}$ (resp. $\widetilde{\mathrm{Mod}}^{r,\hat{G}_s}_{/\mathfrak{S}_\infty}$). See Definition 3.1 of *loc. cit.*

For a free or torsion $(\varphi,\hat{G}_s)$-module $\hat{\mathfrak{M}}=(\mathfrak{M},\varphi,\hat{G}_s)$ over $\mathfrak{S}_s$, we define a $\mathbb{Z}_p[G_s]$-module $\hat{T}_s(\hat{\mathfrak{M}})$ by

$$\hat{T}_s(\hat{\mathfrak{M}})=\begin{cases}\mathrm{Hom}_{\widehat{\mathcal{R}}_s,\varphi}(\widehat{\mathcal{R}}_s\otimes_{\varphi,\mathfrak{S}_s}\mathfrak{M},W(R)) & \text{if } \mathfrak{M} \text{ is free,}\\ \mathrm{Hom}_{\widehat{\mathcal{R}}_s,\varphi}(\widehat{\mathcal{R}}_s\otimes_{\varphi,\mathfrak{S}_s}\mathfrak{M},\mathbb{Q}_p/\mathbb{Z}_p\otimes_{\mathbb{Z}_p}W(R)) & \text{if } \mathfrak{M} \text{ is torsion.}\end{cases}$$

Here the G_s-action on $\hat{T}_s(\hat{\mathfrak{M}})$ is given by $(\sigma.g)(x)=\sigma(g(\sigma^{-1}x))$ for $\sigma\in G_s, g\in\hat{T}_s(\hat{\mathfrak{M}}), x\in\widehat{\mathcal{R}}_s\otimes_{\varphi,\mathfrak{S}_s}\mathfrak{M}$. For a free or torsion $(\varphi,\hat{G}_s)$-module $\hat{\mathfrak{M}}$ over $\mathfrak{S}$, we also define a $\mathbb{Z}_p[G_s]$-module $\hat{T}_s(\hat{\mathfrak{M}})$ in a similar way. Let $\hat{\mathfrak{M}}=(\mathfrak{M},\varphi,\hat{G}_s)$ be a free or torsion $(\varphi,\hat{G}_s)$-module $\hat{\mathfrak{M}}$ over $\mathfrak{S}_s$ (resp. over $\mathfrak{S}$). By (the same proof as) Theorem 2.3.1 (1) of [Li1], we deduce that the map

$$\theta\colon T_{\mathfrak{S}_s}(\mathfrak{M})\to\hat{T}_s(\hat{\mathfrak{M}})\quad(\text{resp. }\theta\colon T_{\mathfrak{S}}(\mathfrak{M})\to\hat{T}_s(\hat{\mathfrak{M}})),$$

defined by $\theta(f)(a\otimes x):=a\varphi(f(x))$ for $a\in\widehat{\mathcal{R}}_s$ and $x\in\mathfrak{M}$, is an isomorphism of representations of G_∞. The following result is one of the most important properties for $(\varphi,\hat{G})$-modules.

Theorem 2.4 ([Li1, Theorem 2.3.1 (2)]). *The contravariant functor $\hat{T}_s$ induces an anti-equivalence between the category $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_s}$ and the category of G_s-stable $\mathbb{Z}_p$-lattices in a semi-stable $\mathbb{Q}_p$-representation of G_s with Hodge-Tate weights in $[0,r]$.*

A criterion for $\hat{T}_s(\hat{\mathfrak{M}})[1/p]$ to be crystalline is known. To describe it, for any integer $m\geq 0$, we define an ideal $I^{[m]}W(R)$ of $W(R)$ as the one consisting of $x\in W(R)$ such that $\varphi^n(x)\in\mathrm{Fil}^m B^+_{\mathrm{cris}}$ for all $n\geq 0$. Fontaine showed in [Fo, Proposition 5.1.3] that $I^{[m]}W(R)$ is a principal ideal which is generated by $([\underline{\epsilon}]-1)^m$. Now we consider the following condition $\langle\mathrm{cris}\rangle$ for a free $(\varphi,\hat{G}_s)$-module $\hat{\mathfrak{M}}=(\mathfrak{M},\varphi,\hat{G}_s)$ over $\mathfrak{S}_s$:

$$\langle\mathrm{cris}\rangle\quad \tau_s(x)-x\in u_s^pI^{[1]}W(R)\otimes_{\varphi,\mathfrak{S}_s}\mathfrak{M}\quad\text{for any } x\in\mathfrak{M}.$$

We define a full subcategory $\mathrm{Mod}^{r,\hat{G}_s,\mathrm{cris}}_{/\mathfrak{S}_s}$ of $\mathrm{Mod}^{r,\hat{G}_s}_{/\mathfrak{S}_s}$ consisting of objects $\hat{\mathfrak{M}}$ which satisfy $\langle\mathrm{cris}\rangle$. Then we have:

Theorem 2.5 ([GLS, Proposition 5.9] and [Oz1, Theorem 19]). *The contravariant functor $\hat{T}_s$ induces an anti-equivalence between the category $\mathrm{Mod}^{r,\hat{G}_s,\mathrm{cris}}_{/\mathfrak{S}_s}$ and the category of G_s-stable $\mathbb{Z}_p$-lattices in a crystalline $\mathbb{Q}_p$-representation of G_s with Hodge-Tate weights in $[0,r]$.*

For $(\varphi,\hat{G}_s)$-modules *defined over* $\mathfrak{S}$, we have

Theorem 2.6 ([Oz2, Theorem 3.4]). *For a free $(\varphi, \hat{G}_s)$-module $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G}_s)$ over $\mathfrak{S}$ of height $\leq r$, $\hat{T}_s(\hat{\mathfrak{M}})[1/p]$ is a crystalline $\mathbb{Q}_p$-representation of G_s if and only if the following condition $\langle \mathrm{cris}' \rangle$ holds:*

$$\langle \mathrm{cris}' \rangle \quad \tau_s(x) - x \in u^p I^{[1]} W(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \quad \text{for any } x \in \mathfrak{M}.$$

§ 3. Keys for the main result

In this section, we define certain classes of $(\varphi, \hat{G})$-modules which are deeply related with our main result and describe important properties of them.

Recall that e is the absolute ramification index of K. Let v_R be the valuation of R normalized by $v_R(\underline{\pi}) = 1/e$. For any real number $x \geq 0$, we define $\mathfrak{m}_R^{\geq x} := \{a \in R; v_R(a) \geq x\}$, which is a principal ideal of R and stable under φ and G_K in R. We consider an ideal J of $W(R)$ which satisfies the following conditions:

(1) $J \not\subset pW(R)$,

(2) J is a principal ideal,

(3) J is φ-stable and G_s-stable in $W(R)$.

By conditions (1) and (2), we know that the image of J under the projection map $W(R) \to R$ is of the form $\mathfrak{m}_R^{\geq c_J}$ for some real number $c_J \geq 0$. Motivated by Theorem 2.5, we define the following notation:

Definition 3.1. We define a full subcategory $\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r, \hat{G}_s, J}$ of $\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r, \hat{G}_s}$ consisting of objects $\hat{\mathfrak{M}} = (\mathfrak{M}, \varphi, \hat{G}_s)$ which satisfy the following:

$$\tau_s(x) - x \in JW(R) \otimes_{\varphi, \mathfrak{S}} \mathfrak{M} \quad \text{for any } x \in \mathfrak{M}.$$

We denote by $\mathrm{Rep}_{\mathrm{tor}}^{r, \hat{G}_s, J}(G_s)$ the essential image of the contravariant functor $\hat{T}_s$ restricted to $\mathrm{Mod}_{/\mathfrak{S}_\infty}^{r, \hat{G}_s, J}$, which is a full subcategory of $\mathrm{Rep}_{\mathrm{tor}}(G_s)$.

Remark 3. In [Oz2], the category $\mathrm{Rep}_{\mathrm{tor}}^{r, \hat{G}_s, J}(G_s)$ is written as $\widetilde{\mathrm{Rep}}_{\mathrm{tor}}^{r, \hat{G}_s, J}(G_s)$. See Definition 4.1 of *loc. cit.*

Relations between the category $\mathrm{Rep}_{\mathrm{tor}}^{r, \hat{G}_s, J}(G_s)$ and three categories defined in the Introduction (e.g., $\mathrm{Rep}_{\mathrm{tor}}^{r, \mathrm{ht}, \mathrm{pcris}(s)}(G_K)$) are summarized as follows.

Proposition 3.2. (1) ([Oz2, Proposition 4.14]) *Suppose that $J = u^p I^{[1]} W(R)$ (thus $c_J = p/e + p/(p-1)$). If T is an object of $\mathrm{Rep}_{\mathrm{tor}}^{r, \mathrm{ht}, \mathrm{pcris}(s)}(G_K)$, then $T|_{G_s}$ is contained in $\mathrm{Rep}_{\mathrm{tor}}^{r, \hat{G}_s, J}(G_s)$.*

(2) ([Oz2, Proposition 4.15]) *Suppose that* $s \geq n-1$ *and* $J = u^p I^{[p^{s-n+1}]}W(R)$ (*thus* $c_J = p/e + p^{s-n+2}/(p-1)$). *If* T *is an object of* $\mathrm{Rep}^{r,\mathrm{cris}}_{\mathrm{tor}}(G_K)$ *which is killed by* p^n, *then* $T|_{G_s}$ *is contained in* $\mathrm{Rep}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s)$.
(3) ([Oz2, Proposition 4.17]) *Suppose that* $s \geq n-1$ *and* $J = I^{[p^{s-n+1}]}W(R)$ (*thus* $c_J = p^{s-n+2}/(p-1)$). *If* T *is an object of* $\mathrm{Rep}^{r,\mathrm{st}}_{\mathrm{tor}}(G_K)$ *which is killed by* p^n, *then* $T|_{G_s}$ *is contained in* $\mathrm{Rep}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s)$.

Remark 4. If we remove the condition "of height $\leq r$" from the definition of $\mathrm{Rep}^{r,\mathrm{ht},\mathrm{pcris}(s)}_{\mathrm{tor}}(G_K)$, then the author does not know whether the statement (1) holds or not. However, the statement (1) without the condition "of height $\leq r$" holds if we replace $u^p I^{[1]}W(R)$ with a larger ideal $u_s^p I^{[1]}W(R)$ by [GLS, Proposition 5.9].

The proposition above essentially follows from previous works of Gee-Liu-Savitt [GLS] and Liu [Li2]. By the proposition, to show our main theorem, it is enough to study the restriction functor $\mathrm{Rep}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s) \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$. For this, we have

Theorem 3.3 ([Oz2, Theorem 4.8]). *Assume that* $J \supset u^p I^{[1]}W(R)$ *or* k *is algebraically closed. Then the restriction functor* $\mathrm{Rep}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s) \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ *is fully faithful if* $p^{s+2}/(p-1) \geq c_J > pr/(p-1)$.

Our main theorem is an immediate consequence of the above proposition and theorem. Hence we may say that to show Theorem 3.3 is the essential result of [Oz2] (cf. §4.1, 4.2 and 4.3 of *loc. cit.*). Since the restriction functor $\mathrm{Rep}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s) \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$ is deeply related with the forgetful functor $\mathrm{Mod}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$, the result below plays an important role.

Proposition 3.4 ([Oz2, Proposition 4.2]). *The forgetful functor* $\mathrm{Mod}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty} \to \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ *is fully faithful if* $c_J > pr/(p-1)$.

§ 4. Outline of the proof of Theorem 3.3

In this section, we give a sketch of the proof of Theorem 3.3. Arguments are written in §4.3 of [Oz2]. We focus on the case where k is algebraically closed since the assumption $J \supset u^p I^{[1]}W(R)$ in Theorem 3.3 is not essentially related with our proof of the main theorems.

At first, we remark that the category $\mathrm{Rep}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s)$ is stable under taking subquotients in $\mathrm{Rep}_{\mathrm{tor}}(G_s)$ (Corollary 4.5 of *loc. cit.*). In particular, it is an exact category. Thus we may consider exact sequences of Hom's and Ext^1's in this category. Therefore, if we admit the following lemma, then we can use a dévissage argument and we obtain the desired result.

Lemma 4.1 (A part of [Oz2, Lemma 4.13]). *Suppose that k is an algebraically closed field. Let $T \in \mathrm{Rep}_{\mathrm{tor}}(G_s)$ and $T' \in \mathrm{Rep}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s)$. Suppose that T is irreducible, $T|_{G_\infty} \simeq T_{\mathfrak{S}}(\mathfrak{M})$ for some $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$ and $p^{s+2}/(p-1) \geq c_J > pr/(p-1)$. Then the injection $\mathrm{Hom}_{G_s}(T,T') \hookrightarrow \mathrm{Hom}_{G_\infty}(T,T')$ is bijective.*

Now we explain the outline of the proof of the lemma. By assumption, $T|_{G_\infty} \simeq T_{\mathfrak{S}}(\mathfrak{M})$ for some $\mathfrak{M} \in \mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. Moreover, we can assume that $\mathfrak{M}$ is maximal in the sense of [CL1, Definition 3.3.1]. The irreducibility of T implies that of $T|_{G_\infty}$, and this implies that $\mathfrak{M}$ is a simple object in the category of maximal objects in $\mathrm{Mod}^r_{/\mathfrak{S}_\infty}$. Then, we know the explicit shape of $\mathfrak{M}$, and so we can construct an object $\hat{\mathfrak{M}} := (\mathfrak{M}, \varphi, \hat{G}_s) \in \mathrm{Mod}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$ which has $\mathfrak{M}$ as a part of data ([Oz2, Definition 4.9, Corollary 4.11]). Then we have isomorphisms $T|_{G_\infty} \simeq T_{\mathfrak{S}}(\mathfrak{M}) \simeq \hat{T}_s(\hat{\mathfrak{M}})|_{G_\infty}$. Since G_s is topologically generated by G_∞ and the wild inertia group of K_s, the restriction functor

$$\mathrm{Rep}_{\mathrm{tor}}(G_s) \to \mathrm{Rep}_{\mathrm{tor}}(G_\infty)$$

is fully faithful when restricted to tame representations, and $T, \hat{T}_s(\hat{\mathfrak{M}})$ are tame because they are irreducible (cf. [CL1, Theorem 3.6.11]). So we have the isomorphism $T \simeq \hat{T}_s(\hat{\mathfrak{M}})$. On the other hand, we can take $\hat{\mathfrak{M}}' = (\mathfrak{M}', \varphi, \hat{G}_s) \in \mathrm{Mod}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$ such that $T' \simeq \hat{T}_s(\hat{\mathfrak{M}}')$ since T' is an object of $\mathrm{Rep}^{r,\hat{G}_s,J}_{\mathrm{tor}}(G_s)$. We then obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{G_s}(T,T') \arrow[rr, hook] & & \mathrm{Hom}_{G_\infty}(T,T') \\
\mathrm{Hom}(\hat{\mathfrak{M}}', \hat{\mathfrak{M}}) \arrow[u, "\hat{T}_s"] \arrow[r, "\text{forgetful}"] & \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}', \mathfrak{M}) \arrow[r, "\mathrm{Max}^r"] & \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathrm{Max}^r(\mathfrak{M}'), \mathfrak{M}) \arrow[u, "T_{\mathfrak{S}}"']
\end{tikzcd}$$

Here $\mathrm{Hom}(\hat{\mathfrak{M}}', \hat{\mathfrak{M}})$ is the set of morphisms $\hat{\mathfrak{M}}' \to \hat{\mathfrak{M}}$ in the category $\mathrm{Mod}^{r,\hat{G}_s,J}_{/\mathfrak{S}_\infty}$. For definitions of the functor Max^r above, see §3.3 of [CL1]. We have the following observations:

1. The first arrow in the bottom line is bijective by Proposition 3.4.
2. The second is also bijective since $\mathfrak{M}$ is maximal.
3. The right vertical arrow is also bijective by [CL1, Corollary 3.3.10].

Therefore, the top horizontal arrow $\mathrm{Hom}_{G_s}(T,T') \hookrightarrow \mathrm{Hom}_{G_\infty}(T,T')$ must be bijective.

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