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A new conjecture for Rubin-Stark elements and its applications

By

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Abstract

We review a new conjecture recently proposed by the author, and also by Mazur and Rubin. We explain that this conjecture is a natural generalization of Darmon's conjecture on cyclotomic units, and Gross's conjectures on Stickelberger elements.

§ 1. Introduction

This article is a research announcement of the paper [17]. The aim of this article is to explain in detail *a new conjecture* (Conjecture 2), which was proposed by the author in [17, Conjecture 3], *generalizing conjectures of Gross and of Darmon* ([10, 6]), and to explain its applications. Mazur and Rubin also proposed essentially the same conjecture as Conjecture 2 in the recent preprint [14].

Conjecture 2 concerns special elements, called Rubin-Stark elements, which are related to special values of L-functions of number fields. We sketch the formulation of Conjecture 2. Let k be a number field, and consider a finite abelian extension K/k. Let S be a finite set of places of k, which contains all infinite places of k and all places which ramify in K. Take a finite set T of places of k satisfying certain conditions (see §2.1 for the precise conditions). Let $\theta_{K/k,S,T}(s)$ denote the *equivariant* (S,T)-L-*function for* K/k (see §2.1 for the definition). Let r be the vanishing order of $\theta_{K/k,S,T}(s)$ at $s=0$. The *Rubin-Stark conjecture* (Conjecture 1), proposed by Rubin in [16, Conjecture B′], predicts that the leading term (namely, the coefficient of s^r) of the Taylor expansion

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of $\theta_{K/k,S,T}(s)$ at $s = 0$ is the image of an integral element $\eta^r_{K/k,S,T}$, which is called the *Rubin-Stark element*, under a regulator map. For any intermediate field $K/L/k$, *Conjecture 2 describes a precise relation between the Rubin-Stark elements* $\eta^r_{K/k,S,T}$ *and* $\eta^{r'}_{L/k,S,T}$, where r' denotes the vanishing order of $\theta_{L/k,S,T}(s)$ at $s = 0$. When $r = r'$, it is known that the norm map $\mathrm{N}_{K/L}$ sends $\eta^r_{K/k,S,T}$ to $\eta^{r'}_{L/k,S,T}$ (see Proposition 2.5). This relation is usually called the *"norm relation"*. In the general case, the author defined in [17] a *"higher norm"* $\mathcal{N}_{K/L}$, which coincides with the norm map $\mathrm{N}_{K/L}$ when $r = r'$, using an idea of Darmon in [6] (see §2.3). Assuming the Rubin-Stark conjecture, Conjecture 2 is formulated as follows:

$$\mathcal{N}_{K/L}(\eta^r_{K/k,S,T}) = \mathrm{Rec}(\eta^{r'}_{L/k,S,T}),$$

where Rec is a map constructed by using local reciprocity maps, and coincides with the identity map when $r = r'$ (see §2.3). We thus generalized the "norm relation" of Rubin-Stark elements.

Conjecture 2 is indeed a generalization of Conjectures of Gross and of Darmon ([10, 6]). Gross's conjecture ([10, Conjecture 4.1]) concerns $\theta_{K/k,S,T}(0)$, which is usually called the *Stickelberger element*, and predicts the equality

$$\theta_{K/k,S,T}(0) = h_{k,S,T} R_{K/k,S,T}, \tag{1.1}$$

where $h_{k,S,T}$ is the (S,T)-modified class number of k, and $R_{K/k,S,T}$ is the "algebraic regulator", constructed by using local reciprocity maps. Observing that $\eta^0_{K/k,S,T} = \theta_{K/k,S,T}(0)$ and that $\eta^{r'}_{L/k,S,T}$ is described explicitly by using $h_{k,S,T}$ when $L = k$, we know that the equation (1.1) is a special case of Conjecture 2. Darmon's conjecture ([6, Conjecture 4.3]) is an analogue of Gross's conjecture for cyclotomic units. Observing that cyclotomic units are examples of Rubin-Stark elements with $r = 1$, we can prove that Darmon's conjecture is also a special case of Conjecture 2.

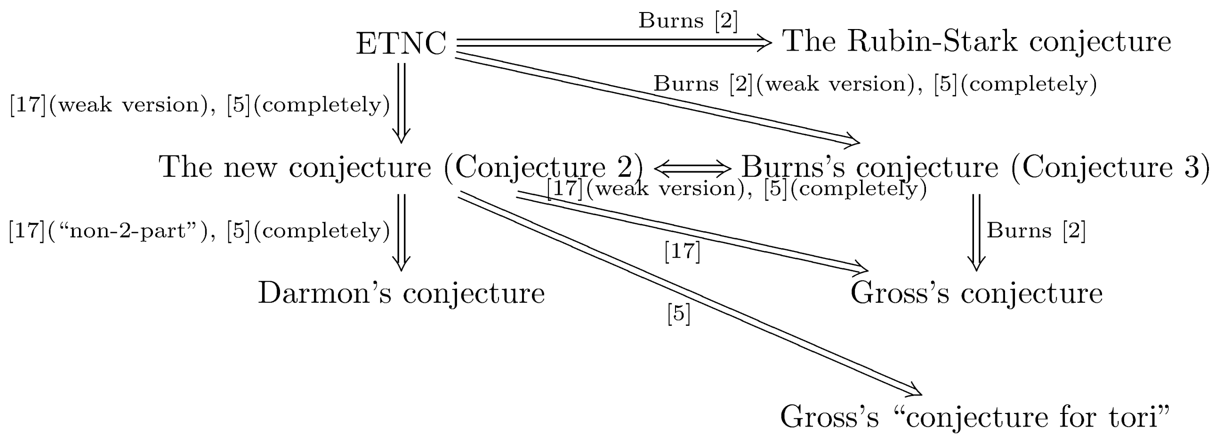
Formulating Conjecture 2, the author was inspired by the work of Burns in [2]. Burns formulated in [2, Theorem 3.1] essentially the following (conjectural) relation between $\eta^r_{K/k,S,T}$ and $\eta^{r'}_{L/k,S,T}$:

$$\Phi(\eta^r_{K/k,S,T}) = \Phi^{K/L}(\mathrm{Rec}(\eta^{r'}_{L/k,S,T})), \tag{1.2}$$

where Φ is any "evaluator", and $\Phi^{K/L}$ is its restriction on L (see §2.4 for the precise definition). In this article, we refer (1.2) as "Burns's conjecture" (Conjecture 3). As explained in the proof of [2, Corollary 4.1], Burns's conjecture is a generalization of Gross's conjecture (see Remark 2.8). The author proved that, *under some assumptions, our new conjecture (Conjecture 2) and Burns's conjecture (Conjecture 3) are equivalent* (see [17, Theorem 3.15] or Proposition 2.12). Burns proved that the Rubin-Stark conjecture and most of Burns's conjecture are deduced from the *"equivariant Tamagawa number conjecture (ETNC)"* ([3, Conjecture 4 (iv)]), a vast generalization of the classical class number formulas, for a particular Tate motive (see [2, Theorem 3.1] or Theorem 2.9). Using this result, the author proved that, *under some assumptions, most of our new conjecture (Conjecture 2) is deduced from the ETNC for a particular Tate motive* (see [17, Theorem 3.21] or Theorem 2.13). This is the main result of [17].

In a recent joint work with Burns and Kurihara ([5]), we were able to prove that *our new conjecture (Conjecture 2) and Burns's conjecture (Conjecture 3) are equivalent under no assumptions.* We also proved that *Conjecture 2 is deduced from the ETNC for a particular Tate motive completely.* Since the ETNC for Tate motives for abelian extensions over $\mathbb{Q}$ is known to be true, by the works of Burns, Greither, and Flach ([4, 7]), we have proved that *Conjecture 2 is true for abelian extensions over* $\mathbb{Q}$. As applications, *we gave a proof of Gross's "conjecture for tori"* ([10, Conjecture 8.8]), which was verified by Greither and Kučera in some particular cases ([8, 9]), and *a full proof of Darmon's conjecture*, whose "non-2-part" was proved by Mazur and Rubin via Kolyvagin systems ([13, Theorem 3.9]). Note that the main result of [17] gives sufficient ingredients to prove the "non-2-part" of Darmon's conjecture, as explained in [17, §4].

Since we mentioned so many conjectures, we illustrate their relations;



Notation

For a finite abelian group G, a $\mathbb{Z}[G]$-module (resp. algebra) is simply called a G-module (resp. algebra). Tensor products, Hom, and exterior powers over $\mathbb{Z}[G]$ are denoted by $\otimes_G$, Hom_G, and $\bigwedge_G$ respectively. For any subgroup $H \subset G$, the norm element is defined by

$$\mathrm{N}_H := \sum_{\sigma \in H} \sigma \in \mathbb{Z}[G].$$

Let M be a G-module, and Q be a G-algebra. For any positive integer r, there is a canonical homomorphism

$$\mathrm{Hom}_G(M,Q) \longrightarrow \mathrm{Hom}_G(\bigwedge_G^r M, (\bigwedge_G^{r-1} M)\otimes_G Q)$$

defined by

$$f \mapsto (f^{(r)} : m_1\wedge\cdots\wedge m_r \mapsto \sum_{i=1}^r (-1)^{i+1} m_1\wedge\cdots\wedge m_{i-1}\wedge m_{i+1}\wedge\cdots\wedge m_r \otimes f(m_i)).$$

For any positive integers r and s with $r\le s$, define a homomorphism

$$\bigwedge_G^r \mathrm{Hom}_G(M,Q) \longrightarrow \mathrm{Hom}_G(\bigwedge_G^s M, (\bigwedge_G^{s-r} M)\otimes_G Q) \tag{1.3}$$

by $f_1\wedge\cdots\wedge f_r \mapsto f_r^{(s-r+1)}\circ\cdots\circ f_2^{(s-1)}\circ f_1^{(s)}$. From this, we often regard an element of $\bigwedge_G^r \mathrm{Hom}_G(M,Q)$ as an element of $\mathrm{Hom}_G(\bigwedge_G^s M, (\bigwedge_G^{s-r} M)\otimes_G Q)$. Note that, when $r=s$, we have

$$(f_1\wedge\cdots\wedge f_r)(m_1\wedge\cdots\wedge m_r) = \det(f_i(m_j))_{1\le i,j\le r}. \tag{1.4}$$

§ 2. Conjectures

In this section, we formulate a new conjecture concerning Rubin-Stark elements (see Conjecture 2). Throughout this section, we fix a finite abelian extension K/k of number fields, and denote its Galois group by G. For any set Σ of places of k, we denote by Σ_K the set of places of K lying above places in Σ.

§ 2.1. The Rubin-Stark conjecture

We review the formulation of Rubin's integral refinement of the Stark conjecture ([16, Conjecture B′]).

Let S and T be sets of places of k satisfying the following:

- S contains all infinite places of k and all places which ramify in K,
- $S\cap T=\emptyset$.

The (S,T)*-unit group of* K is defined by

$$\mathcal{O}_{K,S,T}^\times := \{a\in K^\times : \mathrm{ord}_w(a)=0 \text{ for all } w\notin S_K \text{ and } a\equiv 1 \pmod{w'} \text{ for all } w'\in T_K\},$$

where ord_w denotes the normalized additive valuation at w. *We assume that* $\mathcal{O}_{K,S,T}^\times$ *is torsion-free.* This condition is satisfied when, for example, T contains two primes of unequal residue characteristics.

The *equivariant* (S,T)*-L-function for* K/k is defined by

$$\theta_{K/k,S,T}(s) := \prod_{v\in T}(1-\mathrm{Frob}_v^{-1}\mathrm{N}\,v^{1-s})\prod_{v\notin S}(1-\mathrm{Frob}_v^{-1}\mathrm{N}\,v^{-s})^{-1},$$

where $\mathrm{N}\,v$ denotes the cardinality of the residue field at v, and $\mathrm{Frob}_v \in G$ denotes the Frobenius automorphism at v. $\theta_{K/k,S,T}(s)$ is a $\mathbb{C}[G]$-valued complex function defined on $\mathrm{Re}(s) > 1$. It is well-known that $\theta_{K/k,S,T}(s)$ has a meromorphic continuation on $\mathbb{C}$, and holomorphic at $s = 0$.

Fix an integer r with $0 \le r < |S|$. *Assume that* S *has* r *places which split completely in* K. This assumption ensures that the function

$$\theta^{(r)}_{K/k,S,T}(s) := \frac{1}{s^r}\theta_{K/k,S,T}(s)$$

is holomorphic at $s = 0$ (see [18, Proposition 3.4, Chapitre I]). It is easy to see that $\theta^{(r)}_{K/k,S,T}(0) \in \mathbb{R}[G]$.

Let $Y_{K,S}$ denote the free abelian group on S_K. Define

$$X_{K,S} := \left\{ \sum_{w\in S_K} a_w w \in Y_{K,S} : \sum_{w\in S_K} a_w = 0 \right\}.$$

By Dirichlet's unit theorem, we have the isomorphism of $\mathbb{R}[G]$-modules

$$\lambda_{K,S} : \mathbb{R}\otimes_{\mathbb{Z}} \mathcal{O}^{\times}_{K,S,T} \xrightarrow{\sim} \mathbb{R}\otimes_{\mathbb{Z}} X_{K,S},$$

defined by

$$\lambda_{K,S}(a) := -\sum_{w\in S_K} \log|a|_w w,$$

where $|\cdot|_w$ denotes the normalized absolute value at w.

We set $n := |S|-1$ and *label the elements of* S *as*

$$S = \{v_0, v_1, \dots, v_n\}$$

so that $v_1,\dots,v_r$ *split completely in* K. For each $v_i \in S$, *we fix a place* w_i *of* K *lying above* v_i. Set $V := \{v_1,\dots,v_r\}$.

Define the *Rubin-Stark element for* $(K/k,S,T,V)$

$$\eta_{K/k,S,T,V} \in \mathbb{R}\otimes_{\mathbb{Z}} \bigwedge_G^r \mathcal{O}^{\times}_{K,S,T}$$

as the unique element which corresponds to $\theta^{(r)}_{K/k,S,T}(0)(w_1-w_0)\wedge\cdots\wedge(w_r-w_0) \in \mathbb{R}\otimes_{\mathbb{Z}}\bigwedge_G^r X_{K,S}$ under the isomorphism

$$\mathbb{R}\otimes_{\mathbb{Z}} \bigwedge_G^r \mathcal{O}^{\times}_{K,S,T} \xrightarrow{\sim} \mathbb{R}\otimes_{\mathbb{Z}} \bigwedge_G^r X_{K,S}$$

induced by $\lambda_{K,S}$. Note that $\eta_{K/k,S,T,V}$ depends on the choice of the labeling of the elements of S, and of each place w_i of K lying above v_i.

The Stark conjecture predicts that $\eta_{K/k,S,T,V} \in \mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_G^r \mathcal{O}_{K,S,T}^\times$ (see [16, Proposition 2.3]). The Rubin-Stark conjecture predicts the "integrality" of $\eta_{K/k,S,T,V}$: Rubin defined a lattice

$$\bigcap_G^r \mathcal{O}_{K,S,T}^\times := \left\{ a \in \mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_G^r \mathcal{O}_{K,S,T}^\times : \Phi(a) \in \mathbb{Z}[G] \text{ for all } \Phi \in \bigwedge_G^r \operatorname{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G]) \right\},$$

and conjectured

Conjecture 1 (The Rubin-Stark conjecture for $(K/k, S, T, V)$)**.**

$$\eta_{K/k,S,T,V} \in \bigcap_G^r \mathcal{O}_{K,S,T}^\times.$$

Remark 2.1. Our formulation of the Rubin-Stark conjecture is slightly different from the original formulation of Rubin ([16, Conjecture B′]). By [16, Proposition 2.4 and Lemma 2.6 (ii)], it is not difficult to see that these conjectures are equivalent.

Remark 2.2. As noted in the remark after [16, Conjecture B′], the validity of the Rubin-Stark conjecture does not depend on the choice of the labeling of the elements of S, and of each place w_i of K lying above v_i.

Remark 2.3. Clearly, we have

$$\operatorname{im}(\bigwedge_G^r \mathcal{O}_{K,S,T}^\times \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_G^r \mathcal{O}_{K,S,T}^\times) \subset \bigcap_G^r \mathcal{O}_{K,S,T}^\times.$$

But in general we have

$$\eta_{K/k,S,T,V} \notin \operatorname{im}(\bigwedge_G^r \mathcal{O}_{K,S,T}^\times \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge_G^r \mathcal{O}_{K,S,T}^\times).$$

(See [16, §4.1].) This shows that Rubin-Stark elements have "denominators" in general.

Remark 2.4. The Rubin-Stark conjecture for $(K/k, S, T, V)$ is known to be true in the following cases:

(i) $V = \emptyset$ i.e. $r = 0$ (in this case, the Rubin-Stark element is the "Stickelberger element" $\theta_{K/k,S,T}(0) \in \mathbb{R}[G]$, and the Rubin-Stark conjecture asserts that $\theta_{K/k,S,T}(0) \in \mathbb{Z}[G]$. This is a well-known result, due to Deligne and Ribet, see [16, Theorem 3.3]),

(ii) $[K : k] \leq 2$ ([16, Corollary 3.2 and Theorem 3.5]),

(iii) K is an abelian extension over $\mathbb{Q}$ (Burns, [2, Theorem A]).

§ 2.2. The "norm relation"

As mentioned in [16, Introduction], the study of Rubin's integral refinement of the Stark conjecture was an attempt to relate the Stark conjecture to the theory of Euler systems, initiated by Kolyvagin in [12]. An Euler system is a certain norm compatible system of global units. Rubin-Stark elements have norm compatible relations as follows.

Proposition 2.5 ("norm relation", [16, Proposition 6.1], [17, Proposition 3.5]). *Let L be an intermediate field of K/k, and put $H := \mathrm{Gal}(K/L)$. Then we have*

$$\mathrm{N}_H^r(\eta_{K/k,S,T,V}) = \eta_{L/k,S,T,V},$$

where N_H^r denotes the map

$$\mathbb{R} \otimes_{\mathbb{Z}} \bigwedge_G^r \mathcal{O}_{K,S,T}^\times \longrightarrow \mathbb{R} \otimes_{\mathbb{Z}} \bigwedge_{G/H}^r \mathcal{O}_{L,S,T}^\times$$

induced by $\mathrm{N}_H : \mathcal{O}_{K,S,T}^\times \to \mathcal{O}_{L,S,T}^\times$.

Using this relation, Rubin constructed Euler systems from Rubin-Stark elements, assuming the Rubin-Stark conjecture (see [16, Corollary 6.3 and the following Remark]).

One of the motivations of formulating our new conjecture is to generalize this relation of Rubin-Stark elements. See Remark 2.6 below.

§ 2.3. A new conjecture

We fix an intermediate field L of K/k, and set $H := \mathrm{Gal}(K/L)$. *Assume that $v_1, \ldots, v_{r'}$ split completely in L for some integer r' with $r \leq r' \leq n$*, and set $V' := \{v_1, \ldots, v_{r'}\}$. Our conjecture describes a relation between two Rubin-Stark elements $\eta_{K/k,S,T,V}$ and $\eta_{L/k,S,T,V'}$.

We denote by $I(H)$ the augmentation ideal of $\mathbb{Z}[H]$. Set $I_H := I(H)\mathbb{Z}[G]$. It is easy to see that

$$I_H = \ker(\mathbb{Z}[G] \longrightarrow \mathbb{Z}[G/H]).$$

For any non-negative integer i, put

$$Q_H^i := I_H^i / I_H^{i+1} \quad \text{and} \quad Q(H)^i := I(H)^i / I(H)^{i+1}.$$

Note that Q_H^i is naturally regarded as a G/H-module. There is a natural isomorphism of G/H-modules

$$Q_H^i \simeq \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} Q(H)^i. \tag{2.1}$$

For each $v_i \in S$, we denote by G_i the decomposition group of v_i in G. Note that, if $1 \leq i \leq r'$, then $G_i \subset H$ since v_i splits completely in L. For i with $r < i \leq r'$, define a G/H-homomorphism

$$\mathrm{Rec}_i : \mathcal{O}_{L,S,T}^\times \longrightarrow Q_H^1$$

by

$$\mathrm{Rec}_i(a) := \sum_{\tau \in G/H} \tau^{-1}(\mathrm{rec}_{w_i}(\tau a) - 1),$$

where $\mathrm{rec}_{w_i} : L^\times \to G_i \subset H$ denotes the local reciprocity map at the place of L lying under w_i. The element

$$\bigwedge_{r<i\leq r'} \mathrm{Rec}_i \in \bigwedge_{G/H}^{r'-r} \mathrm{Hom}_{G/H}(\mathcal{O}_{L,S,T}^\times, Q_H^1)$$

defines a map

$$\bigwedge_{G/H}^{r'} \mathcal{O}_{L,S,T}^\times \longrightarrow (\bigwedge_{G/H}^{r} \mathcal{O}_{L,S,T}^\times) \otimes_{G/H} Q_H^{r'-r} \simeq (\bigwedge_{G/H}^{r} \mathcal{O}_{L,S,T}^\times) \otimes_{\mathbb{Z}} Q(H)^{r'-r}.$$

(See (1.3) and (2.1).) One can show that this extends to a map

$$\bigcap_{G/H}^{r'} \mathcal{O}_{L,S,T}^\times \longrightarrow (\bigcap_{G/H}^{r} \mathcal{O}_{L,S,T}^\times) \otimes_{\mathbb{Z}} Q(H)^{r'-r},$$

which we denote by $\mathrm{Rec}_{V,V'}$ (see [17, Proposition 2.7]). Note that, if $V = V'$ i.e. $r = r'$, then $Q(H)^0 = \mathbb{Z}[H]/I(H)$ is identified with $\mathbb{Z}$ and $\mathrm{Rec}_{V,V}$ is the identity map in this case.

When $r > 0$ (resp. $r = 0$), we define the "higher norm"

$$\mathcal{N}_H : \bigcap_{G}^{r} \mathcal{O}_{K,S,T}^\times \longrightarrow (\bigcap_{G}^{r} \mathcal{O}_{K,S,T}^\times) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{r'-r+1}$$

$$(\text{resp. } \mathcal{N}_H : \bigcap_{G}^{0} \mathcal{O}_{K,S,T}^\times = \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]/I_H^{r'+1})$$

by $\mathcal{N}_H(a) := \sum_{\sigma \in H} \sigma a \otimes \sigma^{-1}$ (resp. the natural map). Note that, if $r = r'$, then by the canonical identifications

$$\mathbb{Z}[H]/I(H) \simeq \mathbb{Z} \quad \text{and} \quad \mathbb{Z}[G]/I_H \simeq \mathbb{Z}[G/H],$$

we have

$$\mathcal{N}_H = \begin{cases} \mathrm{N}_H : \bigcap_G^r \mathcal{O}_{K,S,T}^\times \longrightarrow \bigcap_G^r \mathcal{O}_{K,S,T}^\times & \text{if } r > 0, \\ \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G/H] & \text{if } r = 0. \end{cases}$$

Thus, $\mathcal{N}_H$ is related to the usual norm. This is the reason why we call $\mathcal{N}_H$ the "higher norm".

We define an injection

$$\iota : (\bigcap_{G/H}^{r} \mathcal{O}_{L,S,T}^{\times}) \otimes_{\mathbb{Z}} Q(H)^{r'-r} \longrightarrow \begin{cases} (\bigcap_G^r \mathcal{O}_{K,S,T}^{\times}) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{r'-r+1} & \text{if } r > 0, \\ \mathbb{Z}[G]/I_H^{r'+1} & \text{if } r = 0 \end{cases}$$

as follows. When $r > 0$, one can show that there is a canonical injection

$$\iota : \bigcap_{G/H}^{r} \mathcal{O}_{L,S,T}^{\times} \longrightarrow \bigcap_{G}^{r} \mathcal{O}_{K,S,T}^{\times}$$

satisfying $\iota(\mathrm{N}_H^r\, a) = \mathrm{N}_H\, a$ for any $a \in \bigcap_G^r \mathcal{O}_{K,S,T}^{\times}$ (see [17, Lemma 2.11 and Remark 2.12]). It can be shown that the map

$$\begin{aligned}(\bigcap_{G/H}^{r} \mathcal{O}_{L,S,T}^{\times}) \otimes_{\mathbb{Z}} Q(H)^{r'-r} &\longrightarrow (\bigcap_{G}^{r} \mathcal{O}_{K,S,T}^{\times}) \otimes_{\mathbb{Z}} Q(H)^{r'-r} \\ &\longrightarrow (\bigcap_{G}^{r} \mathcal{O}_{K,S,T}^{\times}) \otimes_{\mathbb{Z}} \mathbb{Z}[H]/I(H)^{r'-r+1},\end{aligned}$$

where the first map is induced by ι and the second by the natural injection $Q(H)^{r'-r} \hookrightarrow \mathbb{Z}[H]/I(H)^{r'-r}$, is also injective (see [17, Lemma 2.11]). We denote this injection also by ι. When $r = 0$, we define

$$\iota : (\bigcap_{G/H}^{0} \mathcal{O}_{L,S,T}^{\times}) \otimes_{\mathbb{Z}} Q(H)^{r'} \simeq Q_H^{r'} \hookrightarrow \mathbb{Z}[G]/I_H^{r'+1}$$

to be the natural injection.

Conjecture 2. *Assume that the Rubin-Stark conjecture (Conjecture 1) holds for* $(K/k, S, T, V)$ *and* $(L/k, S, T, V')$. *Then* $\mathcal{N}_H(\eta_{K/k,S,T,V}) \in \operatorname{im} \iota$ *and*

$$\iota^{-1}(\mathcal{N}_H(\eta_{K/k,S,T,V})) = (-1)^{r(r'-r)} \mathrm{Rec}_{V,V'}(\eta_{L/k,S,T,V'}).$$

Remark 2.6. If $V = V'$ i.e. $r = r'$, then one sees easily that $\mathcal{N}_H(\eta_{K/k,S,T,V}) \in \operatorname{im} \iota$ and

$$\iota^{-1}\mathcal{N}_H(\eta_{K/k,S,T,V}) = \mathrm{N}_H^r(\eta_{K/k,S,T,V}).$$

So in this case Conjecture 2 is true by the "norm relation" (Proposition 2.5). In other words, Conjecture 2 is a generalization of the "norm relation".

Remark 2.7. In §3, we will see that Conjecture 2 is a natural generalization of conjectures of Gross and of Darmon ([10, 6]).

§ 2.4. Burns's conjecture and the equivariant Tamagawa number conjecture

In [2, Theorem 3.1], Burns gave a formulation relating two Rubin-Stark elements. We modify his formulation to propose a conjecture (Conjecture 3), and refer it as "Burns's conjecture".

We introduce some notation. For any $\varphi \in \mathrm{Hom}_G(M, \mathbb{Z}[G])$, where M is a G-module, define $\varphi^H \in \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H])$ by

$$\varphi : M^H \xrightarrow{\varphi} \mathbb{Z}[G]^H \xrightarrow{\sim} \mathbb{Z}[G/H],$$

where the last isomorphism is given by $\mathrm{N}_H \mapsto 1$. Let s be a non-negative integer. For any $\Phi \in \bigwedge_G^s \mathrm{Hom}_G(M, \mathbb{Z}[G])$, define $\Phi^H \in \bigwedge_{G/H}^s \mathrm{Hom}_{G/H}(M^H, \mathbb{Z}[G/H])$ to be the image of Φ under the map

$$\begin{cases} \varphi_1 \wedge \cdots \wedge \varphi_s \mapsto \varphi_1^H \wedge \cdots \wedge \varphi_s^H & \text{if } s > 0, \\ \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G/H] & \text{if } s = 0. \end{cases}$$

Note that, by (1.3), for a given $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])$, we have the maps

$$\Phi : \bigcap_G^r \mathcal{O}_{K,S,T}^\times \longrightarrow \mathbb{Z}[G]$$

and

$$\Phi^H : (\bigcap_{G/H}^r \mathcal{O}_{L,S,T}^\times) \otimes_{\mathbb{Z}} Q(H)^{r'-r} \longrightarrow \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} Q(H)^{r'-r} \simeq Q_H^{r'-r}.$$

Conjecture 3 (Burns's conjecture)**.** *Assume that the Rubin-Stark conjecture (Conjecture 1) holds for $(K/k, S, T, V)$ and $(L/k, S, T, V')$. Then we have*

$$\Phi(\eta_{K/k,S,T,V}) = (-1)^{r(r'-r)} \Phi^H(\mathrm{Rec}_{V,V'}(\eta_{L/k,S,T,V'})) \quad \textit{in} \quad Q_H^{r'-r}$$

for every $\Phi \in \bigwedge_G^r \mathrm{Hom}_G(\mathcal{O}_{K,S,T}^\times, \mathbb{Z}[G])$.

Remark 2.8. Burns's conjecture is a natural generalization of Gross's conjecture ([10, Conjecture 4.1]). Indeed, consider the following case:

- $V = \emptyset$ i.e. $r = 0$,
- $L = k$ i.e. $G = H$,
- $r' = n(= |S| - 1)$.

In this case, the Rubin-Stark conjecture for $(K/k, S, T, \emptyset)$ (resp. $(k/k, S, T, S \setminus \{v_0\})$) is true by Remark 2.4 (i) (resp. (ii)), and the associated Rubin-Stark element is given by

$$\eta_{K/k,S,T,\emptyset} = \theta_{K/k,S,T}(0) \in \mathbb{Z}[G]$$

$$(\text{resp. } \eta_{k/k,S,T,S\setminus\{v_0\}} = \pm h_{k,S,T} u_1 \wedge \cdots \wedge u_n \in \bigwedge_{\mathbb{Z}}^{n} \mathcal{O}_{k,S,T}^{\times},$$

where $h_{k,S,T}$ is the (S,T)-class number of k and $\{u_1, \ldots, u_n\}$ is a basis of $\mathcal{O}_{k,S,T}^{\times}$, see [15, Example 3.2.11]). Conjecture 3 reads

$$\theta_{K/k,S,T}(0) = \pm h_{k,S,T} \det(\mathrm{rec}_{v_i}(u_j) - 1)_{1 \le i \le n} \quad \text{in} \quad Q(G)^n.$$

(See (1.4).) This is exactly the formulation of Gross's conjecture ([10, Conjecture 4.1]).

Burns related Conjecture 3 to the equivariant Tamagawa number conjecture ([3, Conjecture 4 (iv)]) as follows.

Theorem 2.9 (Burns, [2, Theorem 3.1], [17, Theorem 3.17 and Proposition 3.20])**.** *Assume that the equivariant Tamagawa number conjecture for the pair $(h^0(\operatorname{Spec} K), \mathbb{Z}[G])$ holds. Then the Rubin-Stark conjecture (Conjecture 1) holds for both $(K/k, S, T, V)$ and $(L/k, S, T, V')$, and for every $\Phi \in \bigwedge_G^r \operatorname{Hom}_G(\mathcal{O}_{K,S,T}^{\times}, \mathbb{Z}[G])$ we have*

$$\Phi(\eta_{K/k,S,T,V}) = (-1)^{r(r'-r)} \Phi^H(\mathrm{Rec}_{V,V'}(\eta_{L/k,S,T,V'})) \quad \text{in} \quad Q_H^{r'-r} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{[L:k]}].$$

By Remark 2.8, we obtain

Corollary 2.10 (Burns, [2, Corollary 4.1])**.** *The equivariant Tamagawa number conjecture for the pair $(h^0(\operatorname{Spec} K), \mathbb{Z}[G])$ implies Gross's conjecture for $(K/k, S, T)$ ([10, Conjecture 4.1]).*

Note that, if K is abelian over $\mathbb{Q}$, then the equivariant Tamagawa number conjecture for the pair $(h^0(\operatorname{Spec} K), \mathbb{Z}[G])$ is known to be true, by the works of Burns, Greither, and Flach ([4, 7]). Using this fact, Burns gave a proof of Gross's conjecture for abelian extensions over $\mathbb{Q}$, which was first proved by Aoki in [1] (see [2, Theorem A]). Note that the result in Remark 2.4 (iii) is also a consequence of this fact, using Theorem 2.9.

§ 2.5. Relations among conjectures

Comparing Conjecture 2 and Conjecture 3, it is natural to guess the following

Conjecture 4. *Assume that the Rubin-Stark conjecture (Conjecture 1) holds for $(K/k, S, T, V)$. If $\mathcal{N}_H(\eta_{K/k,S,T,V}) \in \operatorname{im} \iota$, then for every $\Phi \in \bigwedge_G^r \operatorname{Hom}_G(\mathcal{O}_{K,S,T}^{\times}, \mathbb{Z}[G])$ we have*

$$\Phi(\eta_{K/k,S,T,V}) = \Phi^H(\iota^{-1}(\mathcal{N}_H(\eta_{K/k,S,T,V}))) \quad \text{in} \quad Q_H^{r'-r}.$$

Remark 2.11. If $r = 0$, then Conjecture 4 is clearly true. In [17, Proposition 2.15], the author proved Conjecture 4 when $r = 1$ or $r = r'$. In a recent joint work with Burns and Kurihara ([5]), we proved Conjecture 4 completely.

Proposition 2.12 ([17, Theorem 3.15]). *Assume that* $\mathcal{N}_H(\eta_{K/k,S,T,V}) \in \operatorname{im}\iota$, *and that Conjecture 4 holds. Then Conjectures 2 and 3 are equivalent.*

Proof. It is clear that under the assumptions Conjecture 2 implies Conjecture 3. The converse follows from the fact that the map

$$(\bigcap_{G/H}^{r} \mathcal{O}_{K,S,T}^{\times}) \otimes_{\mathbb{Z}} Q(H)^{r'-r} \longrightarrow \operatorname{Hom}_G(\bigwedge_G^r \operatorname{Hom}_G(\mathcal{O}_{K,S,T}^{\times}, \mathbb{Z}[G]), Q_H^{r'-r})$$

defined by $a \mapsto (\Phi \mapsto \Phi^H(a))$ is injective (see [17, Theorem 2.17]). □

Combining (the proof of) Proposition 2.12 with Burns's result (Theorem 2.9), we obtain the following

Theorem 2.13 ([17, Theorem 3.21]). *Assume that* $\mathcal{N}_H(\eta_{K/k,S,T,V}) \in \operatorname{im}\iota$, *that Conjecture 4 holds, and that the equivariant Tamagawa number conjecture for the pair* $(h^0(\operatorname{Spec} K), \mathbb{Z}[G])$ *holds. Then we have the equality*

$$\iota^{-1}(\mathcal{N}_H(\eta_{K/k,S,T,V})) = (-1)^{r(r'-r)}\operatorname{Rec}_{V,V'}(\eta_{L/k,S,T,V'})$$

in $(\bigcap_{G/H}^{r} \mathcal{O}_{L,S,T}^{\times}) \otimes_{\mathbb{Z}} Q(H)^{r'-r} \otimes_{\mathbb{Z}} \mathbb{Z}[1/[L:k]]$.

Remark 2.14. In the joint work with Burns and Kurihara ([5]), we proved that *Conjectures 2 and 3 are equivalent under no assumptions* (namely, we removed the assumptions of Proposition 2.12). Furthermore, we proved that *the equivariant Tamagawa number conjecture for the pair* $(h^0(\operatorname{Spec} K), \mathbb{Z}[G])$ *implies Conjecture 2 directly.* This result improves both Theorem 2.13 and Theorem 2.9. Since the equivariant Tamagawa number conjecture for the pair $(h^0(\operatorname{Spec} K), \mathbb{Z}[G])$ is known to be true if K is abelian over $\mathbb{Q}$, as we noted before, we have proved that *Conjecture 2 is true if* K *is abelian over* $\mathbb{Q}$.

§ 3. Applications

In this section, we explain that Conjecture 2 is indeed a generalization of conjectures of Gross and of Darmon ([10, 6]). In [17, §4], it was shown that the "non-2-part" of Darmon's conjecture, which had been solved by Mazur and Rubin via Kolyvagin systems ([13, Theorem 3.9]), is deduced from Conjecture 2, and Theorem 2.13 was applied to give another proof of the "non-2-part" of Darmon's conjecture. We formulate a slightly modified version of Darmon's conjecture, and explain that it is deduced from Conjecture 2. (For the difference between the original Darmon's conjecture and our modified version, see Remark 3.1 below.) We also explain that Gross's "conjecture for tori" ([10, Conjecture 8.8]), is deduced from Conjecture 2 by a similar argument. Admitting the assertion in Remark 2.14 that Conjecture 2 is true if K is abelian over $\mathbb{Q}$, we give a full proof of (modified) Darmon's conjecture, and a proof of Gross's "conjecture for tori" for abelian extensions over $\mathbb{Q}$, whose particular cases were verified by Greither and Kučera in [8, 9]. This improvement of the main results of [13, 8, 9] is treated in the joint work with Burns and Kurihara [5].

§ 3.1. Darmon's Conjecture

We formulate a slightly modified version of Damon's conjecture ([6, Conjecture 4.3], [13, Conjecture 3.8]).

Let L be a real quadratic field, and χ be the corresponding Dirichlet character with conductor f. Let K be the maximal real subfield of $L(\mu_n)$, where n is a square-free positive integer coprime to f, and μ_n denotes the group of n-th roots of unity in $\overline{\mathbb{Q}}^{\times}$. Set $G := \mathrm{Gal}(K/\mathbb{Q})$ and $H := \mathrm{Gal}(K/L)$. Put $n_{\pm} := \prod_{\ell | n, \chi(\ell)=\pm 1} \ell$, and $\nu_{\pm} := |\{\ell | n_{\pm}\}|$ (in this section, ℓ always denotes a prime number). We fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Define a cyclotomic unit by

$$\beta_n := \mathrm{N}_{L(\mu_n)/K}\Big(\prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_{nf})/\mathbb{Q}(\mu_n))} \sigma(1-\zeta_{nf})^{\chi(\sigma)} \Big) \in K^{\times},$$

where $\zeta_{nf} = e^{\frac{2\pi i}{nf}}$. Let τ be the generator of $\mathrm{Gal}(L/\mathbb{Q})$. Write $n_+ = \ell_1 \cdots \ell_{\nu_+}$. Let $\mathcal{O}_L$ denote the ring of integers of L. Note that $(1-\tau)\mathcal{O}_L[1/n]^{\times}$ is a free abelian group of rank $\nu_+ + 1$ (see [13, Lemma 3.2 (ii)]). Take $u_0, \ldots, u_{\nu_+} \in \mathcal{O}_L[1/n]^{\times}$ such that $\{u_0^{1-\tau}, \ldots, u_{\nu_+}^{1-\tau}\}$ is a basis of $(1-\tau)\mathcal{O}_L[1/n]^{\times}$ and

$$\det(\log |u_i^{1-\tau}|_{\lambda_j})_{0 \le i,j \le \nu_+} > 0,$$

where each λ_j $(1 \le j \le \nu_+)$ is a (fixed) place of L lying above ℓ_j, and λ_0 is the infinite place of L corresponding to the embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ fixed above. Define

$$R_n := \Big(\bigwedge_{1 \le i \le \nu_+} (\mathrm{rec}_{\lambda_i}(\cdot) - 1) \Big)(u_0^{1-\tau} \wedge \cdots \wedge u_{\nu_+}^{1-\tau}) \in L^{\times} \otimes_{\mathbb{Z}} Q(H)^{\nu_+}.$$

Let h_n denote the order of the Picard group of $\mathcal{O}_L[1/n]$. Our modified Darmon's conjecture is formulated as follows.

Conjecture 5 (Darmon's conjecture)**.**

$$\sum_{\sigma \in H} \sigma\beta_n \otimes \sigma^{-1} = -2^{\nu_-} h_n R_n \quad \text{in} \quad (L^{\times}/\{\pm 1\}) \otimes_{\mathbb{Z}} Q(H)^{\nu_+}.$$

Remark 3.1. Let I_n be the augmentation ideal of $\mathbb{Z}[\mathrm{Gal}(L(\mu_n)/L)]$. Note that the natural map $\mathrm{Gal}(L(\mu_n)/L) \to H$ induces the isomorphism

$$I_n^{\nu_+}/I_n^{\nu_+ +1} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}] \xrightarrow{\sim} Q(H)^{\nu_+} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}].$$

Using this, it is not difficult to see that the following statement is equivalent to [13, Theorem 3.9]:

$$\sum_{\sigma \in H} \sigma\beta_n \otimes \sigma^{-1} = -2^{\nu_-} h_n R_n \quad \text{in} \quad (L^\times/\{\pm 1\}) \otimes_{\mathbb{Z}} Q(H)^{\nu_+} \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{2}].$$

(See [17, Lemma 4.7].) Thus, the "non-2-part" of the original Darmon's conjecture ([13, Conjecture 3.8]) is deduced from our Darmon's conjecture. Note that, in the original Darmon's conjecture, the cyclotomic unit is defined by

$$\alpha_n := \prod_{\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_{nf})/\mathbb{Q}(\mu_n))} \sigma(1-\zeta_{nf})^{\chi(\sigma)},$$

whereas our cyclotomic unit is $\beta_n = \mathrm{N}_{L(\mu_n)/K}(\alpha_n)$. Since cyclotomic units, as Stark elements, lie in real fields, so it is natural to consider β_n.

Theorem 3.2. *Conjecture 2 (with varying T) implies Darmon's conjecture.*

Proof. Set $S := \{\infty \text{ (the infinite place of } \mathbb{Q})\} \cup \{\ell | nf\}$, and label the elements of S as $\{v_0, v_1, \dots, v_m\}$ so that $v_0|f$, $v_1 = \infty$, and $v_i = \ell_{i-1}$ for $2 \le i \le \nu_+ + 1$. Set $V := \{v_1\}$ and $V' := \{v_i : 1 \le i \le \nu_+ + 1\}$. Consider the Rubin-Stark elements

$$\eta_{K/\mathbb{Q},S,T,V} \in \bigcap_G^1 \mathcal{O}_{K,S,T}^\times = \mathcal{O}_{K,S,T}^\times \quad \text{and} \quad \eta_{L/\mathbb{Q},S,T,V'} \in \bigcap_{G/H}^{\nu_+ +1} \mathcal{O}_{L,S,T}^\times,$$

with a some suitable set T. (Note that, by Remark 2.4 (iii), the Rubin-Stark conjecture is true for abelian extensions over $\mathbb{Q}$.) By Conjecture 2, we have

$$\mathcal{N}_H(\eta_{K/\mathbb{Q},S,T,V}) = (-1)^{\nu_+}\mathrm{Rec}_{V,V'}(\eta_{L/\mathbb{Q},S,T,V'}) \quad \text{in} \quad L^\times \otimes_{\mathbb{Z}} Q(H)^{\nu_+}.$$

Using [18, Lemme 1.1, Chapitre IV], we can choose $\mathcal{T}$, a finite family of T, such that

$$2 = \sum_{T \in \mathcal{T}} a_T \prod_{\ell \in T}(1 - \ell \mathrm{Frob}_\ell^{-1}) \quad \text{in} \quad \mathbb{Z}[G]$$

with some $a_T \in \mathbb{Z}[G]$, where $\mathrm{Frob}_\ell \in G$ denotes the Frobenius automorphism at ℓ. We can show that

$$(1-\tau)\sum_{T \in \mathcal{T}} a_T \mathcal{N}_H(\eta_{K/\mathbb{Q},S,T,V}) = \sum_{\sigma \in H} \sigma\beta_n \otimes \sigma^{-1} \quad \text{in} \quad (L^\times/\{\pm 1\}) \otimes_{\mathbb{Z}} Q(H)^{\nu_+}$$

and

$$(1-\tau)\sum_{T\in\mathcal{T}} a_T \mathrm{Rec}_{V,V'}(\eta_{L/\mathbb{Q},S,T,V'}) = (-1)^{\nu_+ +1}2^{\nu_-}h_n R_n \quad \text{in} \quad L^\times \otimes_{\mathbb{Z}} Q(H)^{\nu_+}$$

by using explicit descriptions of the Rubin-Stark elements $\eta_{K/\mathbb{Q},S,T,V}$ and $\eta_{L/\mathbb{Q},S,T,V'}$ (see [17, Lemma 4.6]). This proves the theorem. □

Remark 3.3. By Remark 2.14, Conjecture 2 is true if K is abelian over $\mathbb{Q}$. So the above proof of Theorem 3.2 shows that *Darmon's conjecture is true.* This improves the result of Mazur and Rubin in [13, Theorem 3.9], where the "non-2-part" of Darmon's conjecture was proved.

§ 3.2. Gross's "conjecture for tori"

We review the formulation of Gross's "conjecture for tori" ([10, Conjecture 8.8]). We follow a formulation of Hayward ([11, Conjecture 7.4]). (As Hayward warned in [11, §7.3], the original conjecture [10, Conjecture 8.8] has a slight error.) Let k be a number field, and L/k be a quadratic extension. Let $\widetilde{L}/k$ be a finite abelian extension, which is disjoint to L, and set $K := L\widetilde{L}$. Set $G := \mathrm{Gal}(K/k)$, and $H := \mathrm{Gal}(K/L) = \mathrm{Gal}(\widetilde{L}/k)$. Let τ be the generator of $G/H = \mathrm{Gal}(L/k)$. Let S be a finite set of places of k which contains all infinite places of k and all places which ramify in K. Let T be a finite set of places of k that is disjoint from S and satisfies that $\mathcal{O}_{K,S,T}^\times$ is torsion-free. Let $v_1, \ldots, v_{r'}$ be *all* places in S which split in L. We assume $r' < |S|$. Note that, by [16, Lemma 3.4 (i)], we have

$$\frac{h_{L,S,T}}{h_{k,S,T}} \in \mathbb{Z},$$

where $h_{k,S,T}$ and $h_{L,S,T}$ are the (S,T)-class numbers of k and L respectively (see [11, §2.1]). Take $u_1, \ldots, u_{r'} \in \mathcal{O}_{L,S,T}^\times$ such that $\{u_1^{1-\tau}, \ldots, u_{r'}^{1-\tau}\}$ is a basis of $(1-\tau)\mathcal{O}_{L,S,T}^\times (\simeq \mathbb{Z}^{\oplus r'})$, and

$$\det(-\log|u_i^{1-\tau}|_{w_j})_{1\le i,j\le r'} > 0,$$

where w_j is a (fixed) place of L lying above v_j. Set

$$R_{S,T} := \det(\mathrm{rec}_{w_j}(u_i^{1-\tau})-1)_{1\le i,j\le r'} \in Q(H)^{r'}.$$

We denote $\theta_{K/k,S,T}(0)^-$ for the image of $\theta_{K/k,S,T}(0) \in \mathbb{Z}[G]$ under the map

$$\mathbb{Z}[G] = \mathbb{Z}[H \times G/H] \longrightarrow \mathbb{Z}[H] \quad ; \quad \tau \mapsto -1. \tag{3.1}$$

Conjecture 6 (Gross's "conjecture for tori").

$$\theta_{K/k,S,T}(0)^- = 2^{|S|-1-r'}\frac{h_{L,S,T}}{h_{k,S,T}}R_{S,T} \quad \textit{in} \quad Q(H)^{r'}.$$

Remark 3.4. Conjecture 6 is equivalent to [11, Conjecture 7.4] (if we neglect the sign). This can be seen by noting that

$$R_{S,T} = ((\mathcal{O}_{L,S,T}^{\times})^{-} : (1-\tau)\mathcal{O}_{L,S,T}^{\times})R_H^{-},$$

where R_H^{-} is as in [11, §7.2] (note that our H corresponds to G in [11, §7]).

Theorem 3.5. *Conjecture 2 implies Conjecture 6.*

Proof. Put $V' := \{v_1, \ldots, v_{r'}\}$. We remark that the Rubin-Stark conjecture for $(K/k, S, T, \emptyset)$ and $(L/k, S, T, V')$ is true by Remark 2.4 (i) and (ii) respectively. By Conjecture 2, we have

$$\theta_{K/k,S,T}(0) = \mathrm{Rec}_{\emptyset,V'}(\eta_{L/k,S,T,V'}) \quad \text{in} \quad Q_H^{r'}.$$

We denote $\mathrm{Rec}_{\emptyset,V'}(\eta_{L/k,S,T,V'})^{-}$ for the image of $\mathrm{Rec}_{\emptyset,V'}(\eta_{L/k,S,T,V'})$ under the map

$$Q_H^{r'} \simeq \mathbb{Z}[G/H] \otimes_{\mathbb{Z}} Q(H)^{r'} \longrightarrow Q(H)^{r'}$$

induced by (3.1). It is easy to see that

$$\mathrm{Rec}_{\emptyset,V'}(\eta_{L/k,S,T,V'})^{-} = (\bigwedge_{1\le i\le r'} (\mathrm{rec}_{w_i}(\cdot) - 1))((1-\tau)^{r'}\eta_{L/k,S,T,V'}).$$

We know by the proof of [16, Theorem 3.5] that

$$(1-\tau)^{r'}\eta_{L/k,S,T,V'} = 2^{|S|-1-r'}\frac{h_{L,S,T}}{h_{k,S,T}} u_1^{1-\tau} \wedge \cdots \wedge u_{r'}^{1-\tau}.$$

Hence we have

$$\begin{aligned}
\theta_{K/k,S,T}(0)^{-} &= \mathrm{Rec}_{\emptyset,V'}(\eta_{L/k,S,T,V'})^{-} \\
&= (\bigwedge_{1\le i\le r'} (\mathrm{rec}_{w_i}(\cdot) - 1))((1-\tau)^{r'}\eta_{L/k,S,T,V'}) \\
&= 2^{|S|-1-r'}\frac{h_{L,S,T}}{h_{k,S,T}}(\bigwedge_{1\le i\le r'} (\mathrm{rec}_{w_i}(\cdot) - 1))(u_1^{1-\tau} \wedge \cdots \wedge u_{r'}^{1-\tau}) \\
&= 2^{|S|-1-r'}\frac{h_{L,S,T}}{h_{k,S,T}} R_{S,T}.
\end{aligned}$$

□

Remark 3.6. In [8, 9], Greither and Kučera studied Gross's "conjecture for tori" when $k = \mathbb{Q}$, and obtained partial results on this conjecture. By Remark 2.14, *Theorem 3.5 gives a proof of Gross's "conjecture for tori" for abelian fields (namely, Conjecture 6 for K abelian over $\mathbb{Q}$).* Thus we improve the main results of [8, 9].

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References

[1] Aoki, N., *Gross's conjecture on the special values of abelian L-functions at* $s = 0$, Comment. Math. Univ. St. Paul. **40** (1991), 101-124.
[2] Burns, D., *Congruences between derivatives of abelian L-functions at* $s = 0$, Invent. Math. **169** (2007), 451-499.
[3] Burns, D., and Flach, M., *Tamagawa numbers for motives with (non-commutative) coefficients*, Doc. Math. **6** (2001), 501-570.
[4] Burns, D., and Greither, C., *On the equivariant Tamagawa number conjecture for Tate motives*, Invent. Math. **153** (2003), 303-359.
[5] Burns, D., Kurihara, M., and Sano, T., *On zeta elements for* $\mathbb{G}_m$, preprint (2015) arXiv:1407.6409v2.
[6] Darmon, H., *Thaine's method for circular units and a conjecture of Gross*, Canad. J. Math. **47** (1995), 302-317.
[7] Flach, M., *On the cyclotomic main conjecture for the prime* 2, J. reine angew. Math. **661** (2011), 1-36.
[8] Greither, C., and Kučera, R., *On a conjecture concerning minus parts in the style of Gross*, Acta Arith. **132** (2008), 1-48.
[9] Greither, C., and Kučera, R., *The Minus Conjecture revisited*, J. reine angew. Math. **632** (2009), 127-142.
[10] Gross, B., *On the values of abelian L-functions at* $s = 0$, J. Fac. Sci. Univ. Tokyo **35** (1988), 177-197.
[11] Hayward, A., *A class number formula for higher derivatives of abelian L-functions*, Compositio Math. **140** (2004), 99-129.
[12] Kolyvagin, V. A., *Euler systems*, The Grothendieck Festschrift Vol II (1990), 435-483.
[13] Mazur, B., and Rubin, K., *Refined class number formulas and Kolyvagin systems*, Compos. Math. **147** (2011), 56-74.
[14] Mazur, B., and Rubin, K., *Refined class number formulas for* $\mathbb{G}_m$, preprint (2013) arXiv:1312.4053v1
[15] Popescu, C. D., *Integral and p-adic refinements of the Abelian Stark conjecture*, In *Arithmetic of L-functions* by Popescu, C., Rubin, K., and Silverberg, A., IAS/Park City Mathematics Series **18**, AMS (2011), 45-101.
[16] Rubin, K., *A Stark conjecture "over* $\mathbb{Z}$*" for abelian L-functions with multiple zeros*, Ann. Inst. Fourier (Grenoble) **46** (1996), 33-62.
[17] Sano, T., *Refined abelian Stark conjectures and the equivariant leading term conjecture of Burns*, Compos. Math. **150** (2014), 1809-1835.
[18] Tate, J., *Les conjectures de Stark sur les fonctions L d'Artin en* $s = 0$, vol. 47 of *Progress in Mathematics.*, Boston, Birkhäuser (1984).