RIMS Kôkyûroku Bessatsu
B53 (2015), 219–229

The determinant of a double covering of the projective space and the discriminant of the branch locus (announcement)

By

Yasuhiro TERAKADO*

Abstract

The determinant of the Galois action on the ℓ-adic cohomology of the middle degree of a proper smooth variety of even dimension defines a quadratic character of the absolute Galois group of the base field of the variety. In this announcement, we state that for a double covering of the projective space of even dimension, the character is computed via the square root of the discriminant of the defining polynomial of the covering. As a corollary, we deduce that the parity of a Galois permutation of the exceptional divisors on a del Pezzo surface can be computed by the discriminant.

§ 1. Introduction

In this article, we announce mainly the results of [14].

Let k be a field, $\bar{k}$ an algebraic closure of k and k_s the separable closure of k contained in $\bar{k}$. Let us denote the absolute Galois group of k by $G_k = \mathrm{Gal}(k_s/k) = \mathrm{Aut}_k(\bar{k})$. Further let X be a proper smooth variety of dimension n over k. If ℓ is a prime number invertible in k, the ℓ-adic cohomology $H^r(X_{\bar{k}}, \mathbb{Q}_\ell)$ defines a representation of G_k. First we recall basic facts on the one-dimensional ℓ-adic representation $\det R\Gamma(X_{\bar{k}}, \mathbb{Q}_\ell)$ of G_k defined as the alternating tensor product

$$\det R\Gamma(X_{\bar{k}}, \mathbb{Q}_\ell) = \bigotimes_{r=0}^{2n} \det H^r(X_{\bar{k}}, \mathbb{Q}_\ell)^{(-1)^r}$$

Received April 2, 2014. Revised May 23, 2015.
2010 Mathematics Subject Classification(s): 14J25, 11F80, 14F20, 14J20.
Key Words: *Galois representations, Discriminants, Branched coverings, Del Pezzo surfaces.*
*Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1 Komaba Meguro-ku Tokyo 153-8914, Japan.
e-mail: `terakado@ms.u-tokyo.ac.jp`

of the determinant characters. By Poincaré duality [3, Arcata, VI.Théorème 3.1], we have an isomorphism $\det H^r(X_{\bar{k}}, \mathbb{Q}_\ell) \otimes \det H^{2n-r}(X_{\bar{k}}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-nb_r)$. Here $\mathbb{Q}_\ell(i)$ denotes the i-th power of the ℓ-adic cyclotomic character and $b_r = \dim H^r(X_{\bar{k}}, \mathbb{Q}_\ell)$ denotes the r-th Betti number. Further, if the dimension n of X is odd, the cup product defines a $\mathbb{Q}_\ell(-n)$-valued non-degenerate alternating form on $H^n(X_{\bar{k}}, \mathbb{Q}_\ell)$, and hence the n-th Betti number b_n is even and $\det H^n(X_{\bar{k}}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-\frac{nb_n}{2})$. Thus, we have

$$\det R\Gamma(X_{\bar{k}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-\frac{n\chi}{2}) \otimes \begin{cases} 1 & \text{if } n \text{ is odd} \\ \kappa & \text{if } n \text{ is even} \end{cases} \tag{1.1}$$

where χ denotes the Euler number $\chi = \sum_{r=0}^{2n}(-1)^r \dim H^r(X_{\bar{k}}, \mathbb{Q}_\ell)$ and κ denotes the character of order at most 2 of G_k defined by $\kappa = \det H^n(X_{\bar{k}}, \mathbb{Q}_\ell(\frac{n}{2}))$.

Recently, Suh showed that the following statements on $\det H^r(X_{\bar{k}}, \mathbb{Q}_\ell)$ of each odd degree r for a projective variety extends to a proper smooth case. If X is a *projective* smooth variety, Poincaré duality [3] and the hard Lefschetz theorem [4] equip $H^r(X_{\bar{k}}, \mathbb{Q}_\ell)$ with a non-degenerate bilinear form that is alternating for r odd. In particular, the Betti number b_r is even and $\det H^r(X_{\bar{k}}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-\frac{rb_r}{2})$.

Theorem 1.1 ([12, Corollary 2.2.3, Corollary 3.3.5]). *Let X be a proper smooth variety over any field k. Assume that char $k \neq \ell$. Then for any odd integer $r \geq 1$,*

1. *the r-th Betti number b_r is even, and*
2. *the determinant character $\det H^r(X_{\bar{k}}, \mathbb{Q}_\ell)$ is equal to $\mathbb{Q}_\ell(-\frac{rb_r}{2})$.*

This is shown by Suh using cristalline cohomology. Sun and Zheng [13] showed this by using intersection cohomology.

Now we consider the case that the dimension n of X is even. Further, in this introduction, we assume that the characteristic of k is not 2. Our problem is to determine the quadratic extension (possibly trivial) of k corresponding to the quadratic character $\kappa = \det H^n(X_{\bar{k}}, \mathbb{Q}_\ell(\frac{n}{2}))$ in (1.1).

The cup product on the de Rham cohomology $H^n_{dR}(X/k)$ of the middle degree defines a non-degenerate symmetric bilinear form. Hence its discriminant $\delta_X \in k^\times/(k^\times)^2$ is defined. In this case, Saito showed the following theorem.

Theorem 1.2 ([9, Theorem 2]). *Assume that X is projective and smooth over k, and the dimension n of X is even. Let us denote $b^- = \sum_{i<n} \dim H^i_{dR}(X/k)$. Then the quadratic character $\kappa = \det H^n(X_{\bar{k}}, \mathbb{Q}_\ell(\frac{n}{2}))$ in (1.1) corresponds to the quadratic extension $k\left(\sqrt{(-1)^{\frac{n\chi}{2}+b^-}\delta_X}\right)/k$.*

For a hypersurface of even dimension in the projective space, the determinant is computed by using the discriminant of the defining polynomial of the variety.

Theorem 1.3 ([10, Theorem 3.5, Corollary 4.3]). *Assume that n is even and X is a smooth hypersurface of degree d in the projective space of dimension $n+1$, and let f be a homogeneous polynomial defining it. Let $\mathrm{disc}_d(f)$ be the divided discriminant of f (see Definition 2.1). Then the quadratic character κ corresponds to the quadratic extension $k(\sqrt{\epsilon(n,d)\cdot \mathrm{disc}_d(f)})/k$, where $\epsilon(n,d)$ is $(-1)^{\frac{d-1}{2}}$ if d is odd and is $(-1)^{\frac{d}{2}\cdot\frac{n+2}{2}}$ if d is even.*

In this article, we announce that the determinant of a double covering of the projective space is computed via the discriminant of the defining polynomial of the covering. Assume that the characteristic of k is not 2 and n is even. Let X be the smooth double covering of the projective space of dimension n defined by the equation $y^2 = f$ where f denotes a homogeneous polynomial of $n+1$ variables of even degree d with coefficients in k. The double covering X is branched along the hypersurface defined by f. Let us denote the divided discriminant of the polynomial f by $\mathrm{disc}_d(f)$ (see Definition 2.1).

Theorem 1.4. *The quadratic character κ corresponds to the quadratic extension $k\left(\sqrt{(-1)^{\frac{dn}{4}}\,\mathrm{disc}_d(f)}\right)/k$.*

In the last subsection, we focus on the double covering of the projective plane branched along a quartic curve. In this case, the group G_k acts on $H^2(X_{\bar{k}}, \mathbb{Q}_\ell(1))$ via a subgroup of the Weyl group $W(E_7)$ of the root system E_7. The kernel of the determinant map $W(E_7) \to \{\pm 1\}$ is a simple group of order $2^9\cdot 3^4\cdot 5\cdot 7$ isomorphic to $\mathrm{Sp}_6(\mathbb{F}_2)$ [1, pp.46-47]. We study the discriminant of the branched double covering and its relation to the subgroup of $W(E_7)$ of index two (Corollary 4.3).

§ 2. Discriminant

§ 2.1. The Discriminant of a hypersurface

We review some basic theory concerning the discriminant of a hypersurface studied in [10]. We fix integers $n \geq 0$ and $d \geq 2$. We consider the polynomial ring $\mathbb{Z}[T_0,\dots,T_{n+1}]$ and the free $\mathbb{Z}$-module $E = \bigoplus_{i=0}^{n+1}\mathbb{Z}\cdot T_i$. The d-th symmetric power $S^dE = \mathrm{Sym}^d E$ defined over $\mathbb{Z}$ is identified with the free $\mathbb{Z}$-module of finite rank consisting of homogeneous polynomials of degree d in $\mathbb{Z}[T_0,\dots,T_{n+1}]$ by the map $T_{j_1}\otimes\cdots\otimes T_{j_d} \mapsto T_{j_i}\cdots T_{j_d}$. Note that S^dE is defined over the integer ring, where d is not invertible. If $I = (i_0,\dots,i_{n+1}) \in \mathbb{N}^{n+2}$ is a multi-index, we put $T^I = T_0^{i_0}\cdots T_{n+1}^{i_{n+1}} \in \mathbb{Z}[T_0,\dots,T_{n+1}]$ and $|I| = i_0+\cdots+i_{n+1}$. The monomials T^I of degree $|I| = d$ form a basis of S^dE. Let $(C_I)_{|I|=d}$ be the dual basis of $(S^dE)^\vee$ and define the universal polynomial $F = \sum_{|I|=d} C_I T^I$.

We consider the resultant

$$\operatorname{res}(D_0F, \ldots, D_{n+1}F)$$

of its partial derivatives $D_0F, \ldots, D_{n+1}F$. It is a homogeneous polynomial of degree $m = (n+2)(d-1)^{n+1}$ in $(C_I)_{|I|=d}$ with integral coefficients. If we put

$$a(n,d) = \frac{(d-1)^{n+2} - (-1)^{n+2}}{d},$$

the greatest common divisor of the coefficients is $d^{a(n,d)}$ by [7, Chap. 13.1.D Proposition 1.7].

Definition 2.1. We call

$$\operatorname{disc}_d(F) = \frac{1}{d^{a(n,d)}} \operatorname{res}(D_0F, \ldots, D_{n+1}F)$$

the divided discriminant of F.

The divided discriminant $\operatorname{disc}_d(F)$ is known to be geometrically irreducible in characteristic 0.

Put

$$\epsilon(n,d) = \begin{cases} (-1)^{\frac{d-1}{2}} & \text{if } d \text{ is odd} \\ (-1)^{\frac{d}{2}\frac{n+2}{2}} & \text{if } d \text{ is even.} \end{cases}$$

Theorem 2.2 ([10, Theorem 4.2]). *Assume that n is even. Then, there exist homogeneous polynomials $A \in S^{\frac{m}{2}}((S^dE)^\vee)$ and $B \in S^m((S^dE)^\vee)$ where $m = (n+2)(d-1)^{n+1}$ satisfying $\epsilon(n,d) \cdot \operatorname{disc}_d(F) = A^2 + 4B$.*

Example 2.3 (Quadrics). We assume that the degree d equals to 2. Let $F = \sum_{0 \le i \le j \le n+1} c_{ij} T_i T_j$ be the universal quadric polynomial and $A = (a_{ij})$ be the $(n+2) \times (n+2)$ symmetric matrix with coefficients in $\mathbb{Z}[c_{ij}; 0 \le i \le j \le n+1]$ defined by $a_{ij} = a_{ji} = c_{ij}$ for $i < j$ and $a_{ii} = 2c_{ii}$. We have $TA^tT = 2F$ where T is the row vector $(T_0, \ldots, T_{n+1})$. Then $\operatorname{res}(D_0F, \ldots, D_{n+1}F) = \det A$ and $a(n,2) = (1-(-1)^n)/2$. Thus,

$$\operatorname{disc}_d(F) = \begin{cases} 2^{-1} \det A & \text{if } n \text{ is odd} \\ \det A & \text{if } n \text{ is even,} \end{cases} \tag{2.1}$$

$$\deg(\operatorname{disc}_d(F)) = n+2.$$

Example 2.4 (Binary forms). We assume that $n = 0$ and $d > 1$. Let $F = C_0T_0^d + C_1T_0^{d-1}T_1 + \cdots + C_dT_1^d$ be the universal binary polynomial of degree d. The divided discriminant $\operatorname{disc}_d(F)$ is a homogeneous polynomial in (C_i) of degree $2d-2$ and the sign $\epsilon(0,d) = (-1)^{d(d-1)/2}$. The discriminant $\epsilon(0,d) \cdot \operatorname{disc}_d(F)$ is equal to $\operatorname{dis}_d(F) = \tilde{\Delta}(C_0, \ldots, C_d)$ in the notation of [2, Chap.4, Section 6, n°7, formula(52)] where the subscript d stands for the degree.

§ 2.2. The Discriminant of a complete intersection

Next we review the discriminant of a complete intersection studied in [11]. We fix integers $0 \leq r \leq n$. We consider the polynomial ring $\mathbb{Z}[T_0, \ldots, T_{n+r}]$ and the free $\mathbb{Z}$-module $E = \bigoplus_{i=0}^{n+r} \mathbb{Z} \cdot T_i$. We identify the d-th symmetric power $S^d E = \mathrm{Sym}^d E$ defined over $\mathbb{Z}$ with the free $\mathbb{Z}$-module of finite rank consisting of homogeneous polynomials of degree d in $\mathbb{Z}[T_0, \ldots, T_{n+r}]$. Further we fix integers $d_1, \ldots, d_r > 1$. If $I = (i_0, \ldots, i_{n+r}) \in \mathbb{N}^{n+r+1}$ is a multi-index, we put $T^I = T_0^{i_0} \cdots T_n^{i_{n+r}} \in \mathbb{Z}[T_0, \ldots, T_{n+r}]$ and $|I| = i_0 + \cdots + i_{n+r}$. The monomials T^I of degree $|I| = d_j$ form a basis of $S^{d_j} E$. Let $(C_I^{(j)})_{|I|=d_j}$ be the dual basis of $(S^{d_j} E)^\vee$ and define the universal polynomial $F_j = \sum_{|I|=d_j} C_I^{(j)} T^I$.

We put $\mathbb{P}^{n+r} = \mathbb{P}(E) = \mathrm{Proj}\, \mathbb{Z}[T_0, \ldots, T_{n+r}]$. Further we put $V = \bigoplus_{1 \leq j \leq r} S^{d_j} E$ and let $\check{P} = \mathbb{P}(\check{V})$ be the projective space defined by the dual $\check{V} = \mathrm{Hom}(V, \mathbb{Z})$. Then we define the universal family $X \subset \mathbb{P}^{n+r} \times \check{P}$ of complete intersections by the equations $F_1 = \cdots = F_r = 0$.

Let $\pi : X \subset \mathbb{P}^{n+r} \times \check{P} \to \check{P}$ be the canonical map and $U \subset \check{P}$ be the maximal open subscheme over which π is smooth of relative dimension n. Let $D = \check{P} - U$ be the reduced closed subscheme in $\check{P}$.

Proposition 2.5 ([11]). *There exists a geometrically irreducible polynomial in $(C_I^{(j)})_{1 \leq j \leq r, |I|=d_j}$ with coefficients in $\mathbb{Z}$ uniquely defined up to ± 1, such that it defines the closed subscheme $D \subset \check{P}$.*

Definition 2.6. We call the polynomial up to sign in Proposition 2.5 the discriminant and we denote it by $\mathrm{disc}(F_1, \ldots, F_r)$.

Proposition 2.7 ([11]). *1. Let p be a prime. Except for $p = 2$ and n being even, the polynomial $\mathrm{disc}(F_1, \ldots, F_r) \mod p$ is geometrically irreducible.*

2. Assume that n is even. Then the degree $m = \deg \mathrm{disc}(F_1, \ldots, F_r)$ is even and the sign of $\mathrm{disc}(F_1, \ldots, F_r)$ is uniquely defined by the condition that there exist homogeneous polynomials $A \in S^{\frac{m}{2}}(\check{V})$ and $B \in S^m(\check{V})$ such that $\mathrm{disc}(F_1, \ldots, F_r) = A^2 + 4B$.

The relation between the discriminant of a complete intersection and the divided discriminant of a hypersurface is as follows.

Proposition 2.8 ([10, Proposition 2.3]). *If $r = 1$ and $d_1 = d$, then the discriminant $\mathrm{disc}(F_1)$ defined in Definition 2.6 corresponds to $\mathrm{disc}_d(F_1)$ up to ± 1.*

Now we state that the discriminant of the complete intersection of two quadric hypersurfaces has an explicit presentation by the discriminant of a quadric form (Example 2.3) and one of a binary form (Example 2.4). Let $F_1 = \sum_{0 \leq i \leq j \leq n+2} a_{ij} T_i T_j, F_2 = \sum_{0 \leq i \leq j \leq n+2} b_{ij} T_i T_j$ be the two universal quadric forms. If we regard a polynomial

$t_1F_1+t_2F_2$ as quadric form in $T_0,\dots,T_{n+2}$, we can take the divided discriminant of the quadric form $\mathrm{disc}_d(t_1F_1+t_2F_2)$. Further we see $\mathrm{disc}_d(t_1F_1+t_2F_2)$ as a binary form in t_1,t_2 and we can take the divided discriminant $\mathrm{disc}_d(\mathrm{disc}_d(t_1F_1+t_2F_2))$.

Theorem 2.9 ([15]). *1. If n is even, then the equation*

$$\mathrm{disc}(F_1,F_2)=(-1)^{\frac{n}{2}+1}\,\mathrm{disc}_d(\mathrm{disc}_d(t_1F_1+t_2F_2))$$

holds where the left hand side is the discriminant defined in Proposition 2.7.2.

2. If n is odd, then the equation

$$\mathrm{disc}(F_1,F_2)=2^{-2(n+3)}\,\mathrm{disc}_d(\mathrm{disc}_d(t_1F_1+t_2F_2))$$

holds up to ± 1.

§ 3. Determinant

Let X be a proper smooth variety of even dimension n over a field k. If ℓ is a prime number invertible in k, the ℓ-adic cohomology of middle degree $H^n(X_{\overline{k}},\mathbb{Q}_\ell(\frac{n}{2}))$ defines an orthogonal representation of the absolute Galois group G_k. The determinant

$$\kappa:=\det H^n(X_{\overline{k}},\mathbb{Q}_\ell(\frac{n}{2})):G_k\to\{\pm1\}\subset\mathbb{Q}_\ell^\times$$

is independent of the choice of ℓ.

Saito showed the following theorems.

Theorem 3.1 ([10, Theorem 3.5, Corollary 4.3]). *Let X be a smooth hypersurface of degree d in $\mathbb{P}_k^{n+1}$, and let f be a homogeneous polynomial defining it. Let $\mathrm{disc}_d(f)$ be the divided discriminant of f.*

1. If char $k\neq 2$, then the quadratic character κ is defined by the square root of $\epsilon(n,d)\cdot\mathrm{disc}_d(f)$.

2. If char $k=2$, then the quadratic character κ is defined by the Artin-Schreier equation $t^2+t=B(f)A(f)^{-2}$ where A and B are polynomials occuring in Theorem 2.2.

Theorem 3.2 ([11]). *Let X be a smooth complete intersection of multi-degree $d_1,\dots,d_r$ in $\mathbb{P}_k^{n+r}$, and let $f_1,\dots,f_r$ be homogeneous polynomials with coefficients in k defining X.*

1. If char $k\neq 2$, then the quadratic character κ is defined by the square root of $\mathrm{disc}(f_1,\dots,f_r)$, where $\mathrm{disc}(f_1,\dots,f_r)$ is the discriminant whose sign is defined by the condition in Proposition 2.7.2.

2. If char $k=2$, then the quadratic character κ is defined by the Artin-Schreier equation $t^2+t=B(f_1,\dots,f_r)A(f_1,\dots,f_r)^{-2}$ where A and B are the polynomials occuring in Proposition 2.7.2.

For the complete intersection of two quadrics, we see the following by Theorem 2.9 and Theorem 3.2.

Theorem 3.3 ([15]). *Assume that n is even. Let $X \subset \mathbb{P}_k^{n+2}$ be a smooth complete intersection defined by quadric forms f_1, f_2. If char $k \neq 2$, then the quadratic character $\kappa = \det H^n(X_{\overline{k}}, \mathbb{Q}_\ell(\frac{n}{2}))$ is defined by the square root of $(-1)^{\frac{n}{2}+1} \operatorname{disc}_d(\operatorname{disc}_d(t_1F_1 + t_2F_2))$.*

We state the theorem for a double covering of a projective space of even dimension. Assume that the characteristic of k is not 2. Let f be a homogeneous polynomial of $n+1$ variables of even degree d with coefficients in k and X be the double covering of a projective space of dimension n defined by the equation $y^2 = f$. Let $\operatorname{disc}_d(f)$ be the divided discriminant of f.

Theorem 3.4 ([14]). *The quadratic character $\kappa = \det H^n(X_{\overline{k}}, \mathbb{Q}_\ell(\frac{n}{2}))$ of G_k is defined by the square root of $(-1)^{\frac{dn}{4}} \operatorname{disc}_d(f)$.*

The proof of the Theorem 3.4 consists of two parts. First we follow the method given by T. Saito in [10] and a standard argument on universal family connects the determinant character κ with the square root of the discriminant. Second, we then determine the constant multiple of the discriminant by a specialization argument.

Next we state the theorem in the case characteristic 2. Assume that d is even. We consider the more general defining polynomial $y^2 + ay = b$, where a and b are homogeneous polynomials of degree $\frac{d}{2}$ and d over k. Let $A = \sum_{|I|=\frac{d}{2}} R_I T^I$ and $B = \sum_{|J|=d} S_J T^J$ be the universal polynomials of degree $\frac{d}{2}$ and d. Then the greatest common divisor of the coefficients of the polynomial $\operatorname{disc}_d(A^2 + 4B) \in \mathbb{Z}[(R_I)_{|I|=\frac{d}{2}}, (S_J)_{|J|=d}]$ is $4^{ds(n-1,d)}$, where $s(n-1,d) = \frac{(n+1)(d-1)^n - a(n-1,d)}{d}$. Further, there exist polynomials $C, D \in \mathbb{Z}[(R_I)_{|I|=\frac{d}{2}}, (S_J)_{|J|=d}]$ satisfying $4^{-ds(n-1,d)}(-1)^{\frac{dn}{4}} \operatorname{disc}_d(A^2 + 4B) = C^2 + 4D$.

Theorem 3.5 ([14]). *Assume that d is even. Let k be a field of the characteristic equal to 2 and Z be a smooth branched double covering of the projective space $\mathbb{P}_k^n$ of even dimension n defined by an equation $y^2 + ay = b$ where a and b are homogeneous polynomials of degree $\frac{d}{2}$ and d over k. Let $C(a,b) \in k^\times$ and $D(a,b) \in k$ denote the specializations of the polynomials C and D noted as above. Then, the quadratic character $\det H^n(Z_{\bar{k}}, \mathbb{Q}_\ell(\frac{n}{2}))$ of G_k is defined by the Artin-Schreier equation $t^2 + t = D(a,b) \cdot C(a,b)^{-2}$.*

§ 4. The determinant of a del Pezzo surface

Let k be a field of characteristic $\neq 2$. A del Pezzo surface over k is a smooth projective surface S whose anti-canonical sheaf $(\Omega^2_{S/k})^{-1}$ is ample and whose base change

$S_{\overline{k}}$ is birationally equivalent to $\mathbb{P}^2_{\overline{k}}$. We call the intersection number $d = \omega_S \cdot \omega_S$ its degree, where ω_S is the class of the canonical sheaf $\Omega^2_{S/k}$ in $\mathrm{Pic}(S)$. Manin[8, Theorem 24.3] gives bounds for this degree, namely $1 \leq d \leq 9$.

Let S be a del Pezzo surface of degree $d \leq 6$. Then $H^2(S_{\overline{k}}, \mathbb{Q}_\ell(1))$ is of dimension $10 - d$ and is spanned by the classes of their exceptional curves. The group of automorphisms of the $\mathbb{Z}$-lattice spanned by the classes of these exceptional curves permutating them and preserving the intersection form is isomorphic to the Weyl group $W(R_{9-d})$ of the root system R_{9-d}. Here $R_3, \ldots, R_8$ denotes the type $A_1 \times A_2, A_4, D_5, E_6, E_7, E_8$, respectively.

The action of G_k on the exceptional classes defines a homomorphism

$$G_k \to W(R_{9-d})$$

unique up to conjugation. We consider the composition of this homomorphism and the determinant map $G_k \to W(R_{9-d}) \to \{\pm 1\}$. We can apply the theorems in Section 3 to this map in the case of del Pezzo surfaces of degree $2, 3$ and 4.

§ 4.1. Del Pezzo surface of degree 4

A del Pezzo surface of degree 4 is isomorphic to a complete intersection of two quadric hypersurfaces in $\mathbb{P}^4_k$ after a suitable finite extension of the base field [5, Theorem 8.6.2]. Let $S \subset \mathbb{P}^4_k$ be a smooth complete intersection defined by quadric forms f_1, f_2. Then $H^2(S_{\overline{k}}, \mathbb{Q}_\ell(1))$ is spanned by the classes of 16 lines.

By applying the Theorem 3.3 and Theorem 3.2, we can describe the parity of the Galois permutation of the exceptional divisors as follows.

Corollary 4.1 ([15]). *The composition $G_k \to W(D_5) \to \{\pm 1\}$ is defined by the quadratic extension $k(\sqrt{\mathrm{disc}_d(\mathrm{disc}_d(t_1 f_1 + t_2 f_2))})/k$.*

§ 4.2. Del Pezzo surface of degree 3

Any del Pezzo surface of degree 3 is isomorphic to a cubic surface in $\mathbb{P}^3_k$ [8, §24, Theorem 24.4 and Theorem 24.5]. We note that the base fields of del Pezzo surfaces are assumed to be algebraically closed throughout [8, §24]. However, this result holds true with the same proof as the one given in the reference. Let $S \subset \mathbb{P}^3_k$ be a smooth cubic surface. Then $H^2(S_{\overline{k}}, \mathbb{Q}_\ell(1))$ is spanned by the classes of 27 lines.

It is known that the general enough cubic surface in characteristic $\neq 3$ can be put in the form

$$x + y + z + u + v = 0, \quad ax^3 + by^3 + cz^3 + du^3 + ev^3 = 0 \tag{4.1}$$

in $\mathbb{P}^4$ after a suitable finite extension of the ground field. The corresponding cubic surface is smooth if and only if the Salmon discriminant $\mathrm{disc}_s(a,b,c,d,e)$ is non-zero. The definition of the Salmon discriminant is:

$$\mathrm{disc}_s(a,b,c,d,e) = ((s^2 - 64rt)^2 - 4t^3p)^2 - 2^{11}(8t^6q + t^4s(s^2 - 4rt)),$$

where $p = a+b+c+d+e, q = ab+\cdots, r = abc+\cdots, s = abcd+\cdots, t = abcde$ are the elementary symmetric functions of a,b,c,d,e. By eliminating one variable in (4.1), one obtains a cubic polynomial $F_s = ax^3 + by^3 + cz^3 + du^3 - e(x+y+z+u)^3$. The relation between the Salmon discriminant and the divided one is given in [10]:

$$\mathrm{disc}_s(a,b,c,d,e) = 3^{-27}\,\mathrm{disc}_d(F_s).$$

By applying the above relation and Theorem 3.1, we see the following.

Corollary 4.2 ([6, Theorem 2.12] or [10, Section 5.4])**.** *Let k be the field of characteristic $\neq 2,3$ and $a,b,c,d,e \in k$ such that $\mathrm{disc}_s(a,b,c,d,e) \neq 0$. Further let $G_k \to W(E_6)$ be the homomorphism associated with the cubic surface of type (4.1). Then, the composition of this homomorphism and the determinant map $G_k \to W(E_6) \to \{\pm 1\}$ is defined by the quadratic extension $k(\sqrt{-3\,\mathrm{disc}_s(a,b,c,d,e)})/k$.*

§ 4.3. Del Pezzo surface of degree 2

A del Pezzo surface of degree 2 is isomorphic to a double covering of $\mathbb{P}^2$ branched along a quartic curve [5, Proposition 6.3.9 and Proposition 6.3.11]. Let $S \to \mathbb{P}^2$ be the double covering branched along the curve defined by a quartic form f. Then $H^2(S_{\overline{k}}, \mathbb{Q}_\ell(1))$ is spanned by the classes of 56 exceptional divisors. The kernel of the determinant map $W(E_7) \to \{\pm 1\}$ is a simple group of order $2^9 \cdot 3^4 \cdot 5 \cdot 7$ isomorphic to $\mathrm{Sp}_6(\mathbb{F}_2)$ [1, pp.46-47].

By applying Theorem 3.4, we see the following.

Corollary 4.3 ([14])**.** *The composition $G_k \to W(E_7) \to \{\pm 1\}$ is defined by the quadratic extension $k(\sqrt{\mathrm{disc}_d(f)})/k$.*

Let $F = \sum_{i_0+i_1+i_2=4} C_{(i_0,i_1,i_2)} T_0^{i_0} T_1^{i_1} T_2^{i_2}$ be the universal quartic polynomial. The discriminant of a quartic curve $\mathrm{disc}_d(F)$ is a homogeneous polynomial of degree 27 in 15 variables and $\mathrm{disc}_d(F) = 4^{-7}\,\mathrm{res}(D_0F, D_1F, D_2F)$.

We describe a formula for $\mathrm{res}(f_0,f_1,f_2)$ due to Sylvester. Let $f_0,f_1,f_2 \in k[x,y,z]$ be three homogeneous polynomials of degrees $d \geq 2$. For any three non-negative integers a,b,c such that $a+b+c = d-1$, every monomial of degree d is divisible by at least one of $x^{a+1}, y^{b+1}, z^{c+1}$. Then we can write f_0, f_1, f_2 in the form

$$f_0 = x^{a+1}P_0 + y^{b+1}Q_0 + z^{c+1}R_0$$
$$f_1 = x^{a+1}P_1 + y^{b+1}Q_1 + z^{c+1}R_1$$
$$f_2 = x^{a+1}P_2 + y^{b+1}Q_2 + z^{c+1}R_2$$

where the P_i, Q_i, R_i are homogeneous polynomials of degrees respectively $d-a-1, d-b-1, d-c-1$. Such a representation of the f_i is not unique. We define the polynomial

$$D_{abc} = \det \begin{pmatrix} P_0 & Q_0 & R_0 \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{pmatrix}$$

of degree $2d-2$. Then we consider the equations

$$\begin{aligned} x^\alpha f_0 &= 0, \quad (x^\alpha = x^{\alpha_0}y^{\alpha_1}z^{\alpha_2}, \alpha_0+\alpha_1+\alpha_2 = d-2) \\ x^\alpha f_1 &= 0, \quad (x^\alpha = x^{\alpha_0}y^{\alpha_1}z^{\alpha_2}, \alpha_0+\alpha_1+\alpha_2 = d-2) \\ x^\alpha f_2 &= 0, \quad (x^\alpha = x^{\alpha_0}y^{\alpha_1}z^{\alpha_2}, \alpha_0+\alpha_1+\alpha_2 = d-2) \\ D_{abc} &= 0, \quad (a+b+c = d-1). \end{aligned}$$

Each polynomial on the left hand side has degree $2d-2$, and there are $2d^2-d$ monomials of this degree. Further the number of the equations is $2d^2-d$. Thus the coefficient matrix of the equations is a $(2d^2-d) \times (2d^2-d)$ matrix and we denote it by C_d. Now the result of Sylvester is as follows.

Theorem 4.4 (see e.g. [7, 3.4.D Theorem 4.10])**.** *The resultant of three homogeneous polynomials f_0, f_1, f_2 of degree $d \geq 2$ is given by*

$$\operatorname{res}(f_0, f_1, f_2) = \pm \det C_d.$$

By applying the above theorem for $f_0 = D_0F, f_1 = D_1F$ and $f_2 = D_2F$ where F is the universal quartic polynomial, one has an explicit presentation of the discriminant $\operatorname{disc}_d(F)$ of a quartic curve.

References

[1] J.H. Conway, et al., Atlas of Finite Groups, Clarendon Press, Oxford, 1985.
[2] N. Bourbaki, Algèbre - Chapitres 4 à 7, Paris, Masson, 1981; new printing, Springer-Verlag, 2006; English translation, Algebra II, Springer-Verlag, 1989.
[3] P. Deligne, Cohomologie étale: Seminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2, Lecture Notes in Mathematics, vol. 569 Springer, 1977.

[4] P. Deligne, La conjecture de Weil. II, Publ. Math. Inst. Hautes Études Sci. 52 (1980), 137-252.
[5] I. Dolgachev, Classical algebraic geometry : A modern view, Cambridge University Press, Cambridge, 2012.
[6] A.-S. Elsenhans and J. Jahnel, The discriminant of a cubic surface, Geometriae dedicata 159 (2012), 29-40.
[7] I. M. Gelfand, M. M. Kapranov, A.V. Zelevinsky, Discriminants, Resultants, and Multidimensional Determinants, Birkhäuser, Boston, 1994.
[8] Y. I. Manin, Cubic Forms : Algebra, Geometry, Arithmetic, second edit., North-Holland, Amsterdam, 1986.
[9] T. Saito, Jacobi sum Hecke characters, de Rham discriminant, and the determinant of ℓ-adic cohomologies, J. of Algebraic Geometry 3 (1994), 411-434.
[10] T. Saito, The discriminant and the determinant of a hypersurface of even dimension, Mathematical Research Letters 19 (2012), no. 04, 855-871.
[11] T. Saito, The discriminant and the determinant of a complete intersection, in preparation.
[12] J. Suh, Symmetry and parity in Frobenius action on cohomology, Compos. Math. 148 (2012), no. 1, 295-303.
[13] S. Sun, W. Zheng, Parity and symmetry in intersection and ordinary cohomology, arXiv:1402.1292.
[14] Y. Terakado, The determinant of a double covering of the projective space of even dimension and the discriminant of the branch locus, preprint.
[15] Y. Terakado, The discriminant of the complete intersection of two quadrics, preprint.