The determinant of a double covering of the projective space and the discriminant of the branch locus (announcement)

By

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Abstract

The determinant of the Galois action on the \( \ell \)-adic cohomology of the middle degree of a proper smooth variety of even dimension defines a quadratic character of the absolute Galois group of the base field of the variety. In this announcement, we state that for a double covering of the projective space of even dimension, the character is computed via the square root of the discriminant of the defining polynomial of the covering. As a corollary, we deduce that the parity of a Galois permutation of the exceptional divisors on a del Pezzo surface can be computed by the discriminant.

§1. Introduction

In this article, we announce mainly the results of [14].

Let \( k \) be a field, \( \overline{k} \) an algebraic closure of \( k \) and \( k_s \) the separable closure of \( k \) contained in \( \overline{k} \). Let us denote the absolute Galois group of \( k \) by \( G_k = \text{Gal}(k_s/k) = \text{Aut}_k(\overline{k}) \). Further let \( X \) be a proper smooth variety of dimension \( n \) over \( k \). If \( \ell \) is a prime number invertible in \( k \), the \( \ell \)-adic cohomology \( H^r(X_{\overline{k}}, \mathbb{Q}_\ell) \) defines a representation of \( G_k \). First we recall basic facts on the one-dimensional \( \ell \)-adic representation \( \det R \Gamma(X_{\overline{k}}, \mathbb{Q}_\ell) \) of \( G_k \) defined as the alternating tensor product

\[
\det R \Gamma(X_{\overline{k}}, \mathbb{Q}_\ell) = \bigotimes_{r=0}^{2n} \det H^r(X_{\overline{k}}, \mathbb{Q}_\ell)^{(-1)^r}
\]
of the determinant characters. By Poincaré duality [3, Arcata, VI.Théorème 3.1], we have an isomorphism \( \det H^r(X_{\overline{k}}, \mathbb{Q}_\ell) \otimes \det H^{2n-r}(X_{\overline{k}}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-nb_r) \). Here \( \mathbb{Q}_\ell(i) \) denotes the \( i \)-th power of the \( \ell \)-adic cyclotomic character and \( b_r = \dim H^r(X_{\overline{k}}, \mathbb{Q}_\ell) \) denotes the \( r \)-th Betti number. Further, if the dimension \( n \) of \( X \) is odd, the cup product defines a \( \mathbb{Q}_\ell(-n) \)-valued non-degenerate alternating form on \( H^n(X_{\overline{k}}, \mathbb{Q}_\ell) \), and hence the \( n \)-th Betti number \( b_n \) is even and \( \det H^n(X_{\overline{k}}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-\frac{nb_n}{2}) \). Thus, we have

\[
(1.1) \quad \det R\Gamma(X_{\overline{k}}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell(-\frac{n\chi}{2}) \otimes \begin{cases} 
1 & \text{if } n \text{ is odd} \\
\kappa & \text{if } n \text{ is even}
\end{cases}
\]

where \( \chi \) denotes the Euler number \( \chi = \sum_{r=0}^{2n} (-1)^r \dim H^r(X_{\overline{k}}, \mathbb{Q}_\ell) \) and \( \kappa \) denotes the character of order at most 2 of \( G_{\ell} \) defined by \( \kappa = \det H^n(X_{\overline{k}}, \mathbb{Q}_\ell(\frac{n}{2})) \).

Recently, Suh showed that the following statements on \( \det H^r(X_{\overline{k}}, \mathbb{Q}_\ell) \) of each odd degree \( r \) for a projective variety extends to a proper smooth case. If \( X \) is a projective smooth variety, Poincaré duality [3] and the hard Lefschetz theorem [4] equip \( H^r(X_{\overline{k}}, \mathbb{Q}_\ell) \) with a non-degenerate bilinear form that is alternating for \( r \) odd. In particular, the Betti number \( b_r \) is even and \( \det H^r(X_{\overline{k}}, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-\frac{rb_r}{2}) \).

**Theorem 1.1** ([12, Corollary 2.2.3, Corollary 3.3.5]). Let \( X \) be a proper smooth variety over any field \( k \). Assume that \( \text{char } k \neq \ell \). Then for any odd integer \( r \geq 1 \),

1. the \( r \)-th Betti number \( b_r \) is even, and
2. the determinant character \( \det H^r(X_{\overline{k}}, \mathbb{Q}_\ell) \) is equal to \( \mathbb{Q}_\ell(-\frac{rb_r}{2}) \).

This is shown by Suh using cristalline cohomology. Sun and Zheng [13] showed this by using intersection cohomology.

Now we consider the case that the dimension \( n \) of \( X \) is even. Further, in this introduction, we assume that the characteristic of \( k \) is not 2. Our problem is to determine the quadratic extension (possibly trivial) of \( k \) corresponding to the quadratic character \( \kappa = \det H^n(X_{\overline{k}}, \mathbb{Q}_\ell(\frac{n}{2})) \) in (1.1).

The cup product on the de Rham cohomology \( H^{2n}_{dR}(X/k) \) of the middle degree defines a non-degenerate symmetric bilinear form. Hence its discriminant \( \delta_X \in k^\times/(k^\times)^2 \) is defined. In this case, Saito showed the following theorem.

**Theorem 1.2** ([9, Theorem 2]). Assume that \( X \) is projective and smooth over \( k \), and the dimension \( n \) of \( X \) is even. Let us denote \( b^- = \sum_{i<n} \dim H^i_{dR}(X/k) \). Then the quadratic character \( \kappa = \det H^n(X_{\overline{k}}, \mathbb{Q}_\ell(\frac{n}{2})) \) in (1.1) corresponds to the quadratic extension \( k \left( \sqrt{(-1)^{\frac{n\chi}{2}+b^-} \delta_X} \right) /k \).

For a hypersurface of even dimension in the projective space, the determinant is computed by using the discriminant of the defining polynomial of the variety.
Theorem 1.3 ([10, Theorem 3.5, Corollary 4.3]). Assume that $n$ is even and $X$ is a smooth hypersurface of degree $d$ in the projective space of dimension $n+1$, and let $f$ be a homogeneous polynomial defining it. Let $\text{disc}_d(f)$ be the divided discriminant of $f$ (see Definition 2.1). Then the quadratic character $\kappa$ corresponds to the quadratic extension $k(\sqrt{\epsilon(n,d) \cdot \text{disc}_d(f)})/k$, where $\epsilon(n,d)$ is $(-1)^{\frac{d(n+2)}{4}}$ if $d$ is odd and is $(-1)^{\frac{d-1}{2}} \cdot \frac{n+2}{2}$ if $d$ is even.

In this article, we announce that the determinant of a double covering of the projective space is computed via the discriminant of the defining polynomial of the covering. Assume that the characteristic of $k$ is not 2 and $n$ is even. Let $X$ be the smooth double covering of the projective space of dimension $n$ defined by the equation $y^2 = f$ where $f$ denotes a homogeneous polynomial of $n+1$ variables of even degree $d$ with coefficients in $k$. The double covering $X$ is branched along the hypersurface defined by $f$. Let us denote the divided discriminant of the polynomial $f$ by $\text{disc}_d(f)$ (see Definition 2.1).

Theorem 1.4. The quadratic character $\kappa$ corresponds to the quadratic extension $k\left(\sqrt{(-1)^{\frac dn} \text{disc}_d(f)}\right)/k$.

In the last subsection, we focus on the double covering of the projective plane branched along a quartic curve. In this case, the group $G_k$ acts on $H^2(X, \mathbb{Q}_\ell(1))$ via a subgroup of the Weyl group $W(E_7)$ of the root system $E_7$. The kernel of the determinant map $W(E_7) \to \{\pm 1\}$ is a simple group of order $2^5 \cdot 3^4 \cdot 5 \cdot 7$ isomorphic to $\text{Sp}_6(\mathbb{F}_2)$ [1, pp.46-47]. We study the discriminant of the branched double covering and its relation to the subgroup of $W(E_7)$ of index two (Corollary 4.3).

§ 2. Discriminant

§ 2.1. The Discriminant of a hypersurface

We review some basic theory concerning the discriminant of a hypersurface studied in [10]. We fix integers $n \geq 0$ and $d \geq 2$. We consider the polynomial ring $\mathbb{Z}[T_0, \ldots, T_{n+1}]$ and the free $\mathbb{Z}$-module $E = \bigoplus_{i=0}^{n+1} \mathbb{Z} \cdot T_i$. The $d$-th symmetric power $S^dE = \text{Sym}^dE$ defined over $\mathbb{Z}$ is identified with the free $\mathbb{Z}$-module of finite rank consisting of homogeneous polynomials of degree $d$ in $\mathbb{Z}[T_0, \ldots, T_{n+1}]$ by the map $T_{j_1} \otimes \cdots \otimes T_{j_d} \mapsto T_{j_1} \cdots T_{j_d}$. Note that $S^dE$ is defined over the integer ring, where $d$ is not invertible. If $I = (i_0, \ldots, i_{n+1}) \in \mathbb{N}^{n+2}$ is a multi-index, we put $T^I = T_{i_0} \cdots T_{i_{n+1}} \in \mathbb{Z}[T_0, \ldots, T_{n+1}]$ and $|I| = i_0 + \cdots + i_{n+1}$. The monomials $T^I$ of degree $|I| = d$ form a basis of $S^dE$. Let $(C_I)_{|I|=d}$ be the dual basis of $(S^dE)^\vee$ and define the universal polynomial $F = \sum_{|I|=d} C_I T^I$. 

We consider the resultant
\[ \text{res}(D_0 F, \ldots, D_{n+1} F) \]
of its partial derivatives \( D_0 F, \ldots, D_{n+1} F \). It is a homogeneous polynomial of degree \( m = (n + 2)(d - 1)^{n+1} \) in \((C_I)_{|I|=d}\) with integral coefficients. If we put
\[ a(n, d) = \frac{(d-1)^{n+2} - (-1)^{n+2}}{d}, \]
the greatest common divisor of the coefficients is \( d^{a(n,d)} \) by [7, Chap. 13.1.D Proposition 1.7].

**Definition 2.1.** We call
\[ \text{disc}_d(F) = \frac{1}{d^{a(n,d)}} \text{res}(D_0 F, \ldots, D_{n+1} F) \]
the divided discriminant of \( F \).

The divided discriminant \( \text{disc}_d(F) \) is known to be geometrically irreducible in characteristic 0.

Put
\[ \epsilon(n, d) = \begin{cases} (-1)^{\frac{d-1}{2}} & \text{if } d \text{ is odd} \\ (-1)^{\frac{n+2}{2}} & \text{if } d \text{ is even}. \end{cases} \]

**Theorem 2.2** ([10, Theorem 4.2]). Assume that \( n \) is even. Then, there exist homogeneous polynomials \( A \in S^\frac{m}{2}((S^d E)^\vee) \) and \( B \in S^m((S^d E)^\vee) \) where \( m = (n + 2)(d - 1)^{n+1} \) satisfying \( \epsilon(n, d) \cdot \text{disc}_d(F) = A^2 + 4B \).

**Example 2.3** (Quadrics). We assume that the degree \( d \) equals to 2. Let \( F = \sum_{0 \leq i \leq j \leq n+1} c_{ij} T_i T_j \) be the universal quadric polynomial and \( A = (a_{ij}) \) be the \((n+2) \times (n+2)\) symmetric matrix with coefficients in \( \mathbb{Z}[c_{ij}; 0 \leq i \leq j \leq n+1] \) defined by \( a_{ij} = a_{ji} = c_{ij} \) for \( i < j \) and \( a_{ii} = 2c_{ii} \). We have \( TA^T = 2F \) where \( T \) is the row vector \((T_0, \ldots, T_{n+1})\). Then \( \text{res}(D_0 F, \ldots, D_{n+1} F) = \det A \) and \( a(n, 2) = (1 - (-1)^n)/2 \). Thus,

\[ \text{disc}_d(F) = \begin{cases} 2^{-1} \det A & \text{if } n \text{ is odd} \\ \det A & \text{if } n \text{ is even}, \end{cases} \]

\[ \deg(\text{disc}_d(F)) = n + 2. \]

**Example 2.4** (Binary forms). We assume that \( n = 0 \) and \( d > 1 \). Let \( F = C_0 T_0^d + C_1 T_0^{d-1} T_1 + \cdots + C_d T_1^d \) be the universal binary polynomial of degree \( d \). The divided discriminant \( \text{disc}_d(F) \) is a homogeneous polynomial in \((C_i)\) of degree \( 2d - 2 \) and the sign \( \epsilon(0,d) = (-1)^{d(d-1)/2} \). The discriminant \( \epsilon(0,d) \cdot \text{disc}_d(F) \) is equal to \( \text{dis}_d(F) = \Delta(C_0, \ldots, C_d) \) in the notation of [2, Chap.4, Section 6, n°7, formula(52)] where the subscript \( d \) stands for the degree.
§ 2.2. The Discriminant of a complete intersection

Next we review the discriminant of a complete intersection studied in [11]. We fix integers $0 \leq r \leq n$. We consider the polynomial ring $\mathbb{Z}[T_0, \ldots, T_{n+r}]$ and the free $\mathbb{Z}$-module $E = \bigoplus_{i=0}^{n+r} \mathbb{Z} \cdot T_i$. We identify the $d$-th symmetric power $S^d E = \text{Sym}^d E$ defined over $\mathbb{Z}$ with the free $\mathbb{Z}$-module of finite rank consisting of homogeneous polynomials of degree $d$ in $\mathbb{Z}[T_0, \ldots, T_{n+r}]$. Further we fix integers $d_1, \ldots, d_r > 1$. If $I = (i_0, \ldots, i_{n+r}) \in \mathbb{N}^{n+r+1}$ is a multi-index, we put $T^I = T_0^{i_0} \cdots T_{n+r}^{i_{n+r}} \in \mathbb{Z}[T_0, \ldots, T_{n+r}]$ and $|I| = i_0 + \cdots + i_{n+r}$. The monomials $T^I$ of degree $|I| = d_j$ form a basis of $S^{d_j} E$. Let $(C_I^{(j)})_{|I|=d_j}$ be the dual basis of $(S^{d_j} E)^\vee$ and define the universal polynomial $F_j = \sum_{|I|=d_j} C_I^{(j)} T^I$. We put $\mathbb{P}^{n+r} = \mathbb{P}(E) = \text{Proj} \mathbb{Z}[T_0, \ldots, T_{n+r}]$. Further we put $V = \bigoplus_{1 \leq j \leq r} S^{d_j} E$ and let $\check{P} = \mathbb{P}(\check{V})$ be the projective space defined by the dual $\check{V} = \text{Hom}(V, \mathbb{Z})$. Then we define the universal family $X \subset \mathbb{P}^{n+r} \times \check{P}$ of complete intersections by the equations $F_1 = \cdots = F_r = 0$.

Let $\pi : X \subset \mathbb{P}^{n+r} \times \check{P} \to \check{P}$ be the canonical map and $U \subset \check{P}$ be the maximal open subscheme over which $\pi$ is smooth of relative dimension $n$. Let $D = \check{P} - U$ be the reduced closed subscheme in $\check{P}$.

**Proposition 2.5 ([11]).** There exists a geometrically irreducible polynomial in $(C_I^{(j)})_{1 \leq j \leq r, |I|=d_j}$ with coefficients in $\mathbb{Z}$ uniquely defined up to $\pm 1$, such that it defines the closed subscheme $D \subset \check{P}$.

**Definition 2.6.** We call the polynomial up to sign in Proposition 2.5 the discriminant and we denote it by $\text{disc}(F_1, \ldots, F_r)$.

**Proposition 2.7 ([11]).** 1. Let $p$ be a prime. Except for $p = 2$ and $n$ being even, the polynomial $\text{disc}(F_1, \ldots, F_r)$ mod $p$ is geometrically irreducible.

2. Assume that $n$ is even. Then the degree $m = \text{deg} \text{disc}(F_1, \ldots, F_r)$ is even and the sign of $\text{disc}(F_1, \ldots, F_r)$ is uniquely defined by the condition that there exist homogeneous polynomials $A \in S^m(\check{V})$ and $B \in S^m(\check{V})$ such that $\text{disc}(F_1, \ldots, F_r) = A^2 + 4B$.

The relation between the discriminant of a complete intersection and the divided discriminant of a hypersurface is as follows.

**Proposition 2.8 ([10, Proposition 2.3]).** If $r = 1$ and $d_1 = d$, then the discriminant $\text{disc}(F_1)$ defined in Definition 2.6 corresponds to $\text{disc}_d(F_1)$ up to $\pm 1$.

Now we state that the discriminant of the complete intersection of two quadric hypersurfaces has an explicit presentation by the discriminant of a quadric form (Example 2.3) and one of a binary form (Example 2.4). Let $F_1 = \sum_{0 \leq i \leq j \leq n+2} a_{ij} T_i T_j$, $F_2 = \sum_{0 \leq i \leq j \leq n+2} b_{ij} T_i T_j$ be the two universal quadric forms. If we regard a polynomial
$t_1F_1 + t_2F_2$ as quadric form in $T_0, \ldots, T_{n+2}$, we can take the divided discriminant of the quadric form $\text{disc}_d(t_1F_1 + t_2F_2)$. Further we see $\text{disc}_d(t_1F_1 + t_2F_2)$ as a binary form in $t_1, t_2$ and we can take the divided discriminant $\text{disc}_d(\text{disc}_d(t_1F_1 + t_2F_2))$.

**Theorem 2.9** ([15]).

1. If $n$ is even, then the equation
   \[
   \text{disc}(F_1, F_2) = (-1)^{\frac{n}{2}+1} \text{disc}_d(\text{disc}_d(t_1F_1 + t_2F_2))
   \]
   holds where the left hand side is the discriminant defined in Proposition 2.7.2.

2. If $n$ is odd, then the equation
   \[
   \text{disc}(F_1, F_2) = 2^{-2(n+3)} \text{disc}_d(\text{disc}_d(t_1F_1 + t_2F_2))
   \]
   holds up to $\pm 1$.

§3. Determinant

Let $X$ be a proper smooth variety of even dimension $n$ over a field $k$. If $\ell$ is a prime number invertible in $k$, the $\ell$-adic cohomology of middle degree $H^n(X, \mathbb{Q}_\ell(\frac{n}{2}))$ defines an orthogonal representation of the absolute Galois group $G_k$. The determinant

\[
\kappa := \det H^n(X, \mathbb{Q}_\ell(\frac{n}{2})) : G_k \to \{\pm 1\} \subset \mathbb{Q}_\ell^\times
\]

is independent of the choice of $\ell$.

Saito showed the following theorems.

**Theorem 3.1** ([10, Theorem 3.5, Corollary 4.3]). Let $X$ be a smooth hypersurface of degree $d$ in $\mathbb{P}^{n+1}_k$, and let $f$ be a homogeneous polynomial defining it. Let $\text{disc}_d(f)$ be the divided discriminant of $f$.

1. If $\text{char } k \neq 2$, then the quadratic character $\kappa$ is defined by the square root of $\varepsilon(n, d) \cdot \text{disc}_d(f)$.

2. If $\text{char } k = 2$, then the quadratic character $\kappa$ is defined by the Artin-Schreier equation $t^2 + t = B(f)A(f)^{-2}$ where $A$ and $B$ are polynomials occuring in Theorem 2.2.

**Theorem 3.2** ([11]). Let $X$ be a smooth complete intersection of multi-degree $d_1, \ldots, d_r$ in $\mathbb{P}^{n+r}_k$, and let $f_1, \ldots, f_r$ be homogeneous polynomials with coefficients in $k$ defining $X$.

1. If $\text{char } k \neq 2$, then the quadratic character $\kappa$ is defined by the square root of $\text{disc}(f_1, \ldots, f_r)$, where $\text{disc}(f_1, \ldots, f_r)$ is the discriminant whose sign is defined by the condition in Proposition 2.7.2.

2. If $\text{char } k = 2$, then the quadratic character $\kappa$ is defined by the Artin-Schreier equation $t^2 + t = B(f_1, \ldots, f_r)A(f_1, \ldots, f_r)^{-2}$ where $A$ and $B$ are the polynomials occuring in Proposition 2.7.2.
For the complete intersection of two quadrics, we see the following by Theorem 2.9 and Theorem 3.2.

**Theorem 3.3 ([15]).** Assume that \( n \) is even. Let \( X \subseteq \mathbb{P}^{n+2} \) be a smooth complete intersection defined by quadric forms \( f_1, f_2 \). If \( \text{char} \ k \neq 2 \), then the quadratic character \( \kappa = \det H^n(X_{\overline{k}}, \mathbb{Q}_{\ell}(\frac{n}{2})) \) is defined by the square root of \((-1)^{\frac{n}{2}+1} \text{disc}_d(\text{disc}_d(t_1F_1 + t_2F_2)) \).

We state the theorem for a double covering of a projective space of even dimension. Assume that the characteristic of \( k \) is not 2. Let \( f \) be a homogeneous polynomial of \( n+1 \) variables of even degree \( d \) with coefficients in \( k \) and \( X \) be the double covering of a projective space of dimension \( n \) defined by the equation \( y^2 = f \). Let \( \text{disc}_d(f) \) be the divided discriminant of \( f \).

**Theorem 3.4 ([14]).** The quadratic character \( \kappa = \det H^n(X_{\overline{k}}, \mathbb{Q}_{\ell}(\frac{n}{2})) \) of \( G_k \) is defined by the square root of \((-1)^{\frac{dn}{4}} \text{disc}_d(f) \).

The proof of the Theorem 3.4 consists of two parts. First we follow the method given by T. Saito in [10] and a standard argument on universal family connects the determinant character \( \kappa \) with the square root of the discriminant. Second, we then determine the constant multiple of the discriminant by a specialization argument.

Next we state the theorem in the case characteristic 2. Assume that \( d \) is even. We consider the more general defining polynomial \( y^2 + ay = b \), where \( a \) and \( b \) are homogeneous polynomials of degree \( \frac{d}{2} \) and \( d \) over \( k \). Let \( A = \sum_{|I| = \frac{d}{2}} R_IT^I \) and \( B = \sum_{|J| = d} S_J T^J \) be the universal polynomials of degree \( \frac{d}{2} \) and \( d \). Then the greatest common divisor of the coefficients of the polynomial \( \text{disc}_d(A^2 + 4B) \in \mathbb{Z}[R_I]_{|I| = \frac{d}{2}}, (S_J)_{|J| = d} \) is \( 4^{ds(n-1,d)} \), where \( s(n-1,d) = \frac{(n+1)(d-1)^{n} - a(n-1,d)}{d} \). Further, there exist polynomials \( C, D \in \mathbb{Z}[R_I]_{|I| = \frac{d}{2}}, (S_J)_{|J| = d} \) satisfying \( 4^{-ds(n-1,d)}(-1)^{\frac{dn}{4}} \text{disc}_d(A^2 + 4B) = C^2 + 4D \).

**Theorem 3.5 ([14]).** Assume that \( d \) is even. Let \( k \) be a field of the characteristic equal to 2 and \( Z \) be a smooth branched double covering of the projective space \( \mathbb{P}^n_k \) of even dimension \( n \) defined by an equation \( y^2 + ay = b \) where \( a \) and \( b \) are homogeneous polynomials of degree \( \frac{d}{2} \) and \( d \) over \( k \). Let \( C(a,b) \in k^\times \) and \( D(a,b) \in k \) denote the specializations of the polynomials \( C \) and \( D \) noted as above. Then, the quadratic character \( \det H^n(Z_{\overline{k}}, \mathbb{Q}_{\ell}(\frac{n}{2})) \) of \( G_k \) is defined by the Artin-Schreier equation \( t^2 + t = D(a,b) \cdot C(a,b)^{-2} \).

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**§ 4. The determinant of a del Pezzo surface**

Let \( k \) be a field of characteristic \( \neq 2 \). A del Pezzo surface over \( k \) is a smooth projective surface \( S \) whose anti-canonical sheaf \( (\Omega^2_{S/k})^{-1} \) is ample and whose base change
$S_{\overline{k}}$ is birationally equivalent to $\mathbb{P}_{\overline{k}}^2$. We call the intersection number $d = \omega_S \cdot \omega_S$ its degree, where $\omega_S$ is the class of the canonical sheaf $\Omega^2_{S/k}$ in $\text{Pic}(S)$. Manin [8, Theorem 24.3] gives bounds for this degree, namely $1 \leq d \leq 9$.

Let $S$ be a del Pezzo surface of degree $d \leq 6$. Then $H^2(S_{\overline{k}}, \mathbb{Q}_\ell(1))$ is of dimension $10 - d$ and is spanned by the classes of their exceptional curves. The group of automorphisms of the $\mathbb{Z}$-lattice spanned by the classes of these exceptional curves permuting them and preserving the intersection form is isomorphic to the Weyl group $W(R_{9-d})$. Here $R_3, \ldots, R_8$ denotes the type $A_1 \times A_2, A_4, D_5, E_6, E_7, E_8$, respectively.

The action of $G_k$ on the exceptional classes defines a homomorphism

$$G_k \to W(R_{9-d})$$

unique up to conjugation. We consider the composition of this homomorphism and the determinant map $G_k \to W(R_{9-d}) \to \{\pm 1\}$. We can apply the theorems in Section 3 to this map in the case of del Pezzo surfaces of degree 2, 3 and 4.

§ 4.1. Del Pezzo surface of degree 4

A del Pezzo surface of degree 4 is isomorphic to a complete intersection of two quadric hypersurfaces in $\mathbb{P}_k^4$ after a suitable finite extension of the base field [5, Theorem 8.6.2]. Let $S \subset \mathbb{P}_k^4$ be a smooth complete intersection defined by quadric forms $f_1, f_2$. Then $H^2(S_{\overline{k}}, \mathbb{Q}_\ell(1))$ is spanned by the classes of 16 lines.

By applying the Theorem 3.3 and Theorem 3.2, we can describe the parity of the Galois permutation of the exceptional divisors as follows.

**Corollary 4.1** ([15]). *The composition $G_k \to W(D_5) \to \{\pm 1\}$ is defined by the quadratic extension $k(\sqrt{\text{disc}_d(t_1f_1 + t_2f_2)))/k$.***

§ 4.2. Del Pezzo surface of degree 3

Any del Pezzo surface of degree 3 is isomorphic to a cubic surface in $\mathbb{P}_k^3$ [8, §24, Theorem 24.4 and Theorem 24.5]. We note that the base fields of del Pezzo surfaces are assumed to be algebraically closed throughout [8, §24]. However, this result holds true with the same proof as the one given in the reference. Let $S \subset \mathbb{P}_k^3$ be a smooth cubic surface. Then $H^2(S_{\overline{k}}, \mathbb{Q}_\ell(1))$ is spanned by the classes of 27 lines.

It is known that the general enough cubic surface in characteristic $\neq 3$ can be put in the form

$$x + y + z + u + v = 0, \quad ax^3 + by^3 + cz^3 + du^3 + ev^3 = 0$$

(4.1)
in $\mathbb{P}^4$ after a suitable finite extension of the ground field. The corresponding cubic surface is smooth if and only if the Salmon discriminant $\text{disc}_s(a, b, c, d, e)$ is non-zero. The definition of the Salmon discriminant is:

$$\text{disc}_s(a, b, c, d, e) = ((s^2 - 64rt)^2 - 4t^3p)^2 - 2^{11}(8t^6q + t^4s(s^2 - 4rt)),$$

where $p = a + b + c + d + e$, $q = ab + \cdots$, $r = abc + \cdots$, $s = abcd + \cdots$, $t = abcde$ are the elementary symmetric functions of $a, b, c, d, e$. By eliminating one variable in (4.1), one obtains a cubic polynomial $F_s = ax^3 + by^3 + cz^3 + du^3 - e(x + y + z + u)^3$. The relation between the Salmon discriminant and the divided one is given in [10]:

$$\text{disc}_s(a, b, c, d, e) = 3^{-27} \text{disc}_d(F_s).$$

By applying the above relation and Theorem 3.1, we see the following.

**Corollary 4.2** ([6, Theorem 2.12] or [10, Section 5.4]). Let $k$ be the field of characteristic $\neq 2, 3$ and $a, b, c, d, e \in k$ such that $\text{disc}_s(a, b, c, d, e) \neq 0$. Further let $G_k \to W(E_6)$ be the homomorphism associated with the cubic surface of type (4.1). Then, the composition of this homomorphism and the determinant map $G_k \to W(E_6) \to \{\pm 1\}$ is defined by the quadratic extension $k(\sqrt{-3\text{disc}_s(a, b, c, d, e)})/k$.

§ 4.3. Del Pezzo surface of degree 2

A del Pezzo surface of degree 2 is isomorphic to a double covering of $\mathbb{P}^2$ branched along a quartic curve [5, Proposition 6.3.9 and Proposition 6.3.11]. Let $S \to \mathbb{P}^2$ be the double covering branched along the curve defined by a quartic form $f$. Then $H^2(S_k, \mathbb{Q}_\ell(1))$ is spanned by the classes of 56 exceptional divisors. The kernel of the determinant map $W(E_6) \to \{\pm 1\}$ is a simple group of order $2^9 \cdot 3^4 \cdot 5 \cdot 7$ isomorphic to $\text{Sp}_6(\mathbb{F}_2)$ [1, pp.46-47].

By applying Theorem 3.4, we see the following.

**Corollary 4.3** ([14]). The composition $G_k \to W(E_7) \to \{\pm 1\}$ is defined by the quadratic extension $k(\sqrt{\text{disc}_d(f)})/k$.

Let $F = \sum_{i_0+i_1+i_2=4}C_{(i_0, i_1, i_2)}T_0^{i_0}T_1^{i_1}T_2^{i_2}$ be the universal quartic polynomial. The discriminant of a quartic curve $\text{disc}_d(F)$ is a homogeneous polynomial of degree 27 in 15 variables and $\text{disc}_d(F) = 4^{-7} \text{res}(D_0F, D_1F, D_2F)$.

We describe a formula for $\text{res}(f_0, f_1, f_2)$ due to Sylvester. Let $f_0, f_1, f_2 \in k[x, y, z]$ be three homogeneous polynomials of degrees $d \geq 2$. For any three non-negative integers $a, b, c$ such that $a + b + c = d - 1$, every monomial of degree $d$ is divisible by at least
one of $x^{a+1}, y^{b+1}, z^{c+1}$. Then we can write $f_0, f_1, f_2$ in the form

\begin{align*}
  f_0 &= x^{a+1}P_0 + y^{b+1}Q_0 + z^{c+1}R_0 \\
  f_1 &= x^{a+1}P_1 + y^{b+1}Q_1 + z^{c+1}R_1 \\
  f_2 &= x^{a+1}P_2 + y^{b+1}Q_2 + z^{c+1}R_2
\end{align*}

where the $P_i, Q_i, R_i$ are homogeneous polynomials of degrees respectively $d-a-1, d-b-1, d-c-1$. Such a representation of the $f_i$ is not unique. We define the polynomial

\[ D_{abc} = \det \begin{pmatrix} P_0 & Q_0 & R_0 \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{pmatrix} \]

of degree $2d-2$. Then we consider the equations

\begin{align*}
  x^\alpha f_0 &= 0, \quad (x^\alpha = x^{\alpha_0}y^{\alpha_1}z^{\alpha_2}, \alpha_0 + \alpha_1 + \alpha_2 = d - 2) \\
  x^\alpha f_1 &= 0, \quad (x^\alpha = x^{\alpha_0}y^{\alpha_1}z^{\alpha_2}, \alpha_0 + \alpha_1 + \alpha_2 = d - 2) \\
  x^\alpha f_2 &= 0, \quad (x^\alpha = x^{\alpha_0}y^{\alpha_1}z^{\alpha_2}, \alpha_0 + \alpha_1 + \alpha_2 = d - 2) \\
  D_{abc} &= 0, \quad (a + b + c = d - 1).
\end{align*}

Each polynomial on the left hand side has degree $2d-2$, and there are $2d^2 - d$ monomials of this degree. Further the number of the equations is $2d^2 - d$. Thus the coefficient matrix of the equations is a $(2d^2 - d) \times (2d^2 - d)$ matrix and we denote it by $C_d$. Now the result of Sylvester is as follows.

**Theorem 4.4** (see e.g. [7, 3.4.D Theorem 4.10]). The resultant of three homogeneous polynomials $f_0, f_1, f_2$ of degree $d \geq 2$ is given by

\[ \text{res}(f_0, f_1, f_2) = \pm \det C_d. \]

By applying the above theorem for $f_0 = D_0F, f_1 = D_1F$ and $f_2 = D_2F$ where $F$ is the universal quartic polynomial, one has an explicit presentation of the discriminant $\text{disc}_d(F)$ of a quartic curve.

**References**

The determinant of a double covering of the projective space