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Shape optimization method applied to room design based on an incompressible fluid model

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Abstract

This paper presents numerical results obtained, using a numerical method for a flow field shape optimization, for the design of a room from which people can evacuate smoothly. The domain of interest is a two-dimensional Poiseuille flow with sudden contraction in which a disk is located initially. In fact, the compressible fluid model has been used to describe the dynamics of people for the study of evacuation. However, we set a model involving the incompressible Navier–Stokes equation for the first trial. And the shape optimization to minimize the dissipation energy on the disk is demonstrated under the volume constraint. For reshaping numerically, the traction method is used. Numerical results reveal that the shape in the wake of the disk becomes an acute angle to decrease the dissipation energy monotonically, thereby satisfying the volume constraint. Such a shape has never been inferred from results of earlier studies of the evacuation problem in jamology.

§ 1. Introduction

The shape optimization for the minimizing problem of the dissipation energy on an disk located in two-dimensional Poiseuille flow with a sudden contraction is addressed for the aim to design an optimal room shape from which people can evacuate smoothly in an emergency situation, where the incompressible Navier–Stokes equations are used

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as governing equations. And optimal shapes of the disk, flow fields obtained by changing Reynolds numbers Re and the position of the disk on the initial domain and validities of the model constructed in this paper are discussed.

Numerical techniques to optimize designs of the shape of elastic bodies in a flow field play an important role in machine design in various fields not only of industry but also of science and engineering. In this paper, a shape optimization problem based on the finite element method is demonstrated, and a functional is defined as the sum of a cost function and a constraint function expressed in weak form, where a shape derivative of the functional is taken to obtain a sensitivity used for minimize the functional. Firstly, J. Hadamard took the shape derivative of the functional to maximize the fundamental frequency of a thin membrane using sensitivity analysis [1]. Especially in fluid dynamics, the shape derivatives of functionals for the shape optimizations of an isolated body in an incompressible Newtonian fluid was presented by O. Pironneau[2],[3],[4].

By the way, for numerical calculations it was noticed by H. M. Imam[5] that direct application of the gradient method often results in oscillating shapes. To avoid oscillation, a method using the Laplace operator as a smoother which is called the traction method was proposed by H. Azegami et al. [7] and H. Azegami [8]. In the traction method, from some previous studies [9],[10],[11], it is known that the functionals are minimized safety.

These techniques hold the possibility of application to a room design for safe and orderly evacuation from a room. The room exit is clogged with people in an emergency situation because people rush for the exit to evacuate the room. One solution to decrease the total evacuation time was ascertained from mathematical modeling, analysis, and real experiments in jamology, and especially numerical simulations built on a cellular automata model show that the total evacuation time decreases if a column is set near the exit and is shifted parallel to the exit [12]. However, the best position and best shape to minimize the total evacuation time remain unclear.

Toward a solution of the evacuation problem, we set an objective of our project as demonstrating the applicability of the shape optimization theory. Concretely, we aim to obtain the optimal design of the column which makes the flow smoother. In the study of evacuation, the compressible fluid model has been used to describe the dynamics of people. However, we set a model involving the incompressible Navier–Stokes equation for the first trial. The main problem of this paper is defined as a two-dimensional Poiseuille flow with sudden contraction, and the disk assuming that the column is located in the domain. As an objective cost function, we use the dissipation energy, and the domain volume is used as a constraint cost function. The shape derivative of the functional with respect to the domain variation is evaluated using the solution of the main problem and the adjoint problem. An iterative algorithm based on the traction method for reshaping are used for numerical schemes used to conduct shape optimization.

This paper is organized as follows. In section 2, we introduce formulation of the problem. Particularly in subsection 2.2, the perturbation $\boldsymbol{\psi}$ of the one-to-one mapping $\boldsymbol{\phi}$ from the initial domain to the designed domain is explained. Subsection 2.4 presents formulation of the shape optimization problem. In fact, it is quite difficult to obtain $\boldsymbol{\phi}$ because an optimal domain is not clear generally in shape optimization problem. Therefore, the shape optimization problem formulated in subsection 2.4 is conducted by adding perturbation $\boldsymbol{\psi}$ into the initial domain through iterative reshaping as minimizing the cost function. The traction method is used in section 3 to minimize the functional stably to the greatest degree possible. Thereby, the shape derivative JL of the functional L is obtained. Section 4, introduces specific use of the traction method.

§ 2. Problem formulation

§ 2.1. Set of domains

A Cartesian coordinate system is used. A position vector is generally denoted as $\boldsymbol{x} = (x, y)$. We consider a domain $\Omega = \Omega_{\mathrm{i}} \backslash \bar{\Omega}_{\mathrm{c}} \subset \mathbb{R}^2$ as presented in Fig. 1, where

$$\Omega_{\mathrm{i}} = \{(x, y) \mid ([0,2] \times [0,1]) \cup ([2,3] \times [0.35, 0.65])\}$$
$$\Omega_{\mathrm{c}} = \left\{(x, y) \mid (x - x_c)^2 + (y - 0.5)^2 < 0.15\right\}.$$

We set the inflow boundary as $\Gamma_{\mathrm{in}} = \{(x, y) \mid x = 0,\ 0 \leq y \leq 1\}$ and the outflow boundary as $\Gamma_{\mathrm{out}} = \{(x, y) \mid x = 3,\ 0.35 \leq y \leq 0.65\}$. Moreover, set $\Gamma_{\mathrm{wall}} = \partial\Omega \backslash (\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}})$.

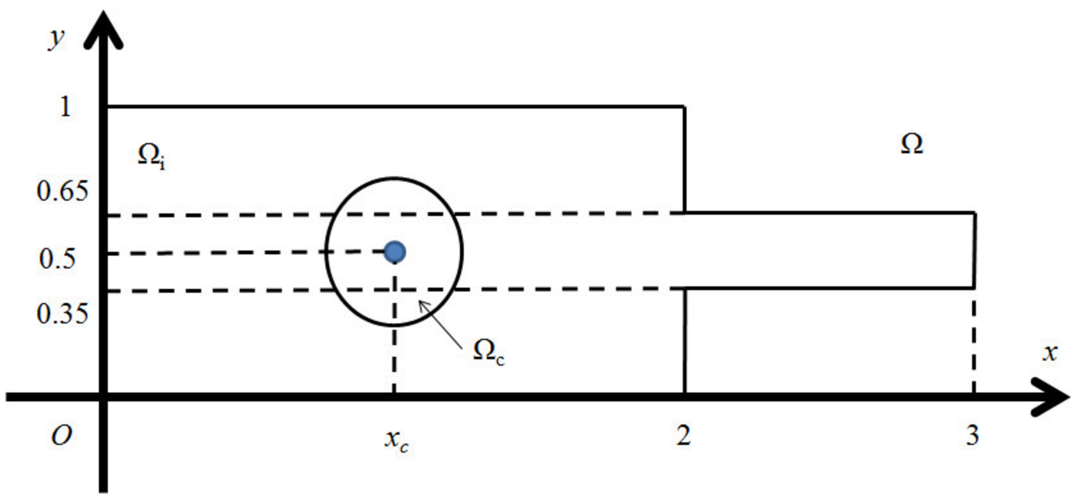


Figure 1. Initial domain.

§ 2.2. Domain variations

Next, we consider domain deformation with Lipschitz transform $\boldsymbol{\phi} : \Omega \to \boldsymbol{\phi}(\Omega)$, where $\boldsymbol{\phi}$ is $\mathbb{R}^2$-valued $W^{1,\infty}$ function. The identity map on $\mathbb{R}^2$ is denoted as $\boldsymbol{\phi}_0(\boldsymbol{x}) = \boldsymbol{x}$. In the following argument, we fix a bounded convex domain $\Omega_0 \subset \mathbb{R}^2$. Then we identify $W^{1,\infty}(\Omega_0, \mathbb{R}^2)$ with $C^{0,1}(\bar{\Omega}_0, \mathbb{R}^2)$. Presuming that $\boldsymbol{\phi} \in W^{1,\infty}(\Omega_0, \mathbb{R}^2)$ satisfies $\|\nabla(\boldsymbol{\phi}^{\mathrm{T}} - \boldsymbol{\phi}_0^{\mathrm{T}})\|_{L^\infty(\Omega_0,\mathbb{R}^2)} < 1$, then $\boldsymbol{\phi}$ is a bi-Lipschitz transform from Ω_0 to $\boldsymbol{\phi}(\Omega_0)$, i.e. $\boldsymbol{\phi}$ is bijective from Ω_0 onto an open set and $\boldsymbol{\phi}$, $\boldsymbol{\phi}^{-1}$ are both uniformly Lipschitz continuous, where $\nabla\boldsymbol{\phi}^{\mathrm{T}} \in \mathbb{R}^{2\times 2}$ is the Jacobian matrix ([13] Proposition 3.1. p.24). We fix an open set Ω which satisfies $\bar{\Omega} \subset \Omega_0$. The deformed domain $\boldsymbol{\phi}(\Omega)$ is denoted by $\Omega(\boldsymbol{\phi})$. The admissible set of $\boldsymbol{\phi}$ is defined as

$$X = \left\{ \boldsymbol{\phi} \in W^{1,\infty}(\Omega_0, \mathbb{R}^2) \mid \bar{\Omega}(\boldsymbol{\phi}) \subset \Omega_0, \|\nabla(\boldsymbol{\phi}^{\mathrm{T}} - \boldsymbol{\phi}_0^{\mathrm{T}})\|_{L^\infty(\Omega_0,\mathbb{R}^2)} < 1, \boldsymbol{\phi} = \mathbf{0} \text{ on } \partial\Omega \right\}.$$

We assume $\boldsymbol{\psi} \in W^{1,\infty}(\Omega_0, \mathbb{R}^2)$ with $\mathrm{supp}(\boldsymbol{\psi}) \in \Omega$. For $\epsilon \in \mathbb{R}$ with $\|\epsilon\nabla\boldsymbol{\psi}^{\mathrm{T}}\|_{L^\infty(\Omega_0,\mathbb{R}^2)} < 1$, we define a bi-Lipschitz transform $\boldsymbol{\phi}(\epsilon) = \boldsymbol{\phi}_0 + \epsilon\boldsymbol{\psi}$ from Ω to itself ([13] Proposition 3.4. p.25). As described in this paper, because the traction method is conducted to reshape the domain Ω in the Hilbert space, we define the admissible set of $\boldsymbol{\psi}$ as

$$D = \left\{ \boldsymbol{\psi} \in W^{1,\infty}(\Omega_0, \mathbb{R}^2) \cap H^1(\Omega_0, \mathbb{R}^2) \; | \boldsymbol{\psi} = \mathbf{0} \text{ on } \partial\Omega \right\}.$$

The shape derivatives of functionals are obtained as follows. We presume that $\varphi \in H^1(\Omega)$ is a scalar valued function which describes a physical state in Ω. For such $\varphi(\boldsymbol{x})$, we introduce the following functional:

$$L(\boldsymbol{\phi}, \boldsymbol{x}, \Omega) = \int_{\Omega(\boldsymbol{\phi})} \zeta(\boldsymbol{\phi}, \boldsymbol{x}, \varphi(\boldsymbol{x}), \nabla\varphi(\boldsymbol{x})) dx,$$

where $\zeta(\boldsymbol{\phi}, \boldsymbol{x}, \varphi(\boldsymbol{x}), \nabla\varphi(\boldsymbol{x})) \in \mathbb{R}$ for $\nabla\varphi(\boldsymbol{x}) \in \mathbb{R}^2$. Using the shape derivative ζ' of ζ, the shape derivative JL of L is given as

$$JL(\boldsymbol{\phi}, \boldsymbol{x}, \Omega) = \int_{\Omega(\boldsymbol{\phi})} \zeta' dx + \int_{\partial\Omega_c(\boldsymbol{\phi})} \zeta \boldsymbol{\nu} \cdot \boldsymbol{\psi} d\gamma, \tag{2.1}$$

where the deformed boundary is denote by $\partial\Omega_c(\boldsymbol{\phi})$ and where $\boldsymbol{\nu}$ denotes an outward unit normal vector on the boundary ([7] Eq. (18), p.274).

§ 2.3. Governing equations

Next we consider a steady-state viscous flow of an incompressible Newtonian fluid in domain $\Omega(\boldsymbol{\phi})$ with boundary $\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}} \cup \Gamma_{\mathrm{wall}} \cup \partial\Omega_{\mathrm{c}}(\boldsymbol{\phi})$. We introduce a set of velocity fields

$$U = \left\{ \boldsymbol{u} = (u, v) \in H^1(\mathbb{R}^2, \mathbb{R}^2) \mid \boldsymbol{u} = \boldsymbol{u}_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{in}}, \; \boldsymbol{u} = \mathbf{0} \text{ on } \Gamma_{\mathrm{wall}} \cup \partial\Omega_{\mathrm{c}}(\boldsymbol{\phi}) \right\},$$

where $\boldsymbol{u}_{\mathrm{D}} = (u_{\mathrm{D}}, v_{\mathrm{D}}) \in \left\{\boldsymbol{u} \in H^1\left(\mathbb{R}^2, \mathbb{R}^2\right) \mid \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0\right\}$ denotes a prescribed fluid velocity field on an inhomogeneous Dirichlet boundary. On the out flow boundary Γ_{out}, we pose

$$\frac{1}{\mathrm{Re}}(\boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{\nu} - p\boldsymbol{\nu} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{out}}. \tag{2.2}$$

Below we always take $u_{\mathrm{D}} = y(1-y)$ and $v_{\mathrm{D}} = 0$ on Γ_{in}. As a set of a pressure fields, we introduce

$$Q = L^2(\Omega).$$

Taking R_1, which is the length of $\Gamma_{\mathrm{in}} = 1$, and $u_{\max}$, which is the maximum value of u_D, as the characteristic length and velocity, the Reynolds number is defined as

$$\mathrm{Re} = \frac{u_{\max} R_1}{d},$$

where d represents the kinematic viscosity. We consider the stationary Navier–Stokes equation in non-dimensional form as

$$\left(\hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\right)\hat{\boldsymbol{u}} = -\boldsymbol{\nabla}\hat{p} + \frac{1}{\mathrm{Re}}\triangle\hat{\boldsymbol{u}}, \tag{2.3}$$

together with the equation of continuity

$$\boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} = 0, \tag{2.4}$$

and the Neumann boundary condition (2.2), where $(\hat{\boldsymbol{u}}, \hat{p}) \in U \times Q$.

The weak forms of Eq. (2.3), (2.4) and (2.2) are written as

$$\int_\Omega \left\{\left(\left(\hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\right)\hat{\boldsymbol{u}}\right) \cdot \hat{\boldsymbol{w}} - \hat{p}\boldsymbol{\nabla} \cdot \hat{\boldsymbol{w}} + \frac{1}{\mathrm{Re}}(\boldsymbol{\nabla}\hat{\boldsymbol{u}}^{\mathrm{T}}) \cdot (\boldsymbol{\nabla}\hat{\boldsymbol{w}}^{\mathrm{T}})\right\} dx - \int_\Omega \hat{q}\boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} dx = 0, \tag{2.5}$$

for all $(\hat{\boldsymbol{w}}, \hat{q}) \in W \times Q$, where $(\hat{\boldsymbol{w}}, \hat{q})$ represents trial functions for velocity $\hat{\boldsymbol{u}}$ and pressure $\hat{p}$, and

$$W = \left\{\boldsymbol{w} \in H^1(\mathbb{R}^2, \mathbb{R}^2) \mid \boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{in}} \cup\ \Gamma_{\mathrm{wall}} \cup \partial\Omega_{\mathrm{c}}(\boldsymbol{\phi})\right\}.$$

As described in this paper, the movement of crowds of person is modeled as the fluid flow governed by incompressible and stationary Navier–Stokes equations for the first trial. However, the population density should be considered using compressible Navier–Stokes equations in fact. To evacuate people as smoothly as possible, we use the minimization problem of the dissipation energy under the volume constraint of the disk, where the dissipation energy is used as the cost function defined as

$$\int_{\Omega(\boldsymbol{\phi})} \frac{2}{\mathrm{Re}}\boldsymbol{E}(\hat{\boldsymbol{u}}) \cdot \boldsymbol{E}(\hat{\boldsymbol{u}}) dx,$$

where $\boldsymbol{E}(\boldsymbol{u})$ depicts the strain tensor as

$$\boldsymbol{E}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{T}} + (\boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{T}})^{\mathrm{T}}\right).$$

People do not move on the boundary Γ_{wall} where the Dirichlet condition $\hat{\boldsymbol{u}} = \boldsymbol{0}$ is defined, but Γ_{wall} is assumed to be just a wall. The Dirichlet condition $\hat{\boldsymbol{u}}_{\mathrm{D}}$ defined on the boundary Γ_{in} depicts the velocity for people to move into the room. On Γ_{out} defined by the Neumann boundary condition (2.2), people evacuate from the room freely. These boundary conditions are used for descriptive purposes, but they are physically unsuitable.

§ 2.4. Shape optimization problem

For the room design to evacuate people as soon as possible, the minimization problem of the dissipation energy under volume constraint M, which is the volume of the initial domain, is formulated in this paper as

$$\begin{aligned}
&\text{Find } \Omega_p \\
&\text{that minimizes } \int_{\Omega(\boldsymbol{\phi})} \frac{2}{\mathrm{Re}} \boldsymbol{E}(\hat{\boldsymbol{u}}) \cdot \boldsymbol{E}(\hat{\boldsymbol{u}}) dx \\
&\text{subject to } (\hat{\boldsymbol{u}}, \hat{p}) \in U \times Q \text{ such that (2.3), (2.4),(2.2) and } \int_{\Omega(\boldsymbol{\phi})} dx = M.
\end{aligned}$$

The problem above is the target of this paper to optimize the boundary shape of $\partial\Omega_{\mathrm{c}}(\boldsymbol{\phi})$. It is expected to suggest the optimal room design for jamology.

§ 3. Evaluation of a shape derivative

As described in this paper, the shape derivative of the Lagrange function L is evaluated by application of the Lagrange multiplier method, where the Lagrange function L is written as

$$\begin{aligned}
L(\boldsymbol{\phi}, \hat{\boldsymbol{u}}, \hat{p}, \hat{\boldsymbol{w}}, \hat{q}, \Lambda) &= \int_{\Omega(\boldsymbol{\phi})} \frac{2}{\mathrm{Re}} \boldsymbol{E}(\hat{\boldsymbol{u}}) \cdot \boldsymbol{E}(\hat{\boldsymbol{u}}) dx \\
&- \int_{\Omega(\boldsymbol{\phi})} \left\{ ((\hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\, \hat{\boldsymbol{u}}) \cdot \hat{\boldsymbol{w}} - \hat{p} \boldsymbol{\nabla} \cdot \hat{\boldsymbol{w}} + \frac{1}{\mathrm{Re}} (\boldsymbol{\nabla} \hat{\boldsymbol{u}}^{\mathrm{T}}) \cdot (\boldsymbol{\nabla} \hat{\boldsymbol{w}}^{\mathrm{T}}) \right\} dx + \int_{\Omega(\boldsymbol{\phi})} \hat{q} \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} dx \\
&+ \Lambda \left(\int_{\Omega(\boldsymbol{\phi})} dx - M \right),
\end{aligned}$$

and $\hat{\boldsymbol{w}}, q, \Lambda$ respectively denote the Lagrange multiplier for velocity $\hat{\boldsymbol{u}}$ and pressure q, the volume constraint. The shape derivative JL of L with respect to arbitrary variation of $(\boldsymbol{\psi}, \hat{\boldsymbol{u}}', \hat{p}', \hat{\boldsymbol{w}}', \hat{q}', \Lambda') \in D \times W \times Q \times W \times Q \times \mathbb{R}$ is obtainable as

$$JL(\boldsymbol{\phi}, \hat{\boldsymbol{u}}, \hat{p}, \hat{\boldsymbol{w}}, \hat{q}, \Lambda) = L_{\boldsymbol{\phi}}[\boldsymbol{\psi}] + L_{\hat{\boldsymbol{u}}, \hat{p}}[\hat{\boldsymbol{u}}', \hat{p}'] + L_{\hat{\boldsymbol{w}}, \hat{q}}[\hat{\boldsymbol{w}}', \hat{q}'] + L_{\Lambda}[\Lambda'], \tag{3.1}$$

where based on the formula for the shape derivative of a functional (2.1),

$$
\begin{aligned}
L_{\boldsymbol{\phi}}[\boldsymbol{\psi}] &= -\int_{\partial\Omega_c(\boldsymbol{\phi})} \left\{ \left((\hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\, \hat{\boldsymbol{u}} \right) \cdot \hat{\boldsymbol{w}} - \hat{p}\boldsymbol{\nabla} \cdot \hat{\boldsymbol{w}} + \frac{1}{\mathrm{Re}} (\boldsymbol{\nabla}\hat{\boldsymbol{u}}^{\mathrm{T}}) \cdot (\boldsymbol{\nabla}\hat{\boldsymbol{w}}^{\mathrm{T}}) \right\} \boldsymbol{\nu} \cdot \boldsymbol{\psi} d\gamma \\
&\quad + \int_{\partial\Omega_c(\boldsymbol{\phi})} \{ \hat{q} \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} \} \boldsymbol{\nu} \cdot \boldsymbol{\psi} d\gamma + \int_{\partial\Omega_c(\boldsymbol{\phi})} \frac{2}{\mathrm{Re}} \{ \boldsymbol{E}(\hat{\boldsymbol{u}}) \cdot \boldsymbol{E}(\hat{\boldsymbol{u}}) \} \boldsymbol{\nu} \cdot \boldsymbol{\psi} d\gamma \\
&\quad + \int_{\partial\Omega_c(\boldsymbol{\phi})} \Lambda \boldsymbol{\nu} \cdot \boldsymbol{\psi} d\gamma, \\
L_{\hat{\boldsymbol{u}},\hat{p}}[\hat{\boldsymbol{u}}', \hat{p}'] & \\
&= -\int_{\Omega(\boldsymbol{\phi})} \left\{ \left((\hat{\boldsymbol{u}}' \cdot \boldsymbol{\nabla})\, \hat{\boldsymbol{u}} \right) \cdot \hat{\boldsymbol{w}} + \left((\hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\, \hat{\boldsymbol{u}}' \right) \cdot \hat{\boldsymbol{w}} - \hat{p}' \boldsymbol{\nabla} \cdot \hat{\boldsymbol{w}} + \frac{1}{\mathrm{Re}} (\boldsymbol{\nabla}\hat{\boldsymbol{u}}'^{\mathrm{T}}) \cdot (\boldsymbol{\nabla}\hat{\boldsymbol{w}}^{\mathrm{T}}) \right\} dx \\
&\quad + \int_{\Omega(\boldsymbol{\phi})} \hat{q} \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}}' dx + 4 \int_{\Omega(\boldsymbol{\phi})} \frac{1}{\mathrm{Re}} \boldsymbol{E}(\hat{\boldsymbol{u}}') \cdot \boldsymbol{E}(\hat{\boldsymbol{u}}) dx, \\
L_{\hat{\boldsymbol{w}},\hat{q}}[\hat{\boldsymbol{w}}', \hat{q}'] &= -\int_{\Omega(\boldsymbol{\phi})} \left\{ \left((\hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla})\, \hat{\boldsymbol{u}} \right) \cdot \hat{\boldsymbol{w}}' - \hat{p} \boldsymbol{\nabla} \cdot \hat{\boldsymbol{w}}' + \frac{1}{\mathrm{Re}} (\boldsymbol{\nabla}\hat{\boldsymbol{u}}^{\mathrm{T}}) \cdot (\boldsymbol{\nabla}\hat{\boldsymbol{w}}'^{\mathrm{T}}) \right\} dx \\
&\quad + \int_{\Omega(\boldsymbol{\phi})} \hat{q}' \boldsymbol{\nabla} \cdot \hat{\boldsymbol{u}} dx, \\
L_{\Lambda}[\Lambda'] &= \Lambda' \left(\int_{\Omega(\boldsymbol{\phi})} dx - M \right).
\end{aligned}
$$

Stationary conditions for the variables $\hat{\boldsymbol{u}}', \hat{\boldsymbol{w}}'$ and $\hat{p}', \hat{q}'$, which are known as the Kuhn–Tucker conditions in the optimization theory ([6], section 4.2 Optimality criteria, p.275), are the following.

$$
L_{\hat{\boldsymbol{u}},\hat{p}}[\hat{\boldsymbol{u}}', \hat{p}'] = 0 \quad \forall (\hat{\boldsymbol{u}}', \hat{p}') \in W \times Q, \tag{3.2}
$$

$$
L_{\hat{\boldsymbol{w}},\hat{q}}[\hat{\boldsymbol{w}}', \hat{q}'] = 0 \quad \forall (\hat{\boldsymbol{w}}', \hat{q}') \in W \times Q, \tag{3.3}
$$

$$
L_{\Lambda}[\Lambda'] = 0. \tag{3.4}
$$

Equation (3.3),(3.4) agrees with the weak form of Eqs. (2.3), (2.4), (2.2) and the volume constraint. Equation (3.2) is obtained as the weak form of the adjoint equations of Eq. (2.3),(2.4), and (2.2). Its strong form is expressed as

$$
-\left(\hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\right) \hat{\boldsymbol{w}} + (\boldsymbol{\nabla}\hat{\boldsymbol{u}}^{\mathrm{T}}) \hat{\boldsymbol{w}} = -\boldsymbol{\nabla}\hat{q} + \frac{1}{\mathrm{Re}} \triangle \hat{\boldsymbol{w}} - \frac{4}{\mathrm{Re}} \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{E}(\hat{\boldsymbol{u}}), \tag{3.5}
$$

$$
\boldsymbol{\nabla} \cdot \hat{\boldsymbol{w}} = 0, \tag{3.6}
$$

and the Neumann boundary condition

$$
\frac{1}{\mathrm{Re}} \left\{ 4\boldsymbol{E}(\hat{\boldsymbol{u}})\boldsymbol{\nu} - (\boldsymbol{\nabla}\hat{\boldsymbol{w}}^{\mathrm{T}})\boldsymbol{\nu} \right\} - \hat{q}\boldsymbol{\nu} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{out}}, \tag{3.7}
$$

for $(\hat{\boldsymbol{w}}, \hat{q}) \in W \times Q$.

By substituting $(\hat{\boldsymbol{u}}, \hat{p})$ and its adjoint variables $(\hat{\boldsymbol{w}}, \hat{q})$ satisfying Eqs. (2.3), (2.4), and (2.2) and Eqs. (3.5), (3.6) and (3.7) into (3.1), the shape derivative JL of the Lagrange function L is obtained as

$$JL(\boldsymbol{\phi}, \hat{\boldsymbol{u}}, \hat{p}, \hat{\boldsymbol{w}}, \hat{q}) = \int_{\partial\Omega_c(\boldsymbol{\phi})} G\boldsymbol{\nu} \cdot \boldsymbol{\psi} d\gamma, \tag{3.8}$$

where $(\hat{\boldsymbol{u}}, \hat{p})$ and its adjoint variables $(\hat{\boldsymbol{w}}, \hat{q})$ meet (3.2) and (3.3) of the Kuhn–Tucker conditions, and the remainder of the Kuhn–Tucker conditions (3.4) are clear because of $\int_{\Omega(\boldsymbol{\phi})} dx = M$. Regarding to $L_{\boldsymbol{\phi}}[\boldsymbol{\psi}]$, (2.4), (3.6), and $\hat{\boldsymbol{w}} = \boldsymbol{0}$ on the designed boundary $\partial\Omega_c(\boldsymbol{\phi})$ are substituted into $L_{\boldsymbol{\phi}}[\boldsymbol{\psi}]$. As described in this paper, $G = G_0 + G_1$ denotes the sensitivity as

$$G_0 = \frac{1}{\mathrm{Re}} \left\{ 2\boldsymbol{E}(\hat{\boldsymbol{u}}) \cdot \boldsymbol{E}(\hat{\boldsymbol{u}}) - (\boldsymbol{\nabla}\hat{\boldsymbol{u}}^{\mathrm{T}}) \cdot (\boldsymbol{\nabla}\hat{\boldsymbol{w}}^{\mathrm{T}}) \right\}, \tag{3.9}$$

$$G_1 = \Lambda. \tag{3.10}$$

G_0 denotes the shape derivative of the dissipation energy.

§ 4. Reshaping scheme

The traction method is applicable to optimize the geometrical domain shape if the shape gradient is obtained. The traction method has been proposed as a procedure for solving the domain variation $\boldsymbol{\psi} \in D$ by

$$c_a \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{E}(\boldsymbol{\psi}) \cdot \boldsymbol{E}(\boldsymbol{y}) dx = -\int_{\partial\Omega_c(\boldsymbol{\phi})} G\boldsymbol{\nu} \cdot \boldsymbol{y} d\gamma, \tag{4.1}$$

for all $\boldsymbol{y} \in W$, where c_a is a positive constant to control the step size. The coerciveness is secured by the Dirichlet condition on Γ_{wall}. Equation (4.1) shows that $\boldsymbol{\psi}$ is obtained as a displacement of a pseudo-elastic body defined in $\Omega(\boldsymbol{\phi})$ by the loading of a pseudo-external force in proportion to $-G\boldsymbol{\nu}$. The reshaping of the boundary can be operated with the degree of freedom discretized with the finite element method for numerical calculations, without reductions of the degree of freedom on the boundary and in the domain.

Taking account of a bilinear equation (4.1) and $G = G_0 + G_1$, the domain variation is expressed as $\boldsymbol{\psi} = \boldsymbol{\psi}_0 + \Lambda\boldsymbol{\psi}_1$. By calculating the following equations (4.2) and (4.3), $\boldsymbol{\psi}_0, \boldsymbol{\psi}_1 \in D$ are obtained.

$$c_a \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{E}(\boldsymbol{\psi}_0) \cdot \boldsymbol{E}(\boldsymbol{y}) dx = -\int_{\partial\Omega_c(\boldsymbol{\phi})} G_0\boldsymbol{\nu} \cdot \boldsymbol{y} d\gamma. \tag{4.2}$$

$$c_a \int_{\Omega(\boldsymbol{\phi})} \boldsymbol{E}(\Lambda \boldsymbol{\psi}_1) \cdot \boldsymbol{E}(\boldsymbol{y}) dx = -\int_{\partial\Omega_{\mathrm{c}}(\boldsymbol{\phi})} G_1 \boldsymbol{\nu} \cdot \boldsymbol{y} d\gamma. \tag{4.3}$$

Based on a previous study ([14], Problem 7, p.7, Eq. (30)), the Lagrange multiplier Λ for the volume constraint is determined by calculating the following expression

$$\int_{\partial\Omega(\boldsymbol{\phi})} \boldsymbol{\nu} \cdot \boldsymbol{\psi}_0 d\gamma + \Lambda \int_{\partial\Omega(\boldsymbol{\phi})} \boldsymbol{\nu} \cdot \boldsymbol{\psi}_1 d\gamma = 0.$$

Finally, because $\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \Lambda$ are given, the domain $\boldsymbol{x} \in \Omega$ is moved into $\boldsymbol{x} + \boldsymbol{\psi} \in \Omega(\boldsymbol{\phi})$.

In the traction method, the domain is reshaped as minimizing the cost function because the sign of the sensitivity G is set as negative. Using the regularity theorem for elliptic boundary value problems in a previous paper [6], the regularity for shape optimization method using the traction method was discussed.

§ 5. Numerical scheme

For the shape optimization problem, the following numerical procedures are performed iteratively, where FreeFEM++ is used for calculations.

First, the stationary Navier–Stokes equation (2.3) and continuity equation (2.4) are solved using the Newton method. In this method, Eqs. (2.3) and (2.4) are rewritten as $\boldsymbol{F}(\boldsymbol{\alpha}) = \boldsymbol{0}$ for $\boldsymbol{\alpha} = (\hat{\boldsymbol{u}}_1, \hat{p}_1, \cdots, \hat{\boldsymbol{u}}_N, \hat{p}_N)^{\mathrm{T}}$, which is the nodal vector of $(\hat{\boldsymbol{u}}, \hat{p})$ for the finite-element model with N nodes. Here, we assume that the derivative $\boldsymbol{B}(\boldsymbol{\alpha})$ of $\boldsymbol{F}(\boldsymbol{\alpha})$, such that $\boldsymbol{F}[\delta\boldsymbol{\alpha}] = \boldsymbol{B}(\boldsymbol{\alpha})\delta\boldsymbol{\alpha}$, with respect to the variation $\delta\boldsymbol{\alpha} = \boldsymbol{\alpha}^{n+1} - \boldsymbol{\alpha}^n$ of $\boldsymbol{\alpha}$ is obtainable. The solution $\boldsymbol{\alpha}$ of $\boldsymbol{F}(\boldsymbol{\alpha}) = \boldsymbol{0}$ is obtained using the iterative calculations

$$\boldsymbol{\alpha}^{n+1} = \boldsymbol{\alpha}^n - \boldsymbol{B}^{-1}(\boldsymbol{\alpha})\boldsymbol{F}(\boldsymbol{\alpha}^n),$$

where n denotes the number of iterations.

Second, the adjoint equations (3.5) and (3.6) are solved by the conjugate gradient method because the adjoint equations become a linear equation with respect to $(\hat{\boldsymbol{w}}, \hat{q})$.

Third, the traction method is used to obtain the domain variation $\boldsymbol{\psi} = \boldsymbol{\psi}_0 + \Lambda\boldsymbol{\psi}_1$ under the volume constraint, where the conjugate gradient method is used for calculations of linear equations (4.2),(4.3) at $c_a = 0.001$. Finally, using calculated $\boldsymbol{\psi}$, the domain is reshaped by $\boldsymbol{x} + \boldsymbol{\psi}$.

In this study, we use a finite-element model with the number of nodes shown in Table 1, which shows the mesh used in the domain corresponding to the x_c coordinate of the center of the circle $\Omega_{\mathrm{c}}(\boldsymbol{\phi})$, where the P2/P1 finite element for the velocity and the pressure is used to discretize equations spatially: (2.3),(2.4),(3.5),(3.6),(4.2), and (4.3).

§ 6. Calculation results

The minimization problem of dissipation energy in two-dimensional Poiseuille flow with sudden contraction was addressed. Based on previous studies of jamology, the

Table 1. Numbers of nodes and elements are $N_{\text{edge}}, N_{\text{element}}$..

x_c	$(N_{\text{edge}}, N_{\text{element}})$
1.5	(17851, 9241)
1.7	(15775, 8203)

kinematic viscosity has not been defined. Therefore, herein, Re and x_c are changed to demonstrate the shape optimization problem and to elucidate the optimum shape, where Re $= 30, 50, 70, 100$ and $x_c = 1.5, 1.7$. Using the traction method, the degree of freedom discretized with the finite element method for numerical calculations is not reduced.

Iteratively conducting numerical procedures described in section 5, iterative histories of f_0 and f_1 are obtained as shown in Fig. 2, using N, M which are the cost function and the volume on the initial domain

$$f_0 = \frac{1}{N} \int_{\Omega(\boldsymbol{\phi})} \frac{2}{\mathrm{Re}} \boldsymbol{E}(\hat{\boldsymbol{u}}) \cdot \boldsymbol{E}(\hat{\boldsymbol{u}}) dx,$$

and

$$f_1 = \frac{1}{M} \int_{\Omega(\boldsymbol{\phi})} dx.$$

Based on the definition above, f_0 and f_1 are equal to 1 at the first step. The numerical procedures are stopped under the convergence condition $|f_0^{m+1} - f_0^m| < 0.01$, where m denotes the number of iterations of the numerical procedures. Fig. 2 shows that iterative histories of f_0 and f_1 for Re $= 30$ at $x_c = 1.5, 1.7$, from which f_0 decreases monotonically while satisfying the volume constraint. From calculations for Re $= 50, 70, 100$ at $x_c = 1.5, 1.7$, it was confirmed that f_0 decreases monotonically while satisfying the volume constraint.

The optimum domains are shown respectively in Fig. 3 for Re $= 30$, Fig. 4 for Re $= 50$, Fig. 5 for Re $= 70$ and Fig. 6 for Re $= 100$. For $x_c = 1.5$, the disk is shaped elliptically, but for $x_c = 1.7$, the shape in the wake of the disk is becoming an acute angle with decreasing Re from 100 to 30. In this study, shape optimization is operated in the domain defined by the incompressible Newtonian fluid. With increasing x_c, the distance between Γ_{wall} and $\partial\Omega_{\text{c}}(\boldsymbol{\phi})$ becomes narrow near the sudden contraction, where the dissipation energy is becoming larger. As a result, the shape in the wake of the disk becomes an acute angle to decrease the dissipation energy. Regarding the flow volume, $\int_{\Gamma_{\text{in}}} u_{\text{D}} d\gamma$ is nearly equal to the flow volume $\int_{\Gamma_{\text{out}}} u d\gamma$ in all case studies.

§ 7. Conclusions

As described in this paper, the shape of a column located near an exit of a room was obtained as a solution of the shape optimization problem of the steady Navier–Stokes flow field of incompressible fluid for which people can evacuate smoothly. Although the movement of people has been assumed as a compressible fluid in previous studies, it was assumed as an incompressible fluid for the studies described herein for the convenience of using a method that has already been developed. Numerical results reveal that the shapes in the wake of the disk are becoming acute angles to decrease the dissipation energy monotonically, thereby satisfying the volume constraint. Such a shape has never been suggested in previous studies of the evacuation problem in Jamology. For the next step, it is necessary that shape optimization of the column should be operated in the domain defined by compressible Navier–Stokes equations under appropriate cost functions and constraint functions, where the kinematic viscosity should be obtained by experimental studies. Moreover, this approach is expected to be applicable to various situations such as determination of the number, the volume, and the position of the columns to achieve smooth and safe evacuation.

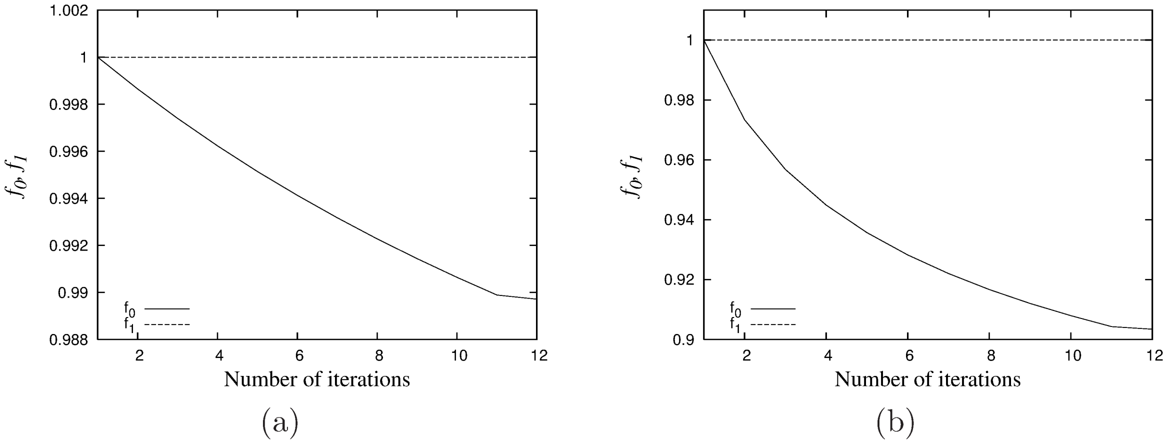


Figure 2. Iterative histories of f_0 and f_1 for Re $= 30$: (a) $x_c = 1.5$, and (b) $x_c = 1.7$.

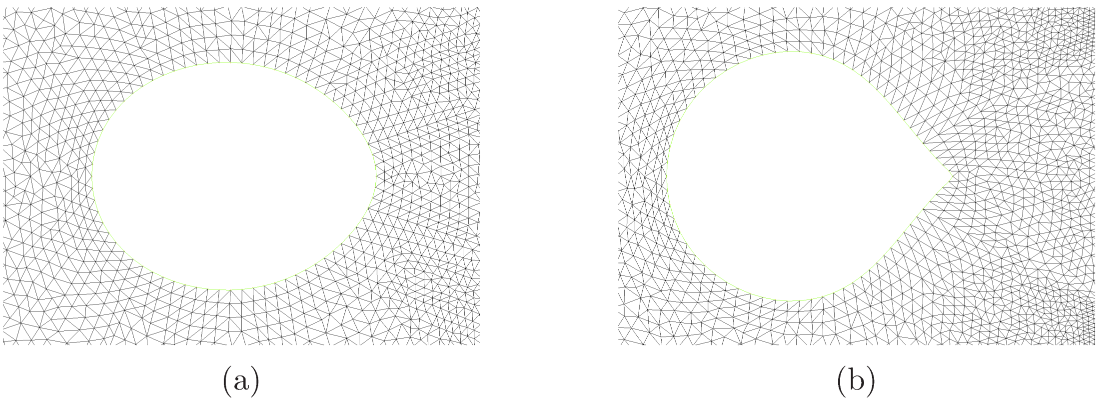
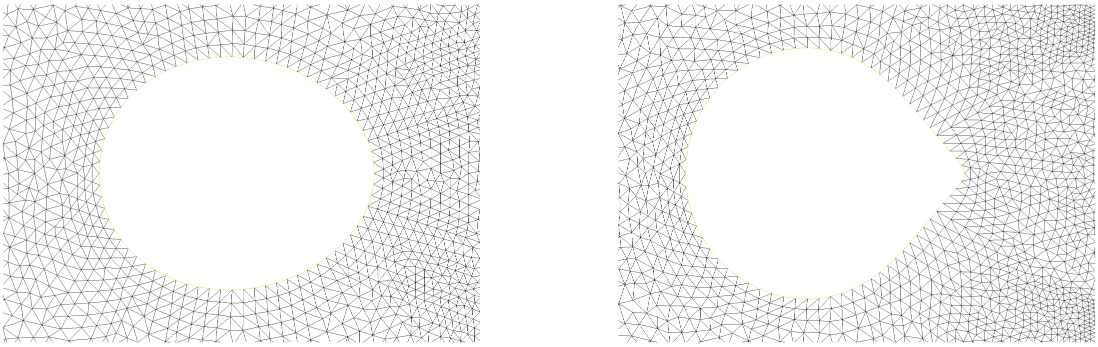
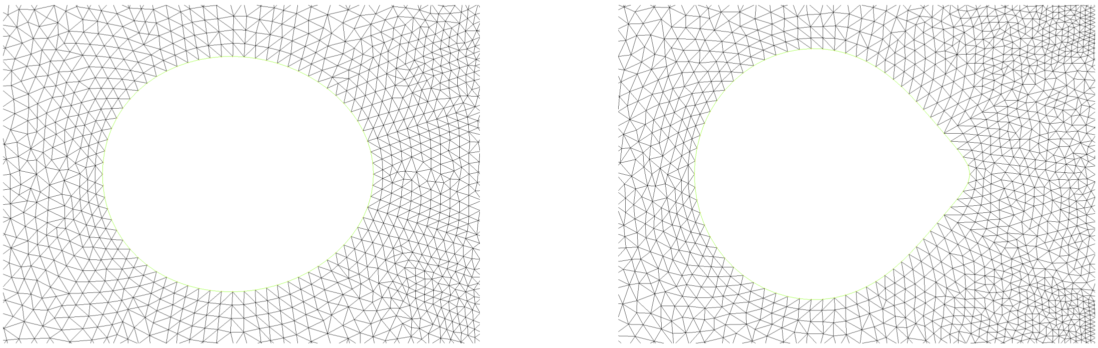


Figure 3. Optimum shapes at $x_c = 1.5$ and $x_c = 1.7$ for Re $= 30$.



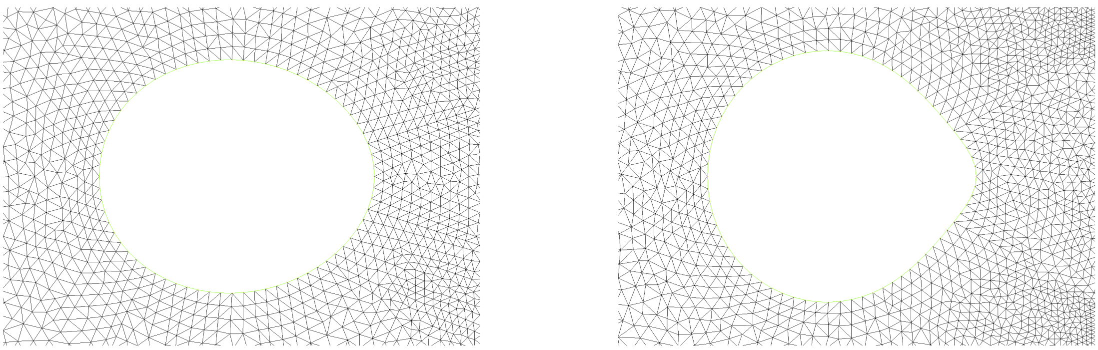
(a) (b)

Figure 4. Optimum shapes at $x_c = 1.5$ and $x_c = 1.7$ for Re = 50.



(a) (b)

Figure 5. Optimum shapes at $x_c = 1.5$ and $x_c = 1.7$ for Re = 70.



(a) (b)

Figure 6. Optimum shapes at $x_c = 1.5$ and $x_c = 1.7$ for Re = 100.

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