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Projections of hypersurfaces in $\mathbb{R}^4$ to planes

By

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Abstract

The aim of this report is to give the singularities of orthogonal projections of a generic embedded hypersurface M in $\mathbb{R}^4$ with or without boundary, to a 2-dimensional plane. The singularities occurring at interior points have been classified in [11] (see also [14]), and the singularities occurring at the boundary points have been classified in [10]. For the first case, we need to classify map germs from $\mathbb{R}^3$ to the plane ($\mathcal{A}$-group), and for the second case we need to classify map germs from $\mathbb{R}^3$ to the plane with the source containing a distinguished plane which is preserved by coordinate changes ($\mathcal{B}$-subgroup). The singularities of such maps measure, for instance, the contact of M with 2-dimensional planes.

§ 1. Introduction

This report is part of our investigation of the flat geometry of generic embedded hypersurfaces in $\mathbb{R}^4$. In [12], we study the contact of M with hyperplanes and lines, and in [11] and [10] we deal with the contact with planes. We consider here the contact with planes. Hypersurfaces M in $\mathbb{R}^4$ are often defined explicitly as the image of a smooth mapping $U \to \mathbb{R}^4$ possibly with singularities, where U is an open subset of $\mathbb{R}^3$. Projections of hypersurfaces M in $\mathbb{R}^4$ into planes are parametrized by the Grassmanian of 2-planes in $\mathbb{R}^4$, $G(2,4)$. Then an orthogonal projection of M to a plane can be represented locally by a germ of a map $(\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$ (see [11]). If a and b are an orthonormal basis of the plane of projection $u \in G(2,4)$ then the family of orthogonal projections to 2-spaces is given by

$$\begin{array}{rcl}\Pi : M \times G(2,4) & \to & \mathbb{R}^2 \\ (p,u) & \mapsto & (\langle p,a\rangle, \langle p,b\rangle),\end{array}$$

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where $\langle , \rangle$ stands for the inner product.

Given $u \in G(2,4)$, the map Π_u measures the contact between M and the plane orthogonal to u, the kernel of Π_u. (Note that Π_u is of corank at most 1). If p is a corank 1 singular point of Π_u then the orthogonal plane to u is a subset of T_pM.

Let $\mathcal{E}_n$ be the local ring of germs of functions $(\mathbb{R}^n,0) \to \mathbb{R}$ and m_n its maximal ideal (which is the subset of germs that vanish at the origin). Denote by $\mathcal{E}(n,p)$ the p-tuples of elements in $\mathcal{E}_n$. Let $\mathcal{A} = \mathcal{R} \times \mathcal{L} =$Diff$(\mathbb{R}^n,0)\times$Diff$(\mathbb{R}^p,0)$ denote the Mather group of right-left equivalences which acts smoothly on $m_n.\mathcal{E}(n,p)$ by $(h,k).f = k \circ f \circ h^{-1}$. Then two map germs $f_1, f_2 : (\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$ are said to be $\mathcal{A}$-equivalent, denoted by $f_1 \sim f_2$, if $f_2 = k \circ f_1 \circ h^{-1}$ for some germs of diffeomorphisms h and k of the source and target, respectively.

The singularities occurring at interior points of the hypersurface were classified by using the group $\mathcal{A}$ in [11]. If the hypersurface has boundary points we also need to use the group $\mathcal{B}$ as described below.

We shall use (x,y,z) coordinates on $\mathbb{R}^3$ and $\mathbb{R}^2 \times \{0\}$ is, naturally, the xy-plane. This corresponds to the boundary of our manifold with boundary, whose interior is taken to be that part of $\mathbb{R}^3$ with $z > 0$.

The group $\mathcal{B}$ is the subgroup of $\mathcal{A}$ consisting of pairs of germs of diffeomorphisms (h,k) in Diff$(\mathbb{R}^3) \times$ Diff$(\mathbb{R}^2)$ with h preserving the manifold as well as its boundary $\mathbb{R}^2 \times \{0\}$ (that is, h takes the variety $V = \{(x,y,z) : z \geq 0\}$ into itself). Then $\mathcal{B}$ acts on the set of map germs $f : (\mathbb{R}^3,0) \to (\mathbb{R}^2,0)$ with $h(x,y,0) = (h_1(x,y,0), h_2(x,y,0), 0)$. Therefore, if $(h,k) \in \mathcal{B}$ we can write $h(x,y,z) = (h_1(x,y,z), h_2(x,y,z), zh_3(x,y,z))$ with $h_3(0,0,0) > 0$, for germs of smooth functions h_i, $i = 1,2,3$.

The method used for the classification in [10] and [11] is that of the complete transversal (Complete transversal, Proposition 2.2 in [4]) together with Mather's Lemma (Lemma 3.1, [9]). The notion of simple germs is defined in [1]. This method of classification of map germs is well known for the group $\mathcal{A}$, but also works for the group $\mathcal{B}$. In fact, the results on finite determinacy and complete transversal were initially proved for the groups $\mathcal{L}$, $\mathcal{R}$, $\mathcal{C}$, $\mathcal{K}$ and $\mathcal{A}$ (see [4, 5]). However, Damon showed that these results are also valid for a larger class of subgroups of $\mathcal{K}$ and $\mathcal{A}$, which he called *geometric subgroups* of $\mathcal{K}$ and $\mathcal{A}$ ([6]). These are subgroups that satisfy some algebraic properties that ensure that all the results on finite determinacy and versal unfoldings are valid for the action of such subgroups on $m_n.\mathcal{E}(n,p)$. The group $\mathcal{B}$ inherits the action of the group $\mathcal{A}$ on $m_3.\mathcal{E}(3,2)$. As it is a Damon geometric subgroup $\mathcal{B}(n)$ of $\mathcal{A}$ ($\mathcal{B}(n) = \mathcal{A}_V$ in the notation of [6]), the determinacy results (see [5]) apply to $\mathcal{B}$.

Given a map-germ $f \in m_n.\mathcal{E}(n,p)$, θ_f denotes the set of germs of vector fields along f (these are sections of the pull-back of the tangent bundle of the target manifold). We set $\theta_n = \theta_{\mathrm{id}_{\mathbb{R}^n,0}}$ and $\theta_p = \theta_{\mathrm{id}_{\mathbb{R}^p,0}}$, where $\mathrm{id}_{\mathbb{R}^n,0}$ and $\mathrm{id}_{\mathbb{R}^p,0}$ denote the germs of the identity maps on $(\mathbb{R}^n, 0)$ and $(\mathbb{R}^p, 0)$ respectively. One can define the homomorphisms $tf : \theta_n \to \theta_f$, by $tf(\psi) = Df.\psi$, and $wf : \theta_p \to \theta_f$, by $wf(\phi) = \phi \circ f$, where Df stands for the differential of f.

The tangent space to the $\mathcal{A}$-orbit of f at the germ f is given by

$$\begin{aligned} L\mathcal{A}(f) &= tf(m_n.\theta_n) + wf(m_p.\theta_p) \\ &= m_n.\{f_{x_1}, \ldots, f_{x_n}\} + f^*(m_p).\{e_1, \ldots, e_p\}, \end{aligned}$$

where f_{x_i} denotes the partial derivative with respect to x_i $(i = 1, \ldots, n)$, $\{e_1, \ldots, e_p\}$ is the standard basis vectors of $\mathbb{R}^p$ considered as elements of $\mathcal{E}(n, p)$, and $f^*(m_p)$ is the pull-back of the maximal ideal in $\mathcal{E}_p$.

The extended tangent space to the $\mathcal{A}$-orbit of f at the germ f is given by

$$\begin{aligned} L_e\mathcal{A}(f) &= tf(\theta_n) + wf(\theta_p) \\ &= \mathcal{E}_n.\{f_{x_1}, \ldots, f_{x_n}\} + f^*(\mathcal{E}_p).\{e_1, \ldots, e_p\}. \end{aligned}$$

Let $k \geq 1$ be an integer. We denote by $J^k(n, p)$ the space of k-th order Taylor expansions without constant terms of elements of $\mathcal{E}(n, p)$ and write $j^k f$ for the k-jet of the map f. A germ f is said to be *k-$\mathcal{A}$-determined* if every map g with $j^k g = j^k f$ is $\mathcal{A}$-equivalent to f (notation: $g \sim f$). The k-jet of f is then called a sufficient jet. (See for example [2, 5, 16] for finite determinacy criteria.)

We follow these germs and carry out the classification inductively on the jet level, using the complete transversal method [4] and the "Transversal" package [7] to help us to apply the method. We observe that we adapted such package to work in the case of hypersurface with boundary. The tangent space calculated by this original package is for the usual groups of Mather, so we changed it for the group $\mathcal{B}$.

The $\mathcal{B}$ (resp. $\mathcal{B}_e$) tangent space of $f \in m_3.\mathcal{E}(3, 2)$ is given by

$$\begin{aligned} L\mathcal{B}.f &= m_3.\{f_x, f_y\} + \mathcal{E}_3\{zf_z\} + f^*m_2.\{e_1, e_2\}, \\ L\mathcal{B}_e.f &= \mathcal{E}_3.\{f_x, f_y\} + \mathcal{E}_3\{zf_z\} + f^*\mathcal{E}_2.\{e_1, e_2\}. \end{aligned}$$

The difference between the tangent spaces $L\mathcal{A}.f$ and $L\mathcal{B}.f$ comes from the tangent of the right action of $\mathcal{B}$ that is represented by the diffeomorfism $h(x, y, z) = (h_1(x, y, z), h_2(x, y, z), zh_3(x, y, z))$ which preserves also the boundary of the manifold as we explain above. Then the difference between applying complete transversal to obtain $\mathcal{A}$ and $\mathcal{B}$ classifications consists in do not use vectors of $m_3\{f_z\}$ distinct from $\mathcal{E}_3\{zf_z\}$ for $L\mathcal{B}.f$, because the $\mathcal{B}$ group needs to preserve the boundary $z = 0$.

The codimension of the (resp. extended) orbit of f is given by

$$\begin{aligned} \mathcal{B}\text{−cod} &= \dim_{\mathbb{R}}(m_n.\mathcal{E}(n, p)/L\mathcal{B}(f)), \\ \mathcal{B}_e\text{−cod} &= \dim_{\mathbb{R}}(\mathcal{E}(n, p)/L_e\mathcal{B}(f)). \end{aligned}$$

As the usual meaning, F is an $\mathcal{B}_e$-s-versal unfolding of f if, and only if, $L\mathcal{B}_e.f + \mathbb{R}\{\dot{F}_1, \dots, \dot{F}_s\} = \mathcal{E}(3,2)$, where $F(x,y,z,u) = (g(x,y,z,u),u)$ with $u \in \mathbb{R}^s$, $g(x,y,z,0) = f(x,y,z)$ and $\dot{F}_i(x,y,z) = \frac{\partial g}{\partial u_i}(x,y,z,0)$.

The situation in this case is similar to that considered in [3], where a classification of codimension ≤ 2 singularities of map-germs $(\mathbb{R}^2,0) \to (\mathbb{R}^2,0)$ up to diffeomorphisms in the source that preserve the variety $\{(x,y) : y \geq 0\}$ and any diffeomorphism in the target is given.

Since the $\mathcal{A}$-equivalence classes of Π_u do not depend on the choice of orthonormal basis a, b of u (see [11]), then we can expect the generic $\mathcal{A}$ and $\mathcal{B}$-equivalence classes of Π_u to be those of $\mathcal{A}_e$ or $\mathcal{B}_e$-codimension ≤ 4.

§ 2. Classification

Germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of corank 1 that can be written in the form $(x, g(x,y) \pm z^2)$ in some coordinate system are classified by Rieger and Ruas in [14].

Theorem 2.1. *[14] The germs $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of corank 1 and $\mathcal{A}_e$-codimension (or codimension of the stratum) ≤ 4, that are $\mathcal{A}$-equivalent to one of the form $(x, g(x,y) \pm z^2)$ are given below, where all the classes are simple with the exception of* $8, 9, 10, 15, 18$ *and* 19.

Table 1: Normal forms of $\mathcal{A}_e$-codimension ≤ 4

Name	*Normal form*	$\mathcal{A}_e$*-cod*
1	(x,y)	0
2 *(fold)*	$(x, y^2 \pm z^2)$	0
3 *(cusp)*	$(x, xy + y^3 \pm z^2)$	0
4_k *(lips/beaks when $k=2$)*	$(x, y^3 + (\pm 1)^{k-1} x^k y \pm z^2), 2 \leq k \leq 5$	$k-1$
5 *(swallowtail)*	$(x, xy + y^4 \pm z^2)$	1
6	$(x, xy + y^5 \pm y^7 \pm z^2)$	2
7	$(x, xy + y^5 \pm z^2)$	3
8	$(x, xy + y^6 \pm y^8 + ay^9 \pm z^2)$	$4(3\dagger)$
9	$(x, xy + y^6 + y^9 \pm z^2)$	4
10	$(x, xy + y^7 \pm y^9 + ay^{10} + by^{11} \pm z^2)$	$6^*(4\dagger)$
11_{2k+1}	$(x, xy^2 + y^4 + y^{2k+1} \pm z^2), 2 \leq k \leq 4$	k
12	$(x, xy^2 + y^5 + y^6 \pm z^2)$	3
13	$(x, xy^2 + y^5 \pm y^9 \pm z^2)$	4
15	$(x, xy^2 + y^6 + y^7 + ay^9 \pm z^2)$	$5(4\dagger)$
16	$(x, x^2y + y^4 \pm y^5 \pm z^2)$	3
17	$(x, x^2y + y^4 \pm z^2)$	4

18	$(x, x^2y + xy^3 + ay^5 + y^6 + by^7 \pm z^2)$	$6^*(4\dagger)$
19	$(x, x^3y + ax^2y^2 + y^4 + x^3y^2 \pm z^2)$	$5(4\dagger)$

† codimension of the stratum

$*$ excluding exceptional values of the moduli.

We complete in [11] the list in Theorem 2.1 and obtain the remaining germs of codimension ≤ 4. These germs are those that can not be written in the form $(x, g(x, y) \pm z^2)$.

Theorem 2.2. *[11] The $\mathcal{A}$-classes of the singularities $\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of corank 1 and $\mathcal{A}_e$-codimension (or codimension of the stratum in the presence of moduli) ≤ 4 that are not listed in Theorem 2.1 are given in Table 2.*

Table 2: More normal forms of $\mathcal{A}_e$-codimension ≤ 4

Name	*Normal form*	$\mathcal{A}_e$-*cod*	δ
N_1	$(x, xy + y^3 + ay^2z + z^3 \pm z^5)$ $a \neq 0,\ 4a^3 \pm 27 \neq 0$	3 (2†)	5
N_2	$(x, xy + y^3 + ay^2z + z^3)$, $a \neq 0,\ 4a^3 + 27 \neq 0$	4 (3†)	5
N_3	$(x, xy + y^3 + z^3 \pm y^3z)$	3	4
N_4	$(x, xy \pm y^2z + z^3 \pm z^5)$	3	5
N_5	$(x, xy \pm y^3 + yz^2 + z^5)$	3	5
N_6	$(x, xy \pm y^2z + z^3)$	4	5
N_7	$(x, xy + y^3 + z^3 \pm y^4z)$	4	5
N_8	$(x, xy + z^3 \pm y^4 + y^3z + ay^4z + b(y^6 + \lambda y^5z)),\ b \neq 0$	6 (4†)	6
N_9	$(x, xy + yz^2 \pm y^4 + z^5 + ay^6)$	5 (4†)	6
N_{10}	$(x, xy + y^2z + yz^3 \pm z^4 + az^6)$	5 (4†)	6
N_{11}	$(x, xy \pm y^3 + yz^2 + z^7)$	4	7
N_{12}	$(x, xyz \pm y^2z + z^3 + ax^2y + bx^2z + cyz^3 + z^4)$	7$*$ (4†)	4

where λ is a constant, and δ denotes the determinacy degree of the germs.

† codimension of the stratum

$*$ $4b - 1 \neq 0$ and $4b - 1 + 6ac \neq 0$.

Boundary points (see [10]): The list of orbits of simple germs of corank at most 1 and $\mathcal{B}_e$-codimension less than or equal to 4 of $\mathcal{B}$ action are given in Theorem 2.3. Note that the germs (x, y) and $(x, z + g(x, y))$ are all submersions.

Theorem 2.3. *[10] The $\mathcal{B}$-simple map-germs $(\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ of corank at most 1 and $\mathcal{B}_e$-codimension ≤ 4 are given in Table 3.*

Table 3: Normal forms of $\mathcal{B}_e$-codimension ≤ 4 ($\epsilon, \epsilon_1 = \pm 1$).

Normal form	$\mathcal{B}_e$-codimension
(x, y)	0
$(x, z + \epsilon y^2)$	0
$(x, z + xy + y^3)$	0
$(x, z + y^3 + \epsilon^{k-1} x^k y)$, $k \geq 2$	$k - 1$
$(x, z + xy + \epsilon y^4)$	1
$(x, z + xy + y^5 + \epsilon y^7)$	2
$(x, z + xy + y^5)$	3
$(x, z + xy + \epsilon y^6 + y^9)$	4
$(x, z + xy^2 + \epsilon y^4 + y^{2k+1})$, $k \geq 2$	k
$(x, z + xy^2 + y^5 + \epsilon y^6)$	3
$(x, z + xy^2 + y^5 + \epsilon y^9)$	4
$(x, z + x^2 y + \epsilon y^4 + \epsilon_1 y^5)$	3
$(x, z + x^2 y + \epsilon y^4)$	4
$(x, y^2 + \epsilon z^2 + {\epsilon_1}^k x^{k-1} z)$, $k \geq 2$	$k - 2$
$(x, y^2 + xz + \epsilon z^3)$	1
$(x, y^2 + xz + \epsilon z^4 + \epsilon_1 z^6)$,	2
$(x, y^2 + xz + \epsilon z^4)$	3
$(x, yz + xy + \epsilon y^3)$	1
$(x, yz + xy + y^4 + \epsilon y^6)$	2
$(x, yz + xy + y^4)$	3
$(x, xy + z^2 + y^3 + \epsilon y^k z)$, $k \geq 2$	k

By studying the normal forms of table 3, we have the following important results.

Theorem 2.4. *[10] The map-germs $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ with 1-jet equivalent to $(z, 0)$ or 2-jet equivalent to (x, y^2), (x, xy), (x, yz), $(x, xz + z^2)$, (x, xz), (x, z^2), $(x, 0)$ or 3-jet $(x, xy + z^2 + \epsilon y^2 z)$, $(x, xy + z^2)$ or 4-jet $(x, y^2 + xz)$, $(x, xy + yz)$ are non-$\mathcal{B}$-simple germs.*

Theorem 2.5. *[10] The map-germs $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ of corank at most 1 and $\mathcal{B}_e$-codimension ≤ 1 are $\mathcal{B}$-simple germs.*

§ 3. The geometry of codimension ≤ 1 singularities

Geometric criteria for recognition of the singularities of $\mathcal{A}_e$-codim ≤ 1 of germs $\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ are given in [8] and [15]. In [11], we adapt these criteria to germs f :

$\mathbb{R}^3, 0 \to \mathbb{R}^2, 0$. When f has corank 1, we can change coordinates and write $f(x, y, z) = (x, g(x, y, z))$. The differential of f at (x, y, z) is then given by

$$df(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{\partial g}{\partial x}(x, y, z) & \frac{\partial g}{\partial y}(x, y, z) & \frac{\partial g}{\partial z}(x, y, z) \end{pmatrix}$$

so that the critical set of f is given by

$$\Sigma = \{(x, y, z) : \frac{\partial g}{\partial y}(x, y, z) = \frac{\partial g}{\partial z}(x, y, z) = 0\}.$$

Let $G : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ such that $G(x, y, z) = (g_y(x, y, z), g_z(x, y, z))$. $\Sigma = G^{-1}(0)$ is locally a smooth curve if G is regular, that is, if G has maximal rank at $(0, 0, 0)$.

It is easy to show that if Σ is smooth, then f can be written in some coordinate systems in the form $(x, g(x, y) \pm z^2)$. In this case $\Sigma = \{(x, y, 0); g_y(x, y) = 0\}$. Let $\phi : I, 0 \to \mathbb{R}^3, 0$ be a parametrization of Σ, where I is a neighbourhood of 0 in $\mathbb{R}$. The order of contact of Σ with the kernel of $df(0, 0, 0)$ ($ker(df(0, 0, 0))$) is the order of the vanishing of the derivatives of $f \circ \phi$ at 0. This order of contact is independent of the parametrization of Σ and is an $\mathcal{A}$-invariant of the map f.

We use this order of contact to recognize geometrically the fold, cusp and swallowtail singularities in the list of Theorem 2.1. When the critical set is smooth, we can set $f(x, y, z) = (x, g(x, y) \pm z^2)$ as the order of contact is invariant.

Considering the above considerations we have the following propositions. We observe that the next two results are the statement that the fold and cusp orbits coincide with the Boardman strata $\Sigma^{1,0}$ and $\Sigma^{1,1,0}$, respectively.

Proposition 3.1. *[11] Let $f : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ a singular germ with a smooth critical set Σ and let ϕ be a parametrization of Σ. Then*

(i) f is a fold if and only if $\frac{\partial}{\partial t}(f \circ \phi)(0) \neq 0$.

(ii) f is a cusp if and only if $\frac{\partial}{\partial t}(f \circ \phi)(0) = 0$ and $\frac{\partial^2}{\partial t^2}(f \circ \phi)(0) \neq 0$.

The conditions in Proposition 3.1 reflect the order of contact of Σ with the kernel of $df(0, 0, 0)$.

Corollary 3.2. *[11] A singular germ $f : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of corank 1 and with a smooth critical set is a fold if and only if Σ is transversal to the set* $\ker(df(0, 0, 0))$ *at the origin. It is a cusp if and only if Σ and* $\ker(df(0, 0, 0))$ *have* 2*-point contact at the origin.*

Proposition 3.3. *[11] Let $f : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ be a singular germ with a smooth critical set Σ and let ϕ be a parametrization of Σ. Then f is a swallowtail if and only if $\frac{\partial}{\partial t}(f \circ \phi)(0) = \frac{\partial^2}{\partial t^2}(f \circ \phi)(0) = 0$ and $\frac{\partial^3}{\partial t^3}(f \circ \phi)(0) \neq 0$.*

Again the conditions in Proposition 3.3 express the order contact of the critical set Σ with the $\ker(df(0,0,0))$.

Corollary 3.4. *[11] A singular germ $f : \mathbb{R}^3, 0 \to \mathbb{R}^2, 0$ of corank 1 and with a smooth critical set is a swallowtail if and only if the critical set Σ has contact 3 with the kernel of $df(0,0,0)$ at the origin.*

When f has a lips/beaks singularity, its critical set Σ is singular. Then we need additional algebraic conditions for recognizing these singularities.

Proposition 3.5. *[11] A singular germ that can be written in the form $f = (x, g(x,y) \pm z^2)$ is a lips/beaks if and only if $\frac{\partial^2 g}{\partial y^2}(0,0) = \frac{\partial^2 g}{\partial x \partial y}(0,0) = 0$, $\frac{\partial^3 g}{\partial y^3}(0,0) \neq 0$ and $(\frac{\partial^3 g}{\partial x \partial y^2}(0,0))^2 - \frac{\partial^3 g}{\partial x^2 \partial y}(0,0).\frac{\partial^3 g}{\partial y^3}(0,0) \neq 0$.*

Proposition 3.6. *[11] The germ $\frac{\partial g}{\partial y} : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ is a germ of a Morse function if and only if $\frac{\partial^2 g}{\partial y^2}(0,0) = \frac{\partial^2 g}{\partial x \partial y}(0,0) = 0$ and $(\frac{\partial^3 g}{\partial x \partial y^2})^2(0,0) - \frac{\partial^3 g}{\partial x^2 \partial y}(0,0).\frac{\partial^3 g}{\partial y^3}(0,0) \neq 0$.*

As $\mathcal{A}$-equivalence preserves the singularity of the critical set, we have the following result.

Corollary 3.7. *[11] Let $f(x,y,z) = (x, g(x,y) \pm z^2))$. If $\frac{\partial^3 g}{\partial y^3}(0,0) \neq 0$, then f is a lips/beaks if and only if the critical set has a non-degenerate curve singularity.*

Remark. The above geometric criteria do not extend to higher codimension singularities. The reason is that when Σ is smooth its order of contact k with the kernel of $df(0,0,0)$ determines the A_{k-1} $\mathcal{K}$-class of the projection. When $k \geq 4$, there are several $\mathcal{A}$-orbits of the projection inside the A_{k-1}-orbit. One needs algebraic conditions to distinguish between these $\mathcal{A}$-orbits.

For the boundary points: we collect together in [10] the normal forms of germs $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ of codimension ≤ 1 which arise from parallel projections of hypersurfaces with boundary (so, they are of corank at most 1 and also simple germs by Theorem 2.5) and, in order to recognize the different cases, we go into some detail on their geometrical properties.

Table 4: Map germs $(\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ of codimension ≤ 1, $\epsilon, \epsilon_1 = \pm 1$.

No.	Normal form	$\mathcal{B}_e$-codimension of orbit
I	(x, y)	0
II	$(x, z + \epsilon y^2)$	0
III	$(x, z + xy + y^3)$	0
IV	$(x, z + \epsilon x^2 y + y^3)$	1
V	$(x, z + xy + \epsilon y^4)$	1
VI	$(x, y^2 + \epsilon z^2 + xz)$	0
VII	$(x, y^2 + \epsilon z^2 + \epsilon_1 x^2 z)$	1
VIII	$(x, y^2 + xz + \epsilon z^3)$	1
IX	$(x, yz + xy + \epsilon y^3)$	1

We give (see [10]), for each non-submersive germ f of codimension 1 in Table 4, a $\mathcal{B}_e$-versal unfolding and a bifurcation diagram to show, for germs close to f in the $\mathcal{B}_e$-versal unfolding, which types of boundary singularities occur. Since the submersions IV and V do not have singular points, then the bifurcation diagrams are given for the restriction of f to the boundary. The notation VI^2 indicates the presence of two singularities of Type VI arbitrarily near the origin for germs in an appropriate region of the diagram. Similarly, VI$_-$ and VI$_+$ mean singularities of Type VI, with $\epsilon = -1$ and $\epsilon = 1$, respectively.

It is also of interest to find, for each germ in Table 4, a single hypersurface M for which the family of parallel projections realizes a versal unfolding of the germ. More precisely, suppose that

$$i : (x, y, z) \mapsto (X(x, y, z), Y(x, y, z), Z(x, y, z), W(x, y, z))$$

is a (germ of an) immersion at $(0, 0, 0)$, so that we can regard the image as a small piece of the smooth hypersurface M in $\mathbb{R}^4$. We are interested in the restriction to $z \geq 0$.

If Π is the family of orthogonal projections to 2-spaces given in §1 and $u \in G(2, 4)$, then the map Π_u measures the contact between M and the plane orthogonal to u. Let $p \in \mathbb{R}^4$ be the origin and let us suppose that

$$T_pM = \langle (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) \rangle.$$

If p is a corank 1 singular point of Π_u then the orthogonal plane to u is a subset of T_pM. So the generators of the plane u can be taken as a vector a of T_pM and a vector b normal to T_pM at p.

The family of planes in $\mathbb{R}^4$ close to the plane u generated by $a = (1, 0, 0, 0)$ and $b = (0, 0, 0, 1)$ may be given taking $(1, \beta_1, \gamma_1, 0)$ and $(0, \beta_2, \gamma_2, 1)$ as generators of those planes, for $\beta_1, \beta_2, \gamma_1, \gamma_2$ close to 0. So, we get the map

$$\Pi_{\beta,\gamma}(X, Y, Z, W) = (X + \beta_1 Y + \gamma_1 Z, W + \beta_2 Y + \gamma_2 Z) \ , \tag{3.1}$$

where β and γ denote the pairs (β_1, β_2) and (γ_1, γ_2). Note that $\Pi_{0,0} = \Pi_u$. To realize a versal unfolding with 1-parameter we take $\beta_1 = \gamma_1 = \beta_2 = 0$ and call $\gamma_2 = \lambda$, where $(x, y, z) \to (X, Y, Z, W)$ is an imersion at $(0, 0, 0)$.

For each normal form we give pictures of the source X together with Σ (i.e. $\Sigma = \Sigma_1$, the critical set of the map f), and the kernel K of $df(0)$. We also draw the critical loci (image of Σ) and of the image of the boundary, for f and for members of the versal family unfolding f. As aids to recognition of the various cases, we give pictures of the fibre $f^{-1}(0)$ and also information about the singularities of the restriction of f to the boundary (see [13]). Parts of Σ or of its image that are virtual, in the sense of corresponding to the part $z < 0$ in the source $\mathbb{R}^3$, appear dashed in the figures. The boundary (plane-xy) and its image are drawn with the gray color. See [10] for the next results.

Geometrical information in Cases I-IX [10].

I-V. These are germs of submersions and so K is a line. Except Case I whose K is transverse to the boundary (that is, on M, the direction of projection is transverse to ∂M), in the other cases K is a subset of the boundary. For I and III to V the image of the boundary is $\mathbb{R}^2$.

- Cases I-III are stable submersions and so realized simply by the following immersions:
I. $i(x, y, z) = (x, 0, z, y)$. See Fig.1(a).
II. $i(x, y, z) = (x, y, 0, z + \epsilon y^2)$. The image of the boundary is, unlike all the other Cases I-V, a semi-plane. Furthermore, the fibre $f^{-1}(0)$ is a curve tangent to the boundary, for $\epsilon = -1$ (see Fig. 1 (b)), and $f^{-1}(0) = 0$, for $\epsilon = 1$ (seeFig.1(c)). The singularity of the restriction of f to the boundary is the fold (x, y^2).
III. $i(x, y, z) = (x, y, 0, z + xy + y^3)$. The set $f^{-1}(0)$ is tangent to the boundary., see Fig.1(d). The singularity of the restriction of f to the boundary $z = 0$ is the cusp $(x, xy + y^3)$.
- Cases IV and V are codimension 1 submersions. For these cases Σ is empty and we can not consider the bifurcation diagram for the germ f. Then the bifurcation diagrams are given for the restriction of f to the boundary that has codimension 1 singularity only in these cases.
IV. A $\mathcal{B}_e$-versal unfolding is $(x, z + y^3 + \epsilon x^2 y + \lambda y)$. A realization is $i(x, y, z) = (x, y, 0, z + y^3 + \epsilon x^2 y)$. The fibre $f^{-1}(0)$ is also tangent to the boundary, as in Case III (Fig.1(d))

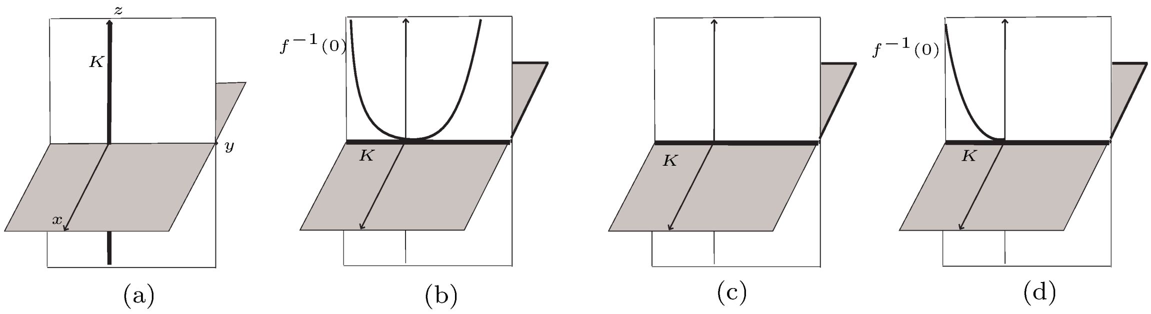

Figure 1. Submersions. (a) Case I. (b) Cases II and V, for $\epsilon = -1$. (c) Cases II and V, for $\epsilon = 1$. (d) Cases III and IV.

but, unlike this case, the singularity of the restriction of f to the boundary $z = 0$ is the lips/beaks $(x, \epsilon x^2 y + y^3)$, which is a codimension 1 singularity.

V. A $\mathcal{B}_e$-versal unfolding is $(x, z + xy + \epsilon y^4 + \lambda y^2)$ and a realization is $i(x, y, z) = (x, y^2, y, z + xy + \epsilon y^4)$. Unlike all Cases I-IV, the boundary $z = 0$ is mapped to a curve with a singularity of type $(3, 4)$ (see Fig.2). The fibre $f^{-1}(0)$ is as in the Case II (Fig.1, (b) and (c)) but, unlike this case, the singularity of the restriction of f to the boundary $z = 0$ is the swallowtail $(x, xy + y^4)$, which is a codimension 1 singularity. We remember that the drawn with the gray color below is the image of the boundary, that is $f(x, y, 0)$.

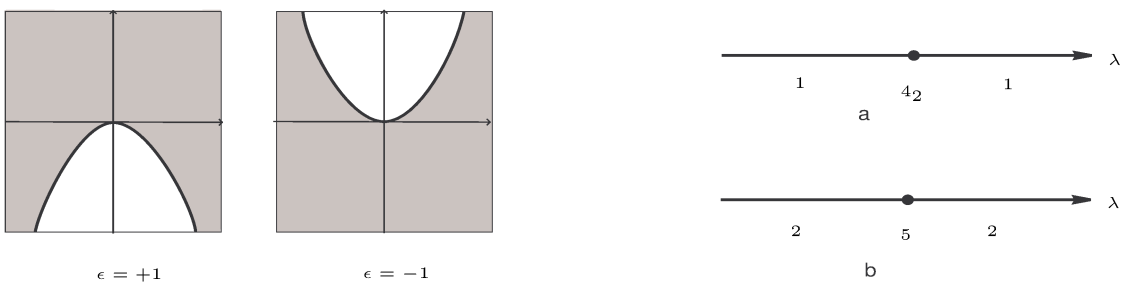

Figure 2. Left: The image of the boundary of Case V. Right: Bifurcation diagrams of the singularities for the restriction of f to the boundary, cases IV and V, respectively (a) and (b). (See Table 1 for notation of Rieger.)

VI-IX. These are those germs with $K = \ker df(0)$ being the plane-yz.

-Case VI. This is stable and so realized by an immersion $i(x, y, z) = (x, y, z, y^2 + \epsilon z^2 + xz)$. The sets K and Σ are transverse to each other and to the boundary, whose image is a semi-plane containing $f|_\Sigma$ for $\epsilon = -1$. The fibre $f^{-1}(0)$ also distinguishes cases $\epsilon = 1$

and $\epsilon = -1$: $f^{-1}(0) = 0$ or it is a set transverse to the boundary, respectively (see Fig. 3). The singularity of the restriction of f to the boundary $z = 0$ is a fold.

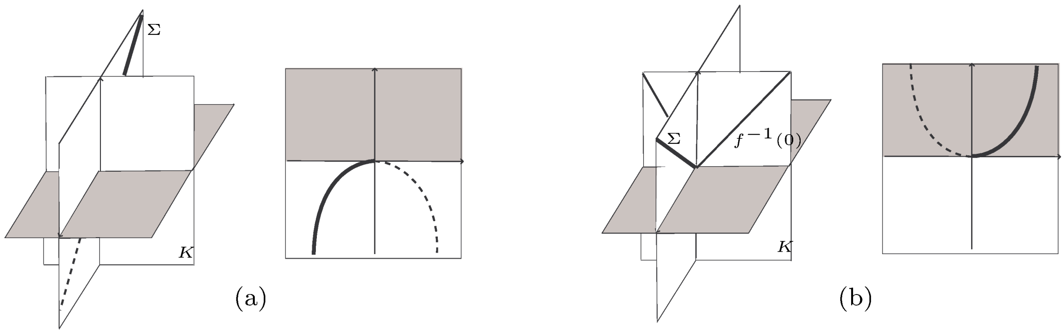

Figure 3. (a) Case VI for $\epsilon = 1$. (b) Case VI for $\epsilon = -1$.

The $\mathcal{B}_e$-versal unfolding given below it that from equation (3.1). Remember that $i(x, y, z) = (X, Y, Z, W)$ needs to be an immersion and that the added term for the second coordinate of the $\mathcal{B}_e$-versal unfolding is λZ. These below $\mathcal{B}_e$-versal unfoldings are $\mathcal{B}_e$-equivalent to the $\mathcal{B}_e$-versal unfolding of the usual meaning (see Introduction for definition).

-Case VII. A $\mathcal{B}_e$-versal unfolding is $(x, y^2 + \epsilon z^2 + \bar{\epsilon} x^2 z + \lambda z)$ and a realization is $i(x, y, z) = (x, y, z, y^2 + \epsilon z^2 + \bar{\epsilon} x^2 z)$. The sets K and Σ are transverse to each other, and Σ is tangent to the boundary (that is, on M, the critical set of the projection is tangent to ∂M) in the region $z \leq 0$ for $\epsilon\bar{\epsilon} = 1$ and in the region $z \geq 0$ for $\epsilon\bar{\epsilon} = -1$. The boundary image is a semi-plane. See Fig. 4 and 5. The fibre $f^{-1}(0)$ is similar to the case VI, acording ϵ, as well as the singularity of the restriction of f to the boundary $z = 0$. Note that the singular set Σ distinguishes this case of all other cases.

-Case VIII. A $\mathcal{B}_e$-versal unfolding is $(x, y^2 + xz + \epsilon z^3 + \lambda(z + z^2))$ and a realization is $i(x, y, z) = (x, y, z + z^2, y^2 + xz + \epsilon z^3)$. The sets K and Σ are tangent to each other, and both of them are transverse to the boundary. The fibre $f^{-1}(0)$ also distinguishes cases $\epsilon = -1$ and $\epsilon = 1$: it is a cuspidal curve or $f^{-1}(0) = 0$, respectively (see Fig. 6). The singularity of the restriction of f to the boundary $z = 0$ is, as the Case VII, the same as Case VI. Note that $f|_\Sigma$, unlike all other cases, has a cusp singularity.

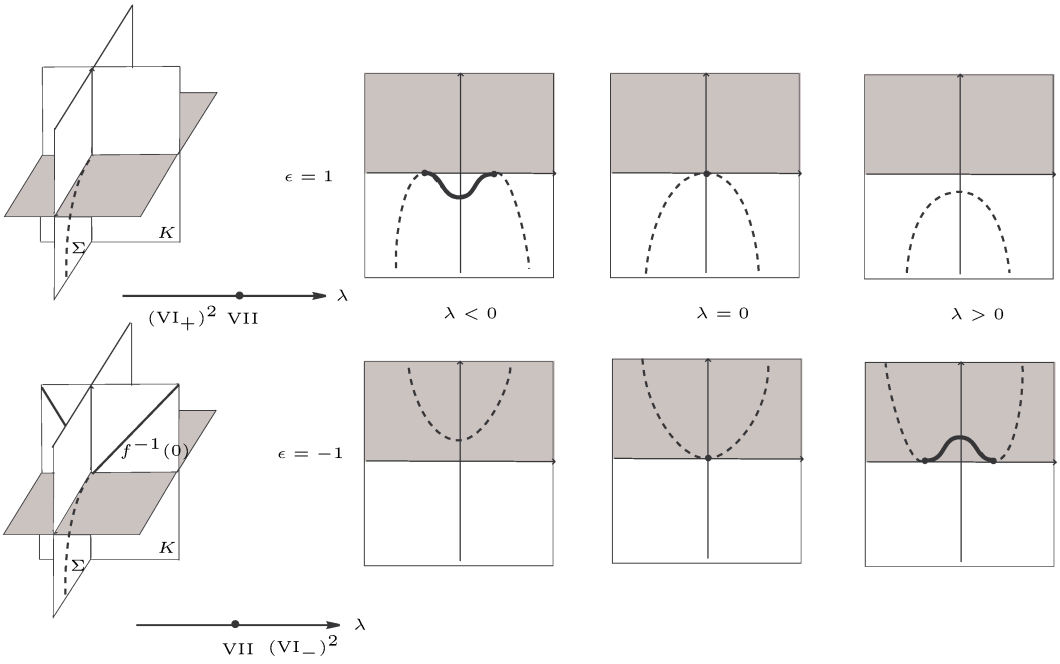

Figure 4. Case VII for $\epsilon\bar{\epsilon} = 1$.

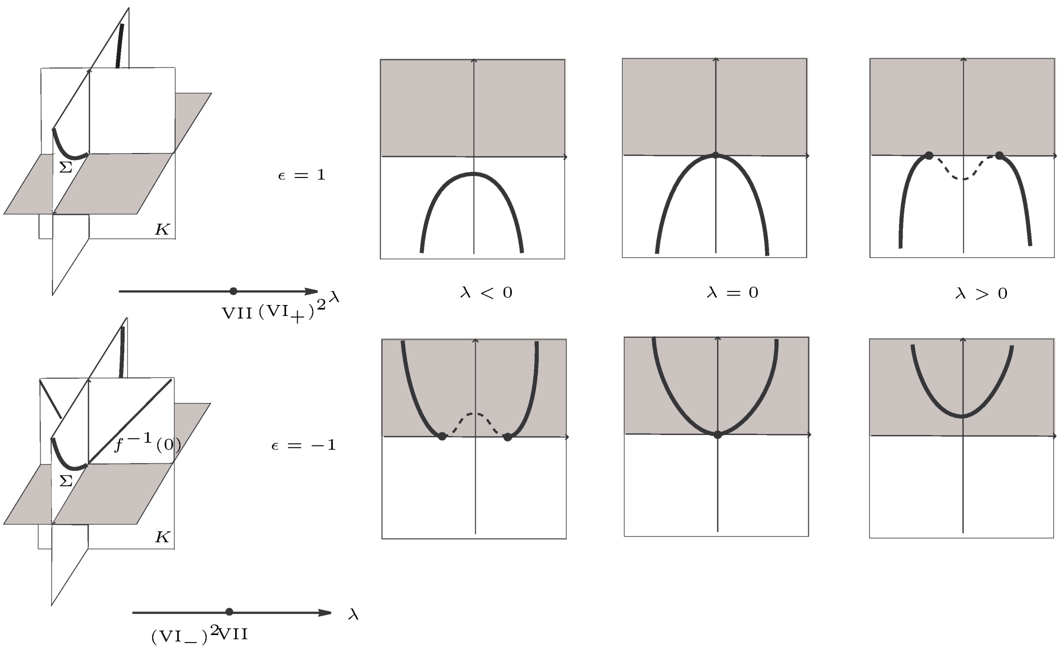

Figure 5. Case VII for $\epsilon\bar{\epsilon} = -1$.

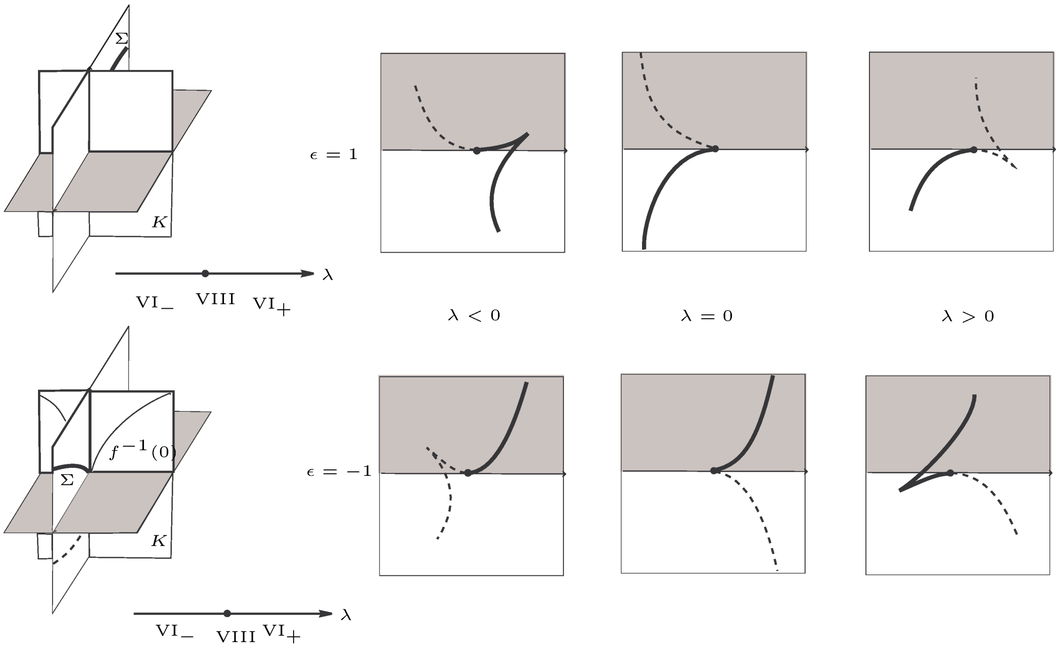

Figure 6. Case VIII ($\epsilon = 1$ and $\epsilon = -1$).

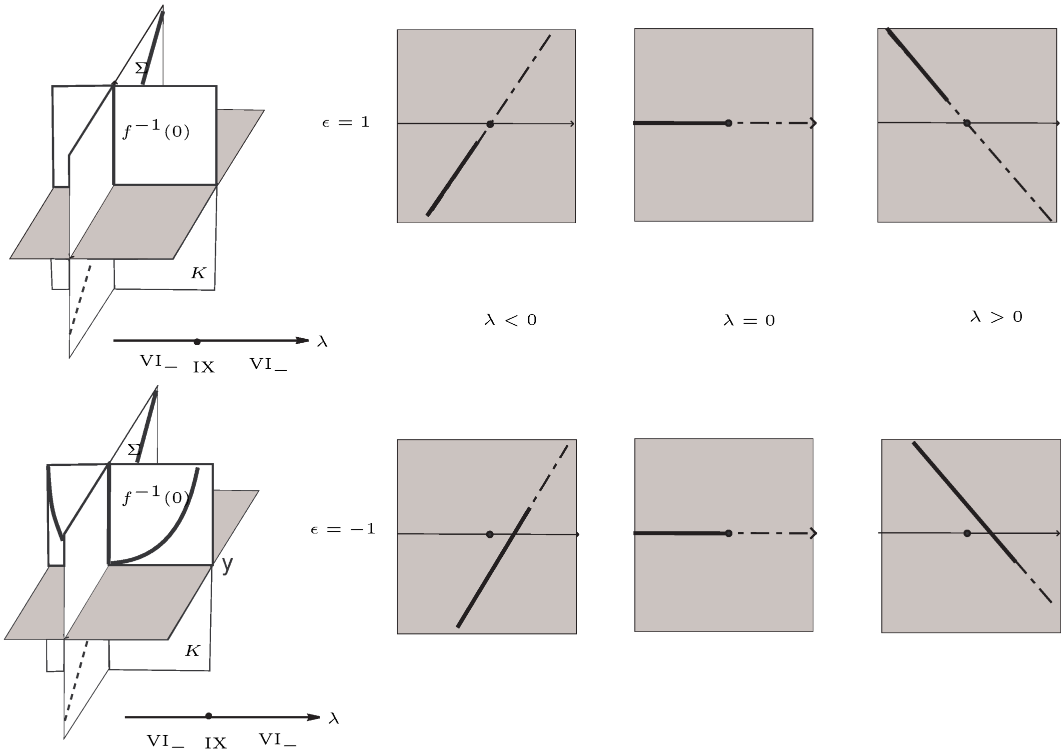

Figure 7. Case IX ($\epsilon = 1$ and $\epsilon = -1$).

-Case IX. A $\mathcal{B}_e$-versal unfolding is $(x, yz + xy + \epsilon y^3 + \lambda(y^2 + z))$ and a realization is $i(x, y, z) = (x, y, y^2 + z, yz + xy + \epsilon y^3)$. The sets K and Σ are transverse to each other, and also to the boundary. Furthermore, the fibre $f^{-1}(0)$ is, unlike all other cases, the semi-line $(0, 0, z)$ for $\epsilon = 1$ and $(0, 0, z) \cup (0, y, y^2)$, for $\epsilon = -1$, with $z \geq 0$ (see Fig. 7). The singularity of the restriction of f to the boundary $z = 0$ is the same as Case III.

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References

[1] Arnol'd,V. I., Guseĭn-Zade S. M. and Varchenko, A. N. *Singularities of differentiable maps*, Vol. I, The classification of critical points, caustics and wave fronts, Monographs in Mathematics, **82**, Birkhäuser Boston, MA, 1985.

[2] Bruce, J. W., Classifications in singularity theory and their applications, New developments in singularity theory (Cambridge, 2000), NATO Sci. Ser. II Math. Phys. Chem., **21**, Kluwer Acad. Publ. (2001), 3–33.

[3] Bruce, J. W. and Giblin, P. J. Projections of surfaces with boundary. *Proc. London Math. Soc.* (3) **60** (1990), 392–416.

[4] Bruce, J. W., Kirk, N. P. and du Plessis, A. A., Complete transversals and the classification of singularities, *Nonlinearity* **10** (1997), no. 1, 253-275.

[5] Bruce, J. W., du Plessis, A. A. and Wall, C. T. C., Determinacy and unipotency, *Invent. Math.* **88** (1987), 521–554.

[6] Damon, J. N., The unfolding and determinacy theorems for subgroups of $\mathcal{A}$ and $\mathcal{K}$, *Mem. Amer. Math. Soc.* **50** (1984), no. 306.

[7] Kirk, N. P., Computational aspects of classifying singularities. LMS J. Comput. Math. 3 (2000), 207-228.

[8] Lu, Y. C., Singularity theory and an introduction to catastrophe theory. Springer-Verlag, 1976.

[9] Mather, J. N., Stability of C^∞ mappings, IV: Classification of stable germs by $\mathcal{R}$-algebras, *Inst. Hautes Etudes Sci. Publ. Math.* **37** (1969), 223–248.

[10] Martins, L. F. and Nabarro, A. C., Projections of hypersurfaces in $\mathbb{R}^4$ with boundary to planes. *Glasg. Math. J.* **56** (2014), no. 1, 149-167

[11] Nabarro, A. C., Projections of hypersurfaces in $\mathbb{R}^4$ to planes, *Lecture Notes in Pure and Applied Maths*, Marcel Dekker - New York, **232** (2003), 283–300.

[12] Nabarro, A. C., Duality and contact of hypersurfaces in $\mathbb{R}^4$ with hyperplanes and lines, *Proc. Edinburgh Math Soc.* **46** (2003), 637–648.

[13] Rieger, J. H. Families of maps from the plane to the plane, *J. London Math. Soc.* (2) **36** (1987), 351–369.

[14] Rieger, J. H. and Ruas, M. A. S., Classification of $\mathcal{A}$-simple germs from k^n to k^2, *Compositio Math.* **79** (1991), 99–108.

[15] Tari, F., Some applications of singularity theory to the geometry of curves and surfaces. Ph.D. Thesis, University of Liverpool, 1990.

[16] Wall, C. T. C., Finite determinacy of smooth map-germs. *Bull. London Math. Soc.* **13** (1981), 481–539.