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Graphs of stable maps between surfaces

We dedicate this paper to the memory of our collaborator and friend Derek Hacon

By

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Abstract

With the aim of studying stable maps from a global viewpoint, we associate weighted graphs to stable maps from closed surfaces to the projective plane and study their properties. This work extends our previous results for the orientable case.

§ 1. Introduction

A well known theorem due to H. Whitney [4] tells us that stable maps between surfaces can only have fold singlarities along curves with isolated cusp singularities on them. A global description of a stable map between two closed surfaces needs both, the determination of the topological type of its regular set in the domain and the description of the behavior of the image of its singular set (i.e. its branch set or apparent contour) in the range surface. There are several interesting works investigating the behaviour of invariants concerning the branch set (see for instance [3], [9], [12], [13]). In order to codify the topology of the regular set, we introduced in [5] a global invariant, known as the *graph of a stable map*. In subsequent papers [6, 7, 8] we studied the properties of these graphs and their behavior through convenient surgeries and isotopies of maps from a closed orientable surface to the plane and to the sphere. In particular, we determined the properties of the graphs that can be associated to stable maps without cusps (i.e.

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fold maps) from closed surfaces to the sphere, for each fixed degree d. Our aim here is to describe how these results can be extended in order to include the non-orientable case. We include a summary of the previously obtained results and introduce a new definition of graph, extending the previous one, that includes the non-orientable case. We see that, under this new definition, any graph with weights in its vertices can be the graph of a stable map with target either the sphere, or the projective plane. We also consider the case of fold maps and provide a sufficient and necessary condition for a graph to be the graph of a fold map with target in the projective plane. The details of the proofs of the new results will be published in a forthcoming paper.

§ 2. Stable maps between closed surfaces and their graphs

Given two closed surfaces M and N with respective genus g_M and g_N let $f : M \to N$ be a stable map between them. The singular set Σf is a finite collection of closed regular simple curves on M made of folds points with possible isolated cusp points. Moreover the image of Σf, known as the *apparent contour* or *branch set* of f, is a collection of closed curves in N with normal crossings and isolated singularities corresponding to the cusp points of f. Topological information of the map f may be conveniently encoded in a weighted graph from which the pair $M, \Sigma f$ may be reconstructed (up to diffeomorphism) ([6], [7]). The edges and vertices of this graph correspond (respectively) to the singular curves and the connected components of the non-singular set. An edge is incident to a vertex if and only if the singular curve corresponding to the edge lies in the frontier of the regular region corresponding to the vertex. In other words, given a stable map $f : M \to N$, its graph $\mathcal{G}(f)$ is the *dual graph* of Σf in M. The weight g_v of a vertex v is defined to be the genus of the corresponding region i.e the genus of the closed surface obtained by adding disk to the region, one for each boundary curve. Clearly, the graph of a stable map is an $\mathcal{A}$-invariant. Figures 7 and 8 show stable maps from different closed surfaces to the projective plane together with their corresponding weighted graphs.

Provided both surfaces M and N are orientable, we can attach a label to each vertex of the graph, $+$ (or $-$) for positive (resp. negative) regions, where we say that a regular region is positive (resp. negative) if the restriction of the map f to it preserves (resp. reverses) orientation. Since each component of Σf is the boundary of a positive and of a negative region, the signs of the vertices are assigned alternatively, that is, the graph $\mathcal{G}(f)$ is bipartite.

Figure 1 displays three examples of stable maps from the torus to the sphere. The singular set in all the cases consists in a unique closed curve. Observe that the maps f and h have degree 1 and g has degree 2. On the other hand, the first map has no cusps, i.e., it is a *fold map.*

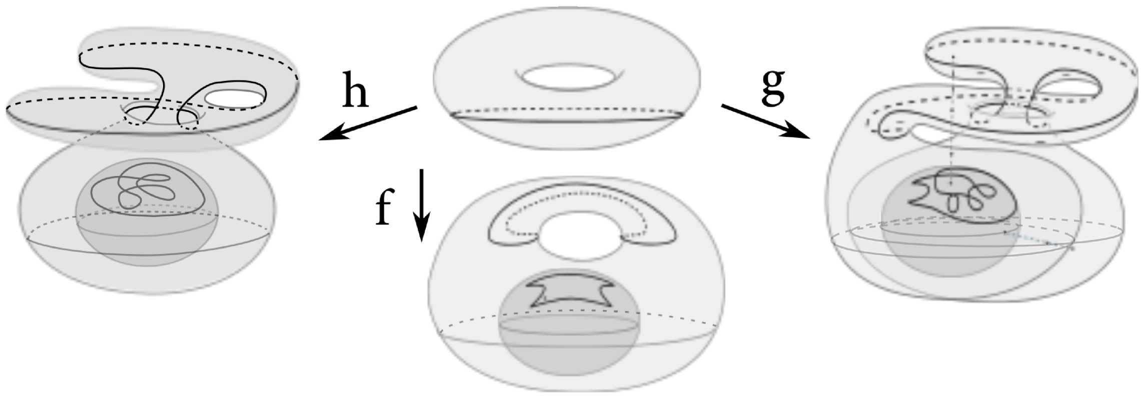

Figure 1. Examples of stable maps from the torus to the sphere.

Some of the basic questions arising in the study of these graphs are the following:

1) Determine necessary and sufficient conditions for a graph to be associated to stable maps between prefixed closed orientable surfaces.

2) Which of these graphs can be associated to fold maps (i.e. stable maps without cusps) to the plane?

3) Given an integer d, which are the graphs that can be associated to fold maps with degree d from a closed oriented surface to the sphere?

In [8] we obtained the following answers to these question for the case of stable maps from orientable surfaces to the sphere:

We first recall that a graph is said to be bipartite if we can attach labels $\pm$ to its vertices in such a way that the vertices at the end of each edge have opposite labels.

1) *Any bipartite graph $\mathcal{G}$ is the graph of a stable map from a convenient orientable surface M to S^2. In fact, if T denotes the sum of all the weights of the vertices of $\mathcal{G}$, we have that the Euler number of M is given by $\chi(M) = 2(\chi(\mathcal{G}) - T)$.*

2) *Given a bipartite graph $\mathcal{G}$, if V^+ and V^- denote the respective numbers of positive and negative vertices of the graph and T^+ and T^- are the respective sums of the weights in the positive and negative vertices, then $\mathcal{G}$ can be associated to some fold map from an orientable closed surface M with $\chi(M) = 2(\chi(\mathcal{G}) - T)$ to the plane if and only if $(V^+ - V^-) - (T^+ - T^-) = 0$.*

3) *Any bipartite graph may be the graph of a fold map from some orientable and closed surface M to S^2. Here we observe that if T denotes the total weight of this graph*

then $\chi(M) = 2(\chi(\mathcal{G}) - T)$. *Moreover, the degree of this map is necessarily given by* $(V^+ - V^-) - (T^+ - T^-)$, *where* V^+ *and* V^- *and* T^+ *and* T^- *are as above.*

Remark. We observe that the results in 1) and 2) can be obtained as a consequence of a general Eliashberg result on fold maps [2]. In [6] we provided an independent constructive proof of these facts.

An interesting particular case is given by the stable Gauss maps on closed orientable surfaces immersed in Euclidean 3-space, for which the following result was obtained in [10]:

Any weighted bipartite graph $\mathcal{G}$ *can be realized as the graph of a stable Gauss map of a closed orientable surface* M, *with*

$$\chi(M) = 2 - 2\big(\beta_1(\mathcal{G}) + \omega(\mathcal{G})\big),$$

where $\beta_1(\mathcal{G})$ *is the number of independent cycles of* $\mathcal{G}$ *and* $\omega(\mathcal{G})$ *is the total weight of* $\mathcal{G}$.

The proofs of the above statements are based in a convenient manipulation of surgeries together with codimension one transitions of types lips and beaks in the space of maps from surfaces to the plane. We give a brief description of these concepts:

a) *Horizontal sum of stable maps:* Given a stable map l between two surfaces M and N, a *bridge* is an embedded arc β in N which meets the set of singular values (or apparent contour) in its two end points (and nowhere else), as in Figure 2 (a). The stable map l_β is constructed as follows. The bridge meets $l(M)$ in its end points, $l(p)$ and $l(q)$, say. Choose small disks in M centered at p and q and replace their interiors by a tube (i.e. an annulus), connected two small disks. As illustrated in Figure 2 (a), the map l may then be extended over the tube to give the required stable map l_β. In particular, if M is the disjoint union of surfaces P and Q and f and g denote the restrictions of l to P and to Q, with $p \in P$ and $q \in Q$ then we call the stable map l_β the *horizontal sum* of f and g, and denote it by $f +_h g$. In other words $l = f \cup g$ and $(f \cup g)_\beta = f +_h g$.

b) *Vertical sum of stable maps:* We take a connected sum by identifying two small non-singular disks in the domain, both positive or both negative (as in the Figure 2 (b)) whose images in N coincide. The disks are replaced by a tube which is mapped into the plane, with a singular curve running around the middle of the tube. Thus the surgery adds a disjoint embedded curve to the branch set. We call the stable map l_β the vertical sum of f and g and it denote it by $f +_v g$.

We observe that when M is connected, the map $f + g$ obtained through one of the above surgeries is defined on a connected surface M' with genus $g(M') = g(M) + 1$.

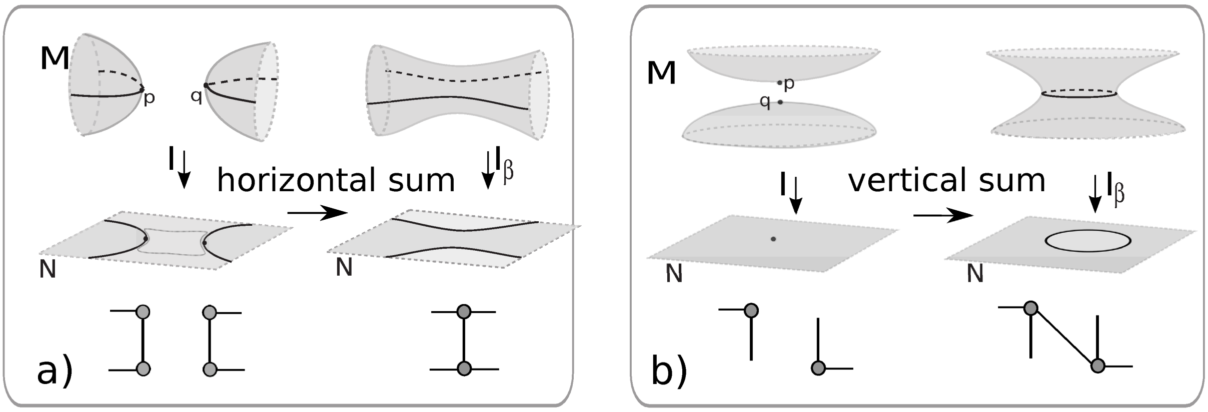

Figure 2. Surgeries of stable maps.

We also notice that a vertical surgery adds a new connected component to the singular set, whereas in the case of a horizontal surgery the number of connected components of the singular set may either increase or decrease by one according to p and q belong to the same connected component or not.

c) *Beaks and lips transitions:* The lips transition creates new singular curves introducing at the same time couples of cusps. On the other hand, the beaks transitions allow us to create and eliminate couples of cusps. Their effects on the graphs are illustrated in Figure 3.

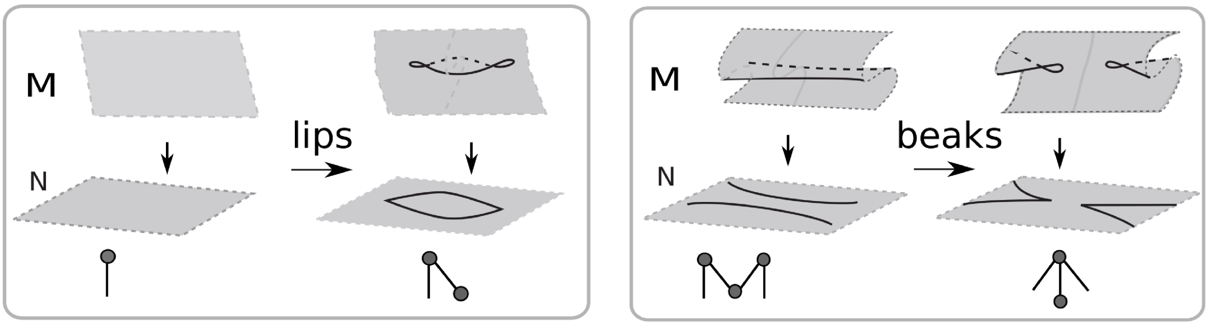

Figure 3. Lips and Beaks transitions

§ 3. Graphs of stable maps from closed surfaces to the projective plane

Our purpose now is to extend the definition of graph to the case stable maps from closed (not necessarily oriented surfaces) to the projective plane $\mathbb{P}$ in order to obtain similar results to the previously stated.

We first observe that when the domain surface M is not orientable, then the graph does no need to be bipartite. Moreover, we have two possibilities for a closed curve γ in M:

a) The curve γ has a small neighbourhood homeomorphic to a cylinder.

b) The curve γ has a small neighbourhood homeomorphic to a Möbius band.

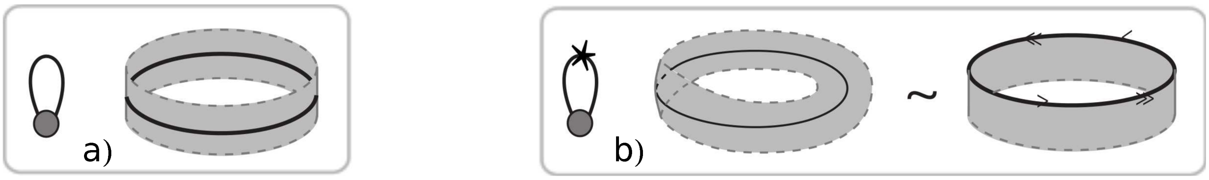

Figure 4. Neighbourhood of a curve.

According to this, for a stable map $f : M \to \mathbb{P}$ of a closed surface, we shall attach a $\star$ to each edge of the graph of f that corresponds to a singular curve of f having a neighbourhood homeomorphic to a Möbius band (Figure 4). Notice that such an edge always determines a loop, i.e. both ends correspond to a unique vertex of the graph.

On the other hand, we observe that in the non-orientable case, the regular set of a stable map may have orientable and non-orientable connected components. We shall reflect this fact in the corresponding graph by attaching a weight $(t, 0)$ to an orientable connected component with genus t and a weight $(0, p)$ to a non-orientable connected component with genus p as illustrated in Figure 6.

The surgeries defined on stable maps defined above lead in a natural way to surgeries between the corresponding graphs: We can perform a *sum of the graphs* $\mathcal{G}_1$ and $\mathcal{G}_2$ either by identifying one edge of $\mathcal{G}_1$ with one edge of $\mathcal{G}_2$ (and their respective vertices) or introducing a new edge that connects a vertex of $\mathcal{G}_1$ with a vertex of $\mathcal{G}_2$. The last surgery can also be performed in a single graph either by introducing a new edge that connects two different vertices, or by adding a loop at some vertex. In connection to this, we observe that

1. The connected sum of two oriented regions with respective genus g_1 and g_2 makes a connected region with genus $g_1 + g_2$.

2. The connected sum of an oriented region of genus t with a non oriented region of genus p is a non oriented region of genus $2t + p$.

With this in mind, we introduce the following concept of surgery between graphs.

Definition 3.1. A *horizontal surgery of two graphs* is carried out by identifying an edge of one of them with an edge of the other one. This gives rise to a new graph and can be done in one of the following ways:

1. The identification of two edges, both of which end at two different vertices (i.e. none of them is a loop) produce an edge that also ends at two different vertices. Then the weights of the involved vertices are added according to the following rules:

   (a) $(s,0)+(t,0)=(s+t,0)$.

   (b) $(0,p)+(0,q)=(0,p+q)$.

   (c) $(t,0)+(0,p)=(0,2t+p)$,

2. The identification of an edge ending at two different vertices with a loop (resp. a $\star$-loop) produce a loop (resp. a $\star$-loop). The weight of the corresponding vertex also follows the above rules.

3. The identification two $\star$-loops produce a loop (without $\star$). The weight of its vertex follows the above rules too.

4. The identification of two loops produce a loop.

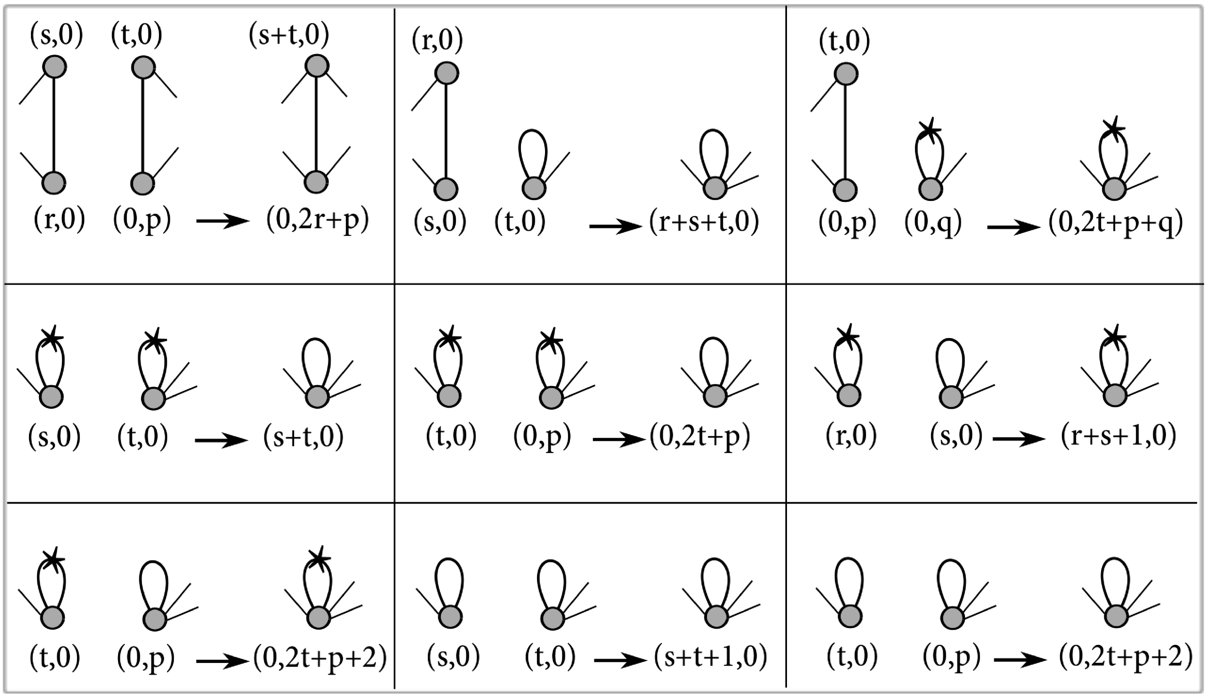

Figure 5. Examples illustrating horizontal surgeries of graphs.

5. The identification of a $\star$-loop and a loop produces a $\star$-loop.

   In cases 4 and 5 the weights of the vertices behave as follows:

   i) $(t,0)+(t',0)=(t+t'+1,0)$,
   ii) $(0,p)+(0,p')=(0,p+p'+1)$
   iii) $(0,p)+(t,0)=(0,2(t+1)+p)$.

Figure 5 illustrates the effects of the horizontal and vertical surgeries on the graphs.

Definition 3.2. A *vertical surgery of graphs* consists in adding an edge in one of the following ways:

i) It joins two different vertices lying in the same graph.

ii) It joins two vertices of different graphs (and thus, connects the two graphs).

iii) It makes a loop in a given vertex of the graph.

In all these cases the weights of the vertices remain unaltered.

Given a graph $\mathcal{G}$ we denote by V its number of vertices, by E its number of edges, by S the number of $\star$ in its loops and by T and P the total sum of the weights of type $(t,0)$ and $(0,p)$ respectively. We have the following result that generalizes the statement 1) given above in order to include the non-orientable case.

Definition 3.3. A graph with a unique edge is said to be *irreducible*. A graph for which $E=1$ and $V=2$ is called an *irreducible tree*.

Figure 6 displays examples of stable maps corresponding to irreducible graphs with total weight ≤ 1.

Theorem 3.4. *Any graph $\mathcal{G}$ with weights type $(t,0)$ and $(0,p)$ in its vertices is the graph of a stable map from a closed surface M to the projective plane. The surface M is orientable if and only if $\mathcal{G}$ is bipartite and $P=S=0$. Moreover, the genus of M is given by $1-V+E+T$ in case that M is orientable and by $2(1-V+E+T)+P-S$ otherwise.*

The proof of this theorem follows an inductive process based in convenient surgeries involving the basic maps shown in Figure 6 ([11]).

§ 4. Fold maps from closed surfaces to the projective plane

We search for a sufficient and necessary condition warranting that some graph may be the graph of some fold map from a closed surface to the projective plane. With this

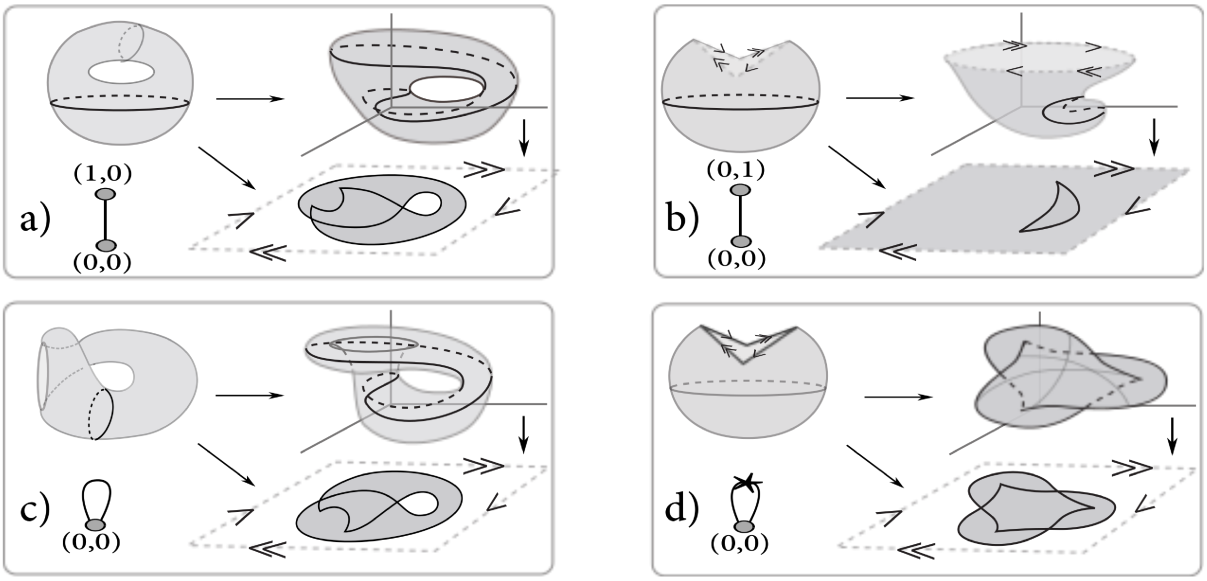

Figure 6. Graphs of stable maps from non-orientable surfaces to the projective plane.

in mind, we consider the possible decompositions of non-orientable regions as connected sums of an orientable region with a non-orientable one. Now, observe that a non-orientable surface with genus p can be decomposed in different ways as the connected sum of an orientable surface of genus q with a non-orientable surface of genus r, for pairs $q, r \in \mathbb{N}$ such that $2q + r = p$. We say that two pairs $(q_1, r_1), (q_2, r_2) \in \mathbb{N} \times \mathbb{N}$ are *equivalent* if it satisfies one of the following conditions

1) $r_1 = r_2 = 0$ and $q_1 = q_2$, or

2) $r_1, r_2 \neq 0$ and $2q_1 + r_1 = 2q_2 + r_2$.

We can consider now the more general case of graphs with weights (q_i, r_i) in its vertices.

Definition 4.1. We say that two graphs are *equivalent* if they are isomorphic as graphs and the corresponding vertices have equivalent weights in the above sense.

Given a bipartite graph $\mathcal{G}$ with weights (q_i, r_i) in its vertices, asign labels $\pm$ to its vertices and denote by V^+ (resp V^-) the total number of vertices labelled with $+$ (resp. with $-$), by Q^+ (resp Q^-) the sum of all the q_i weights corresponding to vertices with positive label (resp. with negative label) and by R^+ (resp. R^-) the sum of all the r_i weights corresponding to vertices with positive label (resp. with negative label).

Definition 4.2. A *maximal bipartite subgraph* of $\mathcal{G}$ is the graph obtained by removing an edge of each non bipartite cycle of $\mathcal{G}$.

Definition 4.3. A bipartite graph $\mathcal{G}$ with weights $\{(q_j, r_j)\}_{j=1}^m$ in its vertices, is a *balanced graph* if it satisfies $(V^+ - V^-) - (Q^+ - Q^-) - (R^+ - R^-) = 0$.

For a given non bipartite graph $\mathcal{G}$ with weights $\{(q_j, r_j)\}_{j=1}^m$ in its vertices, we say that $\mathcal{G}$ is a *balanced graph* if it has some balanced bipartite maximal subgraphs.

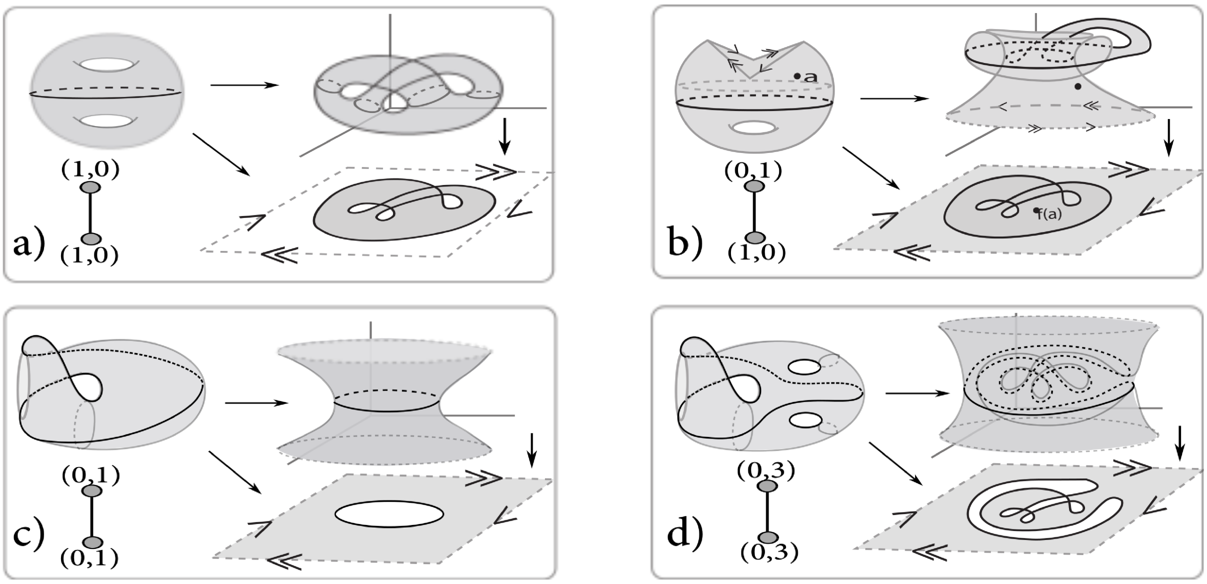

Figure 7. Examples of stable maps with irreducible graph.

Figure 7 shows some examples of fold maps whose balanced graphs are irredutible. We observe that the example d) can be obtained as a horizontal sum of examples a) and c).

Definition 4.4. A graph $\mathcal{G}$ with weights of types $\{(t_i, 0)\}_{i=1}^n$ and $\{(0, p_j)\}_{j=1}^m$ in its vertices is said to be a *quasibalanced graph* if it is equivalent to some balanced graph.

Definition 4.5. A *pre-balanced graph* is a graph that can be obtained as a sum of some quasibalanced graph with suitable balanced irreducible trees.

We show in Figure 8 examples of two pre-balanced graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ with different weights together with their corresponding auxiliary graphs $\mathcal{G}_1^a$ and $\mathcal{G}_2^a$. Their pre-balanced maximal subgraphs are encircled with dashed lines. The numbers below the graph correspond to the values of $(V^+ - V^-, (P^+ + T^+) - (P^- - T^-))$.

Remark. The difficulty in analyzing whether an graph is pre-balanced or not increases exponentially as a function of the number of edges of the graph (see Figure 8).

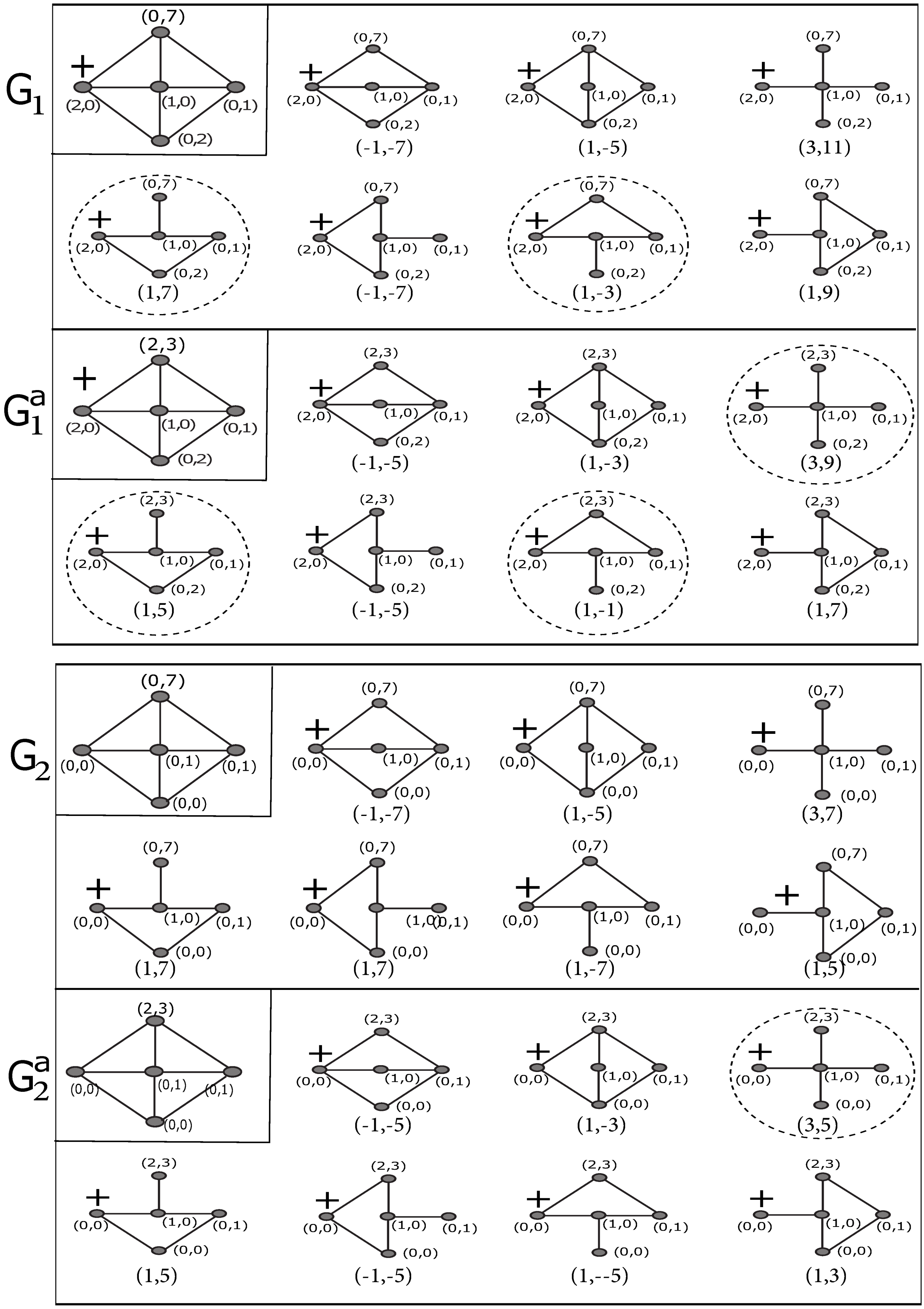

Figure 8. Examples of pre-balanced graphs.

This makes impossible to do it by hand even for graphs with not a very high number of edges. The computing program DFMG [1] verifies if a given graph is pre-balanced or not, providing the possible auxiliary quasibalanced subgraphs. In case that it is pre-balanced it furnishes all the possible decompositions as a sum of a bipartite balanced subgraph together and irreducible trees.

The following result, whose proof will appear in a forthcoming paper [11], gives us a sufficient and necessary condition for a graph which associates to a fold map with target in the projective plane.

Theorem 4.6. *A graph $\mathcal{G}$ with weights of type $(t,0)$ and $(0,p)$ in its vertices is the graph of a fold map from a closed surface to the projective plane if and only if $\mathcal{G}$ is pre-balanced.*

Some particular cases are the following:

Corollary 4.7. *A graph $\mathcal{G}$ can be realized by some fold map $f : S^2 \to \mathbb{R}^2$ if and only if $\mathcal{G}$ is a balanced tree with all weights zero (i. e. $V^+ = V^-$).*

Corollary 4.8. *The graph $\mathcal{G}$ is the graph of a fold map from a closed (non necessarily orientable) surface to the plane if and only if $\mathcal{G}$ is pre-balanced with weights of type $(t,0)$ in its vertices.*

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