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Topology of manifolds and global theory of singularities

By

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Abstract

This is a survey article on topological aspects of the global theory of singularities of differentiable maps between manifolds and its applications. The central focus will be on the notion of the Stein factorization, which is the space of connected components of fibers of a given map. We will give some examples where the Stein factorization plays essential roles in proving important results. Some related open problems will also be presented.

§ 1. Introduction

This is a survey article on the global theory of singularities of differentiable maps between manifolds and its applications, which is based on the author's talk in the RIMS Workshop "Theory of singularities of smooth mappings and around it", held in November 2013. Special emphasis is put on the *Stein factorization*, which is the space of connected components of fibers of a given map between manifolds. We will see that it often gives rise to a good manifold which bounds the original source manifold, and that it is a source of various interesting topological invariants of manifolds.

We start with the study of a class of smooth maps that have the mildest singularities, i.e. the class of *special generic maps* (Section 2). A typical example is a Morse function with only critical points of extremal indices. According to Reeb [33], if a closed connected manifold admits such a Morse function, then it is necessarily homeomorphic to the sphere. However, in higher dimensions, *exotic spheres*, which are smooth manifolds homeomorphic to the sphere but not diffeomorphic to the standard one, are known

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to admit such Morse functions. This means that the existence of such a Morse function contains the information on the topology of the source manifold, but not on the differentiable structure. However, we will see that if we consider special generic maps into higher dimensional Euclidean spaces, then they can give us information on the differentiable structure as well. This will be proved by using the notion of a Stein factorization of a map. A Stein factorization, or more precisely, its quotient space, encodes the connected components of the fibers of a given map. It will be seen that if the difference between the dimensions of the source and the target manifolds is small, then the source manifold of a special generic map is diffeomorphic to the boundary of a disk bundle over the quotient space, and this is the key to the proof of the above-mentioned result on the differentiable structure of the sphere. Basically, the content of Section 2 is a survey of [38, 39, 46].

In Section 3, as another application of the Stein factorization, we consider *stable maps* of codimension -1; i.e. C^∞ stable maps $f : M^n \to \mathbf{R}^{n-1}$. In this case, a regular fiber is a disjoint union of circles and thus bounds a disjoint union of 2-disks. Therefore, there is a good chance to get a 2-disk bundle over the quotient space whose boundary is the source manifold M^n. We will see that this is true for $n = 2$ and $n = 3$ as long as the manifolds are orientable. However, for the case $n = 4$, this is not true: a certain class of *singular fibers* give rise to obstructions to constructing a desired 2-disk bundle. By analyzing the singular fiber that gives the obstruction in detail, we will see that every closed oriented 4-dimensional manifold is oriented cobordant to a disjoint union of some copies of $\mathbf{C}P^2$ or its orientation reversal. As a consequence, the 4-dimensional oriented cobordism group is infinite cyclic generated by the cobordism class of $\mathbf{C}P^2$. This is of course a classical result due to Rohlin [36]; here, we prove this classical result by using the global theory of singularities of differentiable maps. One of the advantages of our proof is that the appearance of $\mathbf{C}P^2$ is quite natural. As a byproduct of this argument, we get the signature formula for 4-dimensional manifolds in terms of the number of certain singular fibers of stable maps on the manifold. This formula implies that the stable maps on a topologically complicated manifold are necessarily complicated. Some related results are also presented. The content of Section 3 is a survey of [45].

In Section 4, we give some ideas for constructing *invariants of manifolds* using stable maps on manifolds or their Stein factorizations. In many situations, we can get plenty of topological invariants using Morse functions on manifolds. For example, in dimension 3, the celebrated Kirby Calculus is now a source of a lot of 3-manifold invariants, and it is based on Morse functions. Unfortunately, not so many invariants are known that are derived from stable maps into Euclidean spaces of dimension ≥ 2. Here, we give several of its examples and their properties, including the quantum invariants of 3-manifolds. The content of Section 4 is a survey of [4, 30].

Finally in Section 5, we present several open problems related to the contents of Sections 2–4. These problems had been posed in the workshop "Singularity Theory and its Applications" held in Oita National College of Technology in 2011. Since then, some of them have been solved. Here we present those which have not been solved until now as far as the author knows.

Throughout the paper, manifolds and maps are smooth of class C^∞ unless otherwise indicated. For a space X, id_X denotes the identity map of X.

§ 2. Special generic maps

In the following, M^n and N^p will denote smooth manifolds of dimensions n and p, respectively.

Definition 2.1. Let $f : M^n \to N^p$, $n \geq p$, be a smooth map. A singular point $q \in M^n$ of f is called a *fold singularity* if f has the normal form

$$(x_1, x_2, \dots, x_n) \mapsto (x_1, x_2, \dots, x_{p-1}, \pm x_p^2 \pm x_{p+1}^2 \pm \cdots \pm x_n^2)$$

with respect to appropriate local coordinates around q and $f(q)$. It is a *definite fold singularity* if all the signs appearing in the above normal form are the same.

Definition 2.2 (Burlet–de Rham [4], Calabi [5]). A smooth map $f : M^n \to N^p$, $n \geq p$, is called a *special generic map* if all of its singular points are definite fold singularities.

The class of special generic maps can be considered to be a class of maps with mildest singularities. Note that every generic map $f : M^n \to N^p$, $n \geq p$, of a closed manifold into an open manifold necessarily has definite fold singularities.

Remark 2.3. Historically, Burlet and de Rham [4] were the first who formulated the notion of a special generic map of a 3-manifold into $\mathbf{R}^2$. Then, Porto and Furuya [31] extended the notion to higher dimensions. However, before all these developments, Calabi [5] had already studied special generic maps in his study of the topology of manifolds admitting certain Riemannian metrics with pinched curvatures.

Example 2.4. A smooth function $f : M^n \to \mathbf{R}$ on a manifold of dimension n is a special generic map if and only if it is a Morse function with only critical points of index 0 or n.

Example 2.5. For the unit sphere $S^n \subset \mathbf{R}^{n+1}$, let $f : S^n \to \mathbf{R}^p$, $n \geq p \geq 1$, be the standard projection $\mathbf{R}^{n+1} \to \mathbf{R}^p$ restricted to S^n. Then, it is an easy exercise to show that it is a special generic map.

Example 2.6. For $b \geq b' \geq 1$, let $f_1 : S^b \to \mathbf{R}^{b'}$ be a special generic map as constructed in Example 2.5. Then, the composition

$$S^a \times S^b \xrightarrow{\mathrm{id}_{S^a} \times f_1} S^a \times \mathbf{R}^{b'} \looparrowright \mathbf{R}^{a+b'}$$

is a special generic map, where the last map is an arbitrary immersion.

We can use special generic maps in order to define an invariant of smooth manifolds as follows.

Definition 2.7. Let M^n be a smooth closed n-dimensional manifold. We define $\mathcal{S}(M^n)$ to be the set of integers p with $1 \leq p \leq n$ such that there exists a special generic map $f : M^n \to \mathbf{R}^p$ of M^n into the p-dimensional Euclidean space.

Note that this is a diffeomorphism invariant of M^n: i.e., if M_0 and M_1 are smooth closed manifolds that are diffeomorphic, then we have $\mathcal{S}(M_0) = \mathcal{S}(M_1)$.

Example 2.8. Examples 2.5 and 2.6 show the following:

(1) $\mathcal{S}(S^n) = \{1, 2, \ldots, n\}$ for $n \geq 1$,

(2) $\mathcal{S}(S^a \times S^b) = \{a+1, a+2, \ldots, a+b\}$ for $b \geq a \geq 1$.

For details, see [38, 39].

Then, using our invariant, we have the following characterization of the standard sphere.

Theorem 2.9 (Saeki [38]). *Let M^n be a smooth closed manifold of dimension n. Then, we have*

$$\mathcal{S}(M^n) = \{1, 2, \ldots, n\}$$

if and only if M^n is diffeomorphic to the standard n-dimensional sphere S^n.

Remark 2.10. Calabi [5] announces a result close to the above theorem: if $f : M^n \to \mathbf{R}^p$ is a special generic map such that the singular point set $S(f)$ is diffeomorphic to the $(p-1)$-dimensional standard sphere and if $f|_{S(f)}$ is an embedding, then M^n is homeomorphic to S^n, and that if $p \geq n-1$ in addition, then M^n is diffeomorphic to the standard sphere S^n, although he did not give a proof.

Note that there exist lots of exotic spheres. More precisely, depending on the dimension n, the topological sphere has differentiable structures other than the standard one [22]. This means that special generic maps can detect the standard differentiable structure on a sphere among those various differentiable structures.

Example 2.11. Let Σ^7 be one of Milnor's exotic 7-spheres [29]. Then, we have

$$\{1, 2, 7\} \subset \mathcal{S}(\Sigma^7) \subset \{1, 2, 3, 7\},$$

which can be proved by using results obtained in [38].

For the proof of Theorem 2.9, we need the following notion.

Definition 2.12. Let $f : M \to N$ be a smooth map. For $x, x' \in M$, define $x \sim_f x'$ if $f(x) = f(x')(= y)$, and x and x' belong to the same connected component of $f^{-1}(y)$. This defines an equivalence relation. We denote by W_f the quotient space $M/\sim_f$, and by $q_f : M \to W_f$ the quotient map. Then, it is easy to see that there exists a unique continuous map $\bar{f} : W_f \to N$ that makes the following diagram commutative:

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ & {}_{q_f}\searrow \quad\quad \nearrow_{\bar{f}} & \\ & W_f. & \end{array}$$

The above diagram or the space W_f is called the *Stein factorization* of f.

It is known that if f is generic enough, then the Stein factorization is triangulable [19]. For special generic maps, we have more structures as follows.

Proposition 2.13. *Let $f : M^n \to N^p$ be a proper special generic map with $n > p$. Then, we have the following.*

(1) *The singular point set $S(f)$ of f is a regular submanifold of M^n of dimension $p-1$.*

(2) *The quotient space W_f has the structure of a smooth p-dimensional manifold possibly with boundary such that $\bar{f} : W_f \to N^p$ is an immersion.*

(3) *The restriction $q_f|_{S(f)} : S(f) \to \partial W_f$ is a diffeomorphism.*

(4) *If M^n is connected and $S(f) \neq \emptyset$, then the restriction $q_f|_{M\setminus S(f)} : M^n \setminus S(f) \to \operatorname{Int} W_f$ is a smooth fiber bundle with fiber the standard $(n-p)$-dimensional sphere S^{n-p}.*

An illustrative example is depicted in Figure 1.

Note that the structure group of the smooth S^{n-p}-bundle in Proposition 2.13 (4) may not necessarily be reduced to an orthogonal group. However, when $n - p \leq 3$, the reduction is possible due to Smale [50] and Hatcher [18], and consequently we have the following disk bundle theorem. Note that a fiber bundle whose structure group can be reduced to an orthogonal group is said to be *linear*.

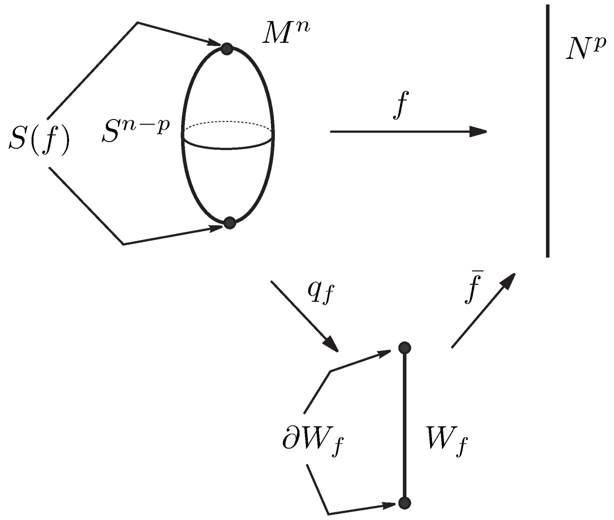

Figure 1. Stein factorization of a special generic map

Theorem 2.14 (Saeki [38]). *Let $f : M^n \to N^p$ be a proper special generic map with $n - p = 1, 2, 3$, where M^n is connected and $S(f) \neq \emptyset$. Then, M^n is diffeomorphic to the boundary of a linear D^{n-p+1}-bundle over W_f.*

Let us now review the outline of the proof of Theorem 2.9.

By Example 2.5, the necessity is clear. Conversely, suppose that we have $\mathcal{S}(M^n) = \{1, 2, \ldots, n\}$. As 1 is contained, there exists a Morse function $f_1 : M^n \to \mathbf{R}$ with only minima and maxima as its critical points. Then, by a theorem of Reeb [33], M^n is homeomorphic to S^n. More precisely, M^n is diffeomorphic to the closed manifold obtained by attaching two copies of the n-dimensional disk D^n along their boundary spheres. In particular, if $n \leq 4$, then M^n is diffeomorphic to the standard S^n (for $n = 4$, we use a result of Cerf [6]). Therefore, we assume $n \geq 5$. As $n - 1 \in \mathcal{S}(M^n)$, there exists a special generic map $f : M^n \to \mathbf{R}^{n-1}$. Then, a standard argument in algebraic topology implies that W_f is contractible. Now by Theorem 2.14, M^n is diffeomorphic to the boundary of a D^2-bundle over W_f, which is contractible. Then, the solution to the generalized Poincaré conjecture due to Smale [51] implies that M^n is diffeomorphic to the standard n-sphere. This completes the proof.

Observe that in the above proof, W_f is the core (or spine) of a "good manifold" whose boundary is diffeomorphic to the given manifold M^n. This was essential in the proof.

In dimension four, we also have the following characterization of the standard $\mathbf{R}^4$. Recall that the differentiable structure on $\mathbf{R}^n$, $n \neq 4$, is unique, while for $n = 4$, $\mathbf{R}^4$ admits more than one differentiable structures. The result in dimension four was a consequence of two big theorems due to Freedman [12] and Donaldson [10] (for details, see [15] or [16]). In fact, there exist uncountably many exotic $\mathbf{R}^4$'s (Taubes [52]).

Theorem 2.15 (Saeki [46]). *Let M^4 be a smooth 4-dimensional manifold homeomorphic to $\mathbf{R}^4$. If there exists a proper special generic map $f : M^4 \to \mathbf{R}^p$ for some p with $1 \leq p \leq 3$, then M^4 is diffeomorphic to the standard $\mathbf{R}^4$.*

Note that the standard $\mathbf{R}^4$ does admit proper special generic maps into $\mathbf{R}$, $\mathbf{R}^2$ and $\mathbf{R}^3$. For example, consider the map defined by $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, x_3^2 + x_4^2)$, etc.

§ 3. Stable maps

Let $f : M^n \to N^{n-1}$ be a smooth map of codimension -1 of a closed n-dimensional manifold M^n into a manifold of dimension $n - 1$, where the *codimension* of a smooth map is defined to be the dimension of the target manifold minus that of the source. In the codimension -1 case, regular fibers are disjoint unions of S^1 and each of their components bounds a 2-dimensional disk. It would be nice to have a "disk bundle" over the quotient space W_f such that the boundary of its total space is diffeomorphic to M^n. This is an optimistic observation derived from the study of special generic maps as surveyed in the previous section. Note that, in this context, obstructions to constructing such a "D^2-bundle" over W_f are concentrated around the singular fibers.

In order to realize such an idea, we need to work with a nice class of smooth maps, i.e. stable maps, which are defined as follows.

For manifolds M and N, we denote by $C^\infty(M, N)$ the space of smooth maps $M \to N$ endowed with the Whitney C^∞ topology. There is a natural action of the groups $\mathrm{Diff}(M)$ and $\mathrm{Diff}(N)$ of self-diffeomorphisms of M and N, respectively, on $C^\infty(M, N)$: for $(\Phi, \varphi) \in \mathrm{Diff}(M) \times \mathrm{Diff}(N)$ and $f \in C^\infty(M, N)$, we define $(\Phi, \varphi) \cdot f = \varphi \circ f \circ \Phi^{-1}$. We say that a smooth map $f \in C^\infty(M, N)$ is C^∞ *stable* (or *stable*, for short) if its orbit is open (see [14] for details). In other words, stable maps are those which do not change their differential topological properties after small perturbations. There are some characterizations of stable maps depending on the dimensions of M and N. For example, a smooth function $f : M^n \to \mathbf{R}$ on a closed manifold M^n of dimension $n \geq 1$ is stable if and only if it is a Morse function (with all the critical values pairwise distinct).

In this section, the notion of a fiber will play an important role. It is formulated as follows.

Definition 3.1. For a smooth map $f : M \to N$ and a point $q \in N$, the map germ along the inverse image

$$f : (M, f^{-1}(q)) \to (N, q)$$

is called the *fiber* over q (see [42]). In particular, if a point $q \in N$ is a regular value of f, then we call the fiber over q a *regular fiber*; otherwise, a *singular fiber*.

An equivalence relation among the fibers is defined as follows. Let $f_i : M_i \to N_i$, $i = 0, 1$, be smooth maps. For $q_i \in N_i$, $i = 0, 1$, we say that the fibers over q_0 and q_1 are *equivalent* if for some open neighborhoods U_i of q_i in N_i, there exist diffeomorphisms $\Psi : f^{-1}(U_0) \to f^{-1}(U_1)$ and $\psi : U_0 \to U_1$ with $\psi(q_0) = q_1$ which make the following diagram commutative:

$$\begin{array}{ccc} (f_0^{-1}(U_0), f_0^{-1}(q_0)) & \xrightarrow{\Psi} & (f_1^{-1}(U_1), f_1^{-1}(q_1)) \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ (U_0, q_0) & \xrightarrow{\psi} & (U_1, q_1). \end{array}$$

Now let us restrict our attention to codimension -1 stable maps. We start with the simplest case, i.e. the case of smooth maps of surfaces into the real line. Let M^2 be a closed oriented surface, and $f : M^2 \to \mathbf{R}$ a Morse function.

In this case, singular fibers are classified as in Figure 2 (for details, see [42]).

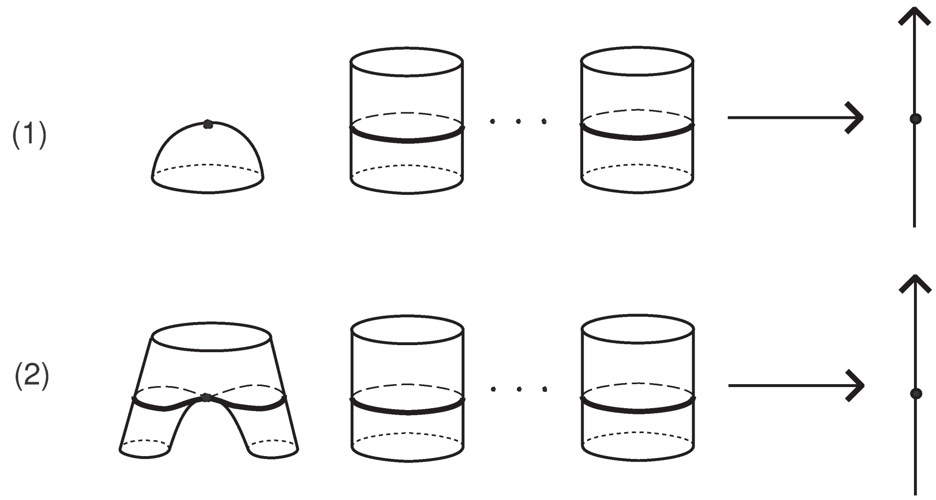

Figure 2. Singular fibers of Morse functions on orientable surfaces

There are no obstructions to filling in the singular fiber neighborhoods. In fact, the leftmost surfaces (with boundary) in (1) and (2) together with the 2-dimensional disks bounded by the boundary circles bound D^3. As a corollary, we get the following well-known result without using the classification theorem of closed surfaces.

Corollary 3.2. *Every closed oriented surface is oriented null-cobordant, i.e., it bounds a compact oriented 3-dimensional manifold.*

Let us now consider the 3-dimensional case. It is known that stable maps of 3-dimensional manifolds into surfaces are characterized as follows. (This can be proved by using the Mather theory of stable maps [27, 28].)

Proposition 3.3. *Let M be a 3-dimensional manifold and N a surface. A proper smooth map $f : M \to N$ is C^∞ stable if and only if the following conditions are satisfied.*

(i) *For every $q \in M$, there exist local coordinates (x, y, z) and (X, Y) around $q \in M$ and $f(q) \in N$, respectively, such that one of the following holds:*

$$(X \circ f, Y \circ f) = \begin{cases} (x, y), & q\text{: regular point,} \\ (x, y^2 + z^2), & q\text{: definite fold point,} \\ (x, y^2 - z^2), & q\text{: indefinite fold point,} \\ (x, y^3 + xy), & q\text{: cusp point.} \end{cases}$$

(ii) *Set $S(f) = \{q \in M : \operatorname{rank} df_q < 2\}$, which is a regular closed 1-dimensional submanifold of M under the above condition* (i). *Then, for each cusp point q, $f^{-1}(f(q)) \cap S(f) = \{q\}$ holds, and $f|_{S(f)\setminus\{\text{cusps}\}}$ is an immersion with normal crossings.*

There is a classification result of singular fibers of stable maps of closed orientable 3-manifolds into surfaces. For details, see [24, 25, 42][1].

Let M^3 be a closed oriented 3-manifold, and $f : M^3 \to \mathbf{R}^2$ a stable map. Then, we can show that the possible obstructions to filling in the singular fiber neighborhoods lie near the singular fibers as in Figure 4 (for details, see [45]). Note that Figures 3 and 4 depict the components of singular fibers containing singular points as inverse images: however, they actually represent map germs along the inverse images up to regular fiber components (for details, see [42]).

Figure 3. Singular fibers for stable maps of 3-manifolds into $\mathbf{R}^2$ which may give obstructions

Costantino–Thurston [9] showed that in fact, there are no obstructions to filling in the neighborhoods of the above singular fibers[2]. Thus, we get the following classical result of Rohlin.

Corollary 3.4 (Rohlin [35]). *Every closed oriented 3-manifold is oriented null-cobordant, i.e., it bounds a compact oriented 4-dimensional manifold.*

[1] In fact, the list consists of those fibers of $\kappa = 1$ and 2 which appear in Figure 4.

[2] Using this idea, Costantino–Thurston [9] showed that every closed oriented 3-manifold efficiently bounds a 4-manifold.

How about the 4-dimensional case? Singular fibers of stable maps of closed orientable 4-manifolds into $\mathbf{R}^3$ are classified as in Figure 4, based on a characterization of stable maps similar to Proposition 3.3 (for details, see [42]). Note that in the list, κ denotes the codimension of the relevant fibers. The *codimension* of a singular fiber is the codimension of the set of points in the target over which lies a singular fiber of the given type. For stable maps, such a set is always a smooth submanifold. In particular, for stable maps of closed 4-manifolds into 3-manifolds, singular fibers of codimension 3 appear discretely and their numbers are finite.

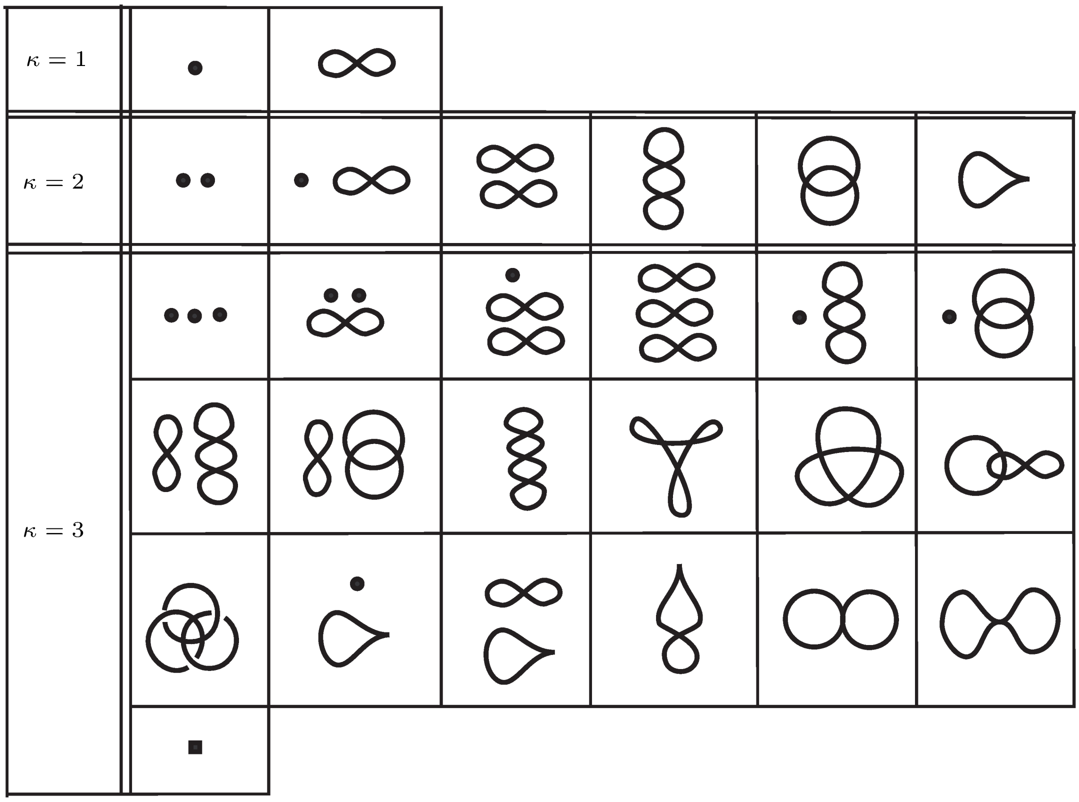

Figure 4. List of singular fibers of proper stable maps of orientable 4-manifolds into 3-manifolds

Then, it has been proved the following.

Theorem 3.5 (Saeki [45]). *There are no obstructions to filling in the singular fiber neighborhoods, except for the* III^8*-type singular fiber as in Figure* 5*. Furthermore, around each singular fiber of type* III^8*, a copy of* $\mathbf{C}P^2$ *or its orientation reversal* $\overline{\mathbf{C}P^2}$ *appears.*

As a corollary, we get a new proof of the following classical result.

Corollary 3.6 (Rohlin [36]). *The oriented* 4*-dimensional cobordism group is in-*

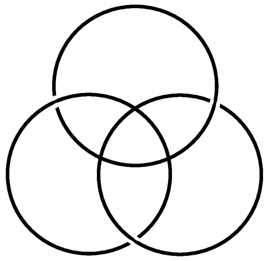

Figure 5. The singular fiber that determines the cobordism class

finite cyclic generated by the class of $\mathbf{C}P^2$.

Proof. By classical algebraic topology, we can show that the signature function $\sigma : \Omega_4 \to \mathbf{Z}$ is a well-defined homomorphism, where Ω_4 denotes the 4-dimensional oriented cobordism group. On the other hand, let $\varphi : \mathbf{Z} \to \Omega_4$ be the homomorphism defined by sending the generator $1 \in \mathbf{Z}$ to the class of $\mathbf{C}P^2$. By Theorem 3.5, this is surjective. Since the signature of $\mathbf{C}P^2$ is equal to 1, the composition $\sigma \circ \varphi$ is the identity, and hence φ is also injective. This completes the proof. □

The above proof shows that the appearance of $\mathbf{C}P^2$ is something natural, and not artificial. There are many representative manifolds for the generator of Ω_4: however, among these, $\mathbf{C}P^2$ is the most natural one.

As a byproduct of the above proof, we get the following signature formula.

Corollary 3.7 (Saeki–T. Yamamoto [49]). *For a* C^∞ *stable map* $f : M^4 \to \mathbf{R}^3$ *of a smooth closed oriented* 4*-manifold* M^4*, the signature of* M^4 *is equal to the number of* III^8*-type singular fibers counted with signs.*

Recall that the original proof of the above signature formula was based on the cohomology calculus of a Vassiliev type complex for singular fibers. The above proof using the Stein factorization for trying to find a 5-dimensional manifold bounding a given 4-manifold is more geometric and intuitive.

The above result has a further consequence about the complexity of a stable map. Recall that for a stable map $f : M^4 \to \mathbf{R}^3$, its set of singular points $S(f)$ is a smooth closed surface embedded in M^4. Furthermore, $f|_{S(f)} : S(f) \to \mathbf{R}^3$ is an immersion with cuspidal edges and swallowtails.

Corollary 3.8. *Let* $f : M^4 \to \mathbf{R}^3$ *be a* C^∞ *stable map as above. Then,* $f|_{S(f)}$ *has at least* $|\sigma(M^4)|$ *triple points, where* $\sigma(M^4)$ *stands for the signature of* M^4.

We can observe that the complexity of a stable map reflects the topology of M^4.

We also have the following related results.

Theorem 3.9 (Gromov [17]). *Let $f : M^n \to \mathbf{R}^2$ be a C^∞ stable map of a closed n-dimensional manifold and $S(f)$ the set of its singular points. Then, we have*

$$\operatorname{rank} H_*(M^n; \mathbf{Z}) \leq 2N_2 + N_{\mathrm{cusp}} + 2N_{\mathrm{comp}},$$

where N_2 is the number of double points of the plane curve $f|_{S(f)} : S(f) \to \mathbf{R}^2$, N_{cusp} is the number of cusps of f, and N_{comp} is the number of components of $S(f)$.

Theorem 3.10 (Costantino–Thurston [9], Gromov [17]). *Let $f : M^3 \to \mathbf{R}^2$ be a C^∞ stable map of a closed orientable 3-manifold M^3. Then, we have*

$$||M^3||_\Delta \leq 10N_{\mathrm{sf}} \leq 10N_2,$$

where $||M^3||_\Delta$ is the simplicial volume, N_{sf} is the number of singular fibers as in Figure 3, and N_2 is the number of double points of $f|_{S(f)}$.

Note that $||M^3||_\Delta$ is less than or equal to the minimal number of 3-simplices of any triangulation of M^3, and that $||M^3||_\Delta = 0$ for graph manifolds M^3. Here, a compact orientable 3-manifold is a *graph manifold* if it is the union of S^1-bundles over surfaces attached along some of their torus boundaries. Recall that every closed orientable graph manifold admits a stable map into $\mathbf{R}^2$ with $N_{\mathrm{sf}} = 0$ (see [41]).

§ 4. Invariants of manifolds

Let M^3 be a smooth closed oriented connected 3-manifold. Let us consider Morse functions $f : M^3 \to \mathbf{R}$. Such a Morse function is not unique: however, every pair of such functions can be connected by a "generic path" in the space of smooth functions (see Cerf [7]). This singularity theoretical fact is the basis of the celebrated Kirby Calculus [23], which has provided a lot of topological invariants. They are in fact still under investigation.

Then, we have the following natural question: how about using stable maps $M^3 \to \mathbf{R}^2$?

Reshetikhin–Turaev [34] defined a quantum invariant for 3-manifolds, just after Witten's celebrated proposal to use (yet mathematically unjustified) path-integral in order to define invariants of 3-manifolds (associated to each Lie algebra). This can, in fact, be interpreted as an invariant derived from stable maps $M^3 \to \mathbf{R}^2$. More precisely, Turaev [53] invented the notion of a *shadow*, which is a 2-dimensional compact connected polyhedron locally modelled on the cone over the edges of a tetrahedron such that to each non-singular surface component is associated a half integer, called a *gleam*. Such a shadow gives rise to a closed 3-manifold, and two shadows corresponding to the same 3-manifold can be connected to each other by a set of certain moves (see also [8, 20]).

For example, the quotient space in the Stein factorization of a stable map gives a shadow as long as it has no singular fiber corresponding to $(1 \cdot 2 \cdot 2 \cdot 1)$-type in Levine's terminology [25], or no singular fiber of type II^3 in the author's terminology [42]. In fact, it is known that such a singular fiber can be replaced by other singular fibers (see [20, Remark 2.3]). Starting from such a shadow, Turaev defined a certain state sum invariant for 3-manifolds, which is closely related to the Turaev–Viro invariant (defined as a state sum on 3-manifold triangulations [54]). As a corollary, Turaev proved that the Turaev–Viro invariant coincides with the absolute square of the Reshetikhin–Turaev invariant.

Let us consider another possibility. For a smooth closed connected orientable 3-manifold M^3 and a stable map $f : M^3 \to \mathbf{R}^2$ of M^3 into the plane, the quotient space W_f is a compact 2-dimensional polyhedron whose local structures have been completely classified (see [24, 25, 30]). Define $\mathcal{W}(M^3)$ to be the set of the homeomorphism classes of the 2-dimensional polyhedrons which appear as the quotient space of some stable map $f : M^3 \to \mathbf{R}^2$. This is clearly a diffeomorphism invariant of M^3. Note that we ignore the gleams associated to the regular surface components of W_f.

Concerning this invariant, the following has been known.

Theorem 4.1 (Motta–Porto–Saeki [30]). *For any finite set $M_1^3, M_2^3, \ldots, M_k^3$ of closed connected orientable 3-manifolds, we have*

$$\bigcap_{i=1}^{k} \mathcal{W}(M_i^3) \neq \emptyset.$$

On the other hand, the intersection over all such 3-manifolds is empty:

$$\bigcap_{M^3} \mathcal{W}(M^3) = \emptyset.$$

Then, we have the following natural question.

Problem 4.2. Let M_i^3, $i = 0, 1$, be smooth closed connected orientable 3-manifolds. If $\mathcal{W}(M_0^3) = \mathcal{W}(M_1^3)$, then are M_0^3 and M_1^3 diffeomorphic?

We have a partial answer as follows.

Theorem 4.3 (Burlet–de Rham [4]). *Suppose $M_0^3 = S^3$ or $\sharp^k(S^1 \times S^2)$ for some $k \geq 1$. Then, for a smooth closed connected orientable 3-manifold M_1^3, if $\mathcal{W}(M_0^3) = \mathcal{W}(M_1^3)$, then M_0^3 and M_1^3 are diffeomorphic.*

The above result is implicit in Burlet–de Rham's work. In fact, they showed that for a stable map which is a special generic map, the topology of the quotient space completely determines the diffeomorphism type of the source 3-manifold. In such a case, gleams are unnecessary.

§ 5. Problems

In this section, we give some open problems in the global theory of singularities of differentiable maps that are related to the topics covered in this paper. These problems were, in fact, presented in the workshop "Singularity Theory and its Applications" held in Oita National College of Technology in 2011, and those problems which we present here are still unsolved, as far as the author knows.

Problem 5.1. Determine $\mathcal{S}(\Sigma^7)$ for an exotic sphere Σ^7 of Milnor (see Example 2.11 of Section 2).

Problem 5.2. Let M and N be smooth manifolds and $f : M \to N$ a smooth map. Define the notion of the "most natural map" (or the "simplest map", or the "standard map", or anything similar) among the *generic* smooth maps in the homotopy class of f, and study such maps (existence, uniqueness, their topological properties, etc.).

Problem 5.3 ([48])**.** Describe the Euler class e of an oriented S^1-bundle in terms of the space $C^\infty(S^1, \mathbf{R}^2)$. Note that Kazarian [21] has obtained some results in terms of $C^\infty(S^1, \mathbf{R})$.

For example, for an oriented S^1-bundle E, if there exists a map $E \to \mathbf{R}^2$ that is an immersion of rotation number ± 1 on each fiber, then the S^1-bundle is necessarily trivial, i.e. $e(E) = 0$.

Problem 5.4 ([26])**.** It is known that for a generic function $f : M^2 \to \mathbf{R}$ on a closed surface, the quotient space in its Stein factorization is a graph, which is often called a *Reeb graph* (see [32]). Let G be an arbitrary finite graph without loops or isolated vertices.

(1) Is there an *embedding* $\eta : M^2 \to \mathbf{R}^3$ of a closed orientable surface such that the Reeb graph of the associated height function is homeomorphic to G ?

(2) Is there an *embedding* $\eta : M^2 \to \mathbf{R}^3 \setminus \{0\}$ of a closed orientable surface such that the Reeb graph of the associated distance function from the origin is homeomorphic to G ?

Problem 5.5. A link in a 3-manifold is a *graph link* if its exterior is a graph manifold. A stable map $f : M \to N$ between manifolds is *simple* if each connected component of every fiber of f contains at most one singular point.

It is known that every closed orientable graph 3-manifold M^3 admits a simple stable map into $\mathbf{R}^2$ and that the singular set of such a map is a graph link in M^3 [41]. Characterize those graph links which appear as the singular set of a simple stable map.

Problem 5.6. Let M^3 be a closed orientable graph 3-manifold. Determine the smallest number of singular set components for simple stable maps $M^3 \to \mathbf{R}^2$.

Problem 5.7 ([44])**.** For a smooth closed connected orientable 3-manifold M^3 and a positive integer g, are the following two conditions equivalent?

(1) There exists a Morse function $f : M^3 \to \mathbf{R}$ such that the genus of every component of every regular fiber is at most g.

(2) The 3-manifold M^3 is diffeomorphic to the connected sum of finitely many closed orientable 3-manifolds of Heegaard genus at most g.

It is known that they are equivalent for $g = 1$.

Note that every closed connected orientable 3-manifold has a Heegaard decomposition, i.e., it is always a union of two handlebodies attached along their boundaries. The *Heegaard genus* of such a 3-manifold is the minimum possible genus of the handlebodies that give its Heegaard decomposition.

Problem 5.8. Let M^4 be a closed oriented 4-dimensional manifold. For a stable map $f : M^4 \to \mathbf{R}^3$, it is known that the number of singular fibers of type III^8, counted with signs, coincides with the signature $\sigma(M^4)$ of M^4 (see [49] or Corollary 3.7 of the present paper). Does there always exist a stable map $M^4 \to \mathbf{R}^3$ such that the number of singular fibers of type III^8 (counted without signs) coincides with $|\sigma(M^4)|$?

Problem 5.9 ([37])**.** A smooth map is a *fold map* if its singularities are all fold singularities (see Definition 2.1). Let M^4 be a simply connected smooth closed manifold of dimension 4. If M^4 admits a simple fold map into $\mathbf{R}^2$, then does it admit a special generic map into $\mathbf{R}^3$?

Problem 5.10. Let G be a finitely presentable group. Does there exist a closed orientable 4-dimensional manifold M^4 and a simple stable map $M^4 \to \mathbf{R}^3$ such that $\pi_1(M^4) \cong G$? Or does there exist a closed orientable 4-dimensional manifold M^4 with $\pi_1(M^4) \cong G$ and a stable map $f : M^4 \to \mathbf{R}^2$ such that every component of every regular fiber is diffeomorphic to S^2 ? (See [47].)

Problem 5.11. Let M_1^4 and M_2^4 be smooth (not necessarily closed) manifolds of dimension 4. We suppose that they are homeomorphic. If there exist proper special generic maps $f_1 : M_1^4 \to \mathbf{R}^3$ and $f_2 : M_2^4 \to \mathbf{R}^3$, then are M_1^4 and M_2^4 diffeomorphic? (This would mean that the differentiable structure on a topological 4-manifold that allows the existence of a proper special generic map into $\mathbf{R}^3$ is unique.) See [46].

Problem 5.12. Does every closed non-orientable 4-dimensional manifold admit a fold map into $\mathbf{R}^3$?

Problem 5.13. It is known that closed manifolds whose tangent bundles satisfy certain conditions admit fold maps for which all the fold indices appear [1, 11]. Study the existence of fold maps with restricted set of fold indices. The extremal case corresponds to that of special generic maps.

Problem 5.14 (Gay–Kirby [13])**.** Let M^n be a closed connected n-dimensional manifold ($n \geq 3$). It is known that every smooth map $M^n \to S^2$ is homotopic to an excellent map (i.e. a smooth map with only folds and cusps as its singularities) without definite folds [43]. If M^n is 1-connected, is every smooth map $M^n \to S^2$ homotopic to an excellent map without folds of index $0, 1, n-2, n-1$?

Problem 5.15. Characterize those surface links that appear as the singular set of a stable map $S^4 \to \mathbf{R}^3$. (Compare with [40].)

Problem 5.16. Let C be a plane projective curve in $\mathbf{C}P^2$. Study the condition for C to be topologically equivalent to a plane projective curve defined by a polynomial of *real* coefficients.

Problem 5.17. Let

$$f(z) = \sum_{j=1}^{n+1} z_j^{a_j} \quad \text{and} \quad g(z) = \sum_{j=1}^{n+1} z_j^{b_j}$$

be Brieskorn–Pham type polynomials. It is known that if their associated algebraic knots are cobordant, then their Seifert forms are Witt equivalent over $\mathbf{R}$. Furthermore, their Seifert forms are Witt equivalent over $\mathbf{R}$ if and only if

$$\prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2a_j} = \prod_{j=1}^{n+1} \cot \frac{\pi \ell}{2b_j}$$

holds for every odd integer ℓ (see [2]). Does it imply that $a_i = b_j$ up to renumbering the indices?

Problem 5.18. Let $f(z)$ be a Brieskorn–Pham type polynomial as above. Describe the condition on the exponents a_j such that $H_{n-1}(K_f; \mathbf{Z})$ is torsion free, where $K_f = f^{-1}(0) \cap S^{2n+1}$ is the $(2n-1)$-dimensional closed manifold called the *link* of f. The condition for the vanishing of $H_{n-1}(K_f; \mathbf{Z})$ has been described in [3].

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