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Criteria for Morin singularities into higher dimensions

Dedicated to Professor Yasutaka Nakanishi on the occasion of his 60*th birthday*

By

Kentaro SAJI*

Abstract

We give criteria for Morin singularities into higher dimensions. As an application, we study the number of $\mathcal{A}$-isotopy classes of Morin singularities.

§ 1. Introduction

A map-germ $f : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ $(m < n)$ is called an *r-Morin singularity* $(m \geq r(m - n + 1))$ if it is $\mathcal{A}$-equivalent to the following map-germ at the origin:

$$h_{0,r} : x \mapsto \bigl(x_1, \dots, x_{m-1}, h_1(x), \dots, h_{n-m+1}(x)\bigr), \tag{1.1}$$

where $x = (x_1, \dots, x_m)$, and

$$\begin{aligned} h_i(x) &= \sum_{j=1}^{r} x_{(i-1)r+j} x_m^j \quad (i = 1, \dots, n - m), \\ h_{n-m+1}(x) &= \sum_{j=1}^{r-1} x_{(n-m)r+j} x_m^j + x_m^{r+1}. \end{aligned} \tag{1.2}$$

We say that two map-germs $f, g : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ are *$\mathcal{A}$-equivalent* if there exist diffeomorphism-germs $\varphi : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^m, 0)$ and $\Phi : (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ such that

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*Department of Mathematics, Graduate School of Science, Kobe University, Rokkodai 1-1, Nada, Kobe 657-8501, Japan.
e-mail: `saji@math.kobe-u.ac.jp`

$\Phi \circ f \circ \varphi = g$ holds. Morin singularities are stable, and conversely, corank one and stable germs are Morin singularities. This means that Morin singularities are fundamental and frequently appear as singularities of maps from a manifold to another. Morin gave a characterization of them by a transversality of the Thom-Boardman singularity set and also gave criteria for germs of a normalized form $\big(x_1, \dots, x_{m-1}, g_1(x), \dots, g_{n-m+1}(x)\big)$. Morin singularities are also characterized using the intrinsic derivative due to Porteous ([18] see also [1, 6]). Criteria for singularities without using normalization are not only more convenient but also indispensable in some cases. We call criteria without normalizing *general criteria*. In fact, in the case of wave front surfaces in 3-space, general criteria for cuspidal edges and swallowtails were given in [13], where we studied the local and global behavior of flat fronts in hyperbolic 3-space using them. Recently general criteria for other singularities and several applications of them have been given (see [9, 10, 12, 17, 27, 28, 30]). In this paper, we give general criteria for Morin singularities. Using them, we give applications to singularities of ruling maps and $\mathcal{A}$-isotopy of Morin singularities. See [4, 5, 20, 21, 32, 33] for other investigations of Morin singularities.

§ 2. Singular set and restriction of a map to the singular set

Let $f : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ $(m < n)$ be a map-germ. We assume $\operatorname{rank} df_0 = m - 1$. Then one can take a coordinate system satisfying

$$\operatorname{rank} d(f_1, \dots, f_{m-1}) = m - 1, \quad \text{and} \quad (df_{m-1+i})_0 = 0 \quad (i = 1, \dots, n - m + 1), \tag{2.1}$$

where $f = (f_1, \dots, f_n)$. We set

$$\begin{aligned} &\lambda_i = \det(df_1, \dots, df_{m-1}, df_{m-1+i}) \quad (i = 1, \dots, n - m + 1), \quad \text{and} \\ &\Lambda = (\lambda_1, \dots, \lambda_{n-m+1}) : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^{n-m+1}, 0). \end{aligned} \tag{2.2}$$

Let 0 be a singular point of f. The singular point 0 of f is said to be *non-degenerate* if $\operatorname{rank} d\Lambda_0 = n - m + 1$ holds. This definition does not depend on the choice of coordinate system on the source, nor on the target:

Lemma 2.1. *Non-degeneracy does not depend on the choice of coordinate system on the source, nor on the target satisfying* (2.1). *Furthermore, if* 0 *is a non-degenerate singular point of* f, *then the set of singular points* $S(f)$ *is a manifold.*

Proof. Let $\varphi : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^m, 0)$ be a diffeomorphism-germ. Then

$$\det\big(d(f_1 \circ \varphi), \dots, d(f_{m-1} \circ \varphi), d(f_{m-1+i} \circ \varphi)\big) = (\lambda_i \circ \varphi) \det d\varphi \ (i = 1, \dots, n - m + 1)$$

holds. Thus non-degeneracy does not depend on the choice of coordinate systems on the source satisfying (2.1). Next, let us assume that $\operatorname{rank} d(f_1, \dots, f_{m-1}) = m - 1$ and

$(df_{m-1+i})_0 = 0$ $(i = 1, \dots, n-m+1)$. Since non-degeneracy does not depend on the choice of coordinate systems on the source, we may assume f is written as

(2.3)
$$f(x) = \big(x_1, \dots, x_{m-1}, f_m(x), \dots, f_n(x)\big), \quad (df_{m-1+i})_0 = 0, \quad (i = 1, \dots, n-m+1),$$

where $x = (x_1, \dots, x_m)$. Let us take a diffeomorphism-germ $\Phi = (\Phi_1, \dots, \Phi_n) : (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$. By assumption, we may assume that

$$d\Phi_0 = \left(\begin{array}{ccc|ccc} (\Phi_1)_{X_1} & \cdots & (\Phi_1)_{X_{m-1}} & (\Phi_1)_{X_m} & \cdots & (\Phi_1)_{X_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (\Phi_{m-1})_{X_1} & \cdots & (\Phi_{m-1})_{X_{m-1}} & (\Phi_{m-1})_{X_m} & \cdots & (\Phi_{m-1})_{X_n} \\ \hline & & & (\Phi_m)_{X_m} & \cdots & (\Phi_m)_{X_n} \\ & O & & \vdots & \vdots & \vdots \\ & & & (\Phi_n)_{X_m} & \cdots & (\Phi_n)_{X_n} \end{array}\right)(0) =: \left(\begin{array}{c|c} M_1 & M_2 \\ \hline O & M_4 \end{array}\right).$$

Let us set

$$\bar{\lambda}_i = \det\big(\Phi_1(f), \dots, \Phi_{m-1}(f), \Phi_{m-1+i}(f)\big) \quad (i = 1, \dots, n-m+1).$$

Then by a direct calculation we see

$$\begin{aligned} &\left(\bar{\lambda}_i\right)_{x_k}(0) = \det M_1 \left(\sum_{j=1}^{n-m+1} (\Phi_{m-1+i})_{X_j} (\lambda_j)_{x_k}\right)(0) \quad (k = 1, \dots, m), \\ &{}^t\big((\bar{\lambda}_1)_{x_k}, \dots, (\bar{\lambda}_{n-m+1})_{x_k}\big)(0) = \Big(\det M_1 \ M_4 {}^t\big((\lambda_1)_{x_k}, \dots, (\lambda_{n-m+1})_{x_k}\big)\Big)(0) \end{aligned} \tag{2.4}$$

for any $i = 1, \dots, n-m+1$, where ${}^t(\ )$ is transposition. Since $\det M_1 \det M_4 \neq 0$ holds at 0, this shows that non-degeneracy does not depend on the choice of coordinate systems satisfying (2.3). We now show the second part. It is easily seen that $S(f) = \Lambda^{-1}(0)$ and non-degeneracy implies that 0 is a regular value of Λ. Hence $S(f)$ is a manifold. $\square$

Let $f : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ satisfies that $\operatorname{rank} df_0 = m - 1$. Then there exists a vector field η on $(\boldsymbol{R}^m, 0)$ such that $\langle \eta_p \rangle_{\boldsymbol{R}} = \ker df_p$ holds for $p \in S(f)$. We call η the *null vector field.* In fact, since $\operatorname{rank} df_0 = m - 1$, we may assume that f is written as (2.3). Since $S(f) = \{(f'_m, \dots, f'_n) = 0\}$ $(\ ' = \partial/\partial x_m)$ holds, ∂x_m satisfies the condition of the null vector field.

Now we discuss higher order non-degeneracy and singularities, by considering restriction of a map-germ to its singular set. The procedure is similar to that of the case of the equidimensional Morin singularities given in [26]. Let $f : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ be a map-germ and 0 a non-degenerate singular point. Let us assume that $f = (f_1, \dots, f_n)$ satisfies (2.1). Let λ_i and Λ be as in (2.2). Since $S(f)$ is a manifold, the condition

$\eta_0 \in T_0S(f)$ is well-defined. A non-degenerate singular point 0 of $f:(\boldsymbol{R}^m,0)\to(\boldsymbol{R}^n,0)$ is 2-*singular* if $\eta_0 \in T_0S(f)$ holds. This condition is equivalent to $\eta\Lambda(0)=0$, where $\eta\Lambda$ stands for the directional derivative. Set $S_2(f)=\{p\in S(f)\,|\,\eta_p\in T_pS(f)\}$. The direction of η is unique on $S(f)$, the definition of $S_2(f)$ does not depend on the choice of η. Moreover, we have the following lemma.

Lemma 2.2. *The equality $S_2(f)=S(f|_{S(f)})$ holds.*

Proof. Since the conclusion does not depend on the choice of coordinate systems and choice of η, we may assume that f is written in the form (2.3), and $\eta=\partial x_m$. Transposition of the matrix representation of $d\Lambda_0$ is

$$\begin{pmatrix} (f'_m)_{x_1} & \cdots & (f'_n)_{x_1} \\ \vdots & \vdots & \vdots \\ (f'_m)_{x_{m-1}} & \cdots & (f'_n)_{x_{m-1}} \\ f''_m & \cdots & f''_n \end{pmatrix}(0) = \begin{pmatrix} (f'_m)_{x_1} & \cdots & (f'_n)_{x_1} \\ \vdots & \vdots & \vdots \\ (f'_m)_{x_{m-1}} & \cdots & (f'_n)_{x_{m-1}} \\ 0 & \cdots & 0 \end{pmatrix}(0), \tag{2.5}$$

where $'=\partial/\partial x_m$. We remark that the assumption 2-singular implies $\eta\Lambda(0)=0$, thus the last row vanishes. Since the rank of the matrix (2.5) is $n-m+1$ by non-degeneracy, we may assume

$$\operatorname{rank}\begin{pmatrix} (f'_m)_{x_1} & \cdots & (f'_n)_{x_1} \\ \vdots & \vdots & \vdots \\ (f'_m)_{x_{n-m+1}} & \cdots & (f'_n)_{x_{n-m+1}} \end{pmatrix}(0) = n-m+1$$

by a numbering change. By the implicit function theorem, there exist functions

$$x_1(x_{n-m+2},\dots,x_m),\dots,x_{n-m+1}(x_{n-m+2},\dots,x_m)$$

such that

$$\Lambda(x_1(x_{n-m+2},\dots,x_m),\dots,x_{n-m+1}(x_{n-m+2},\dots,x_m),x_{n-m+2},\dots,x_m)\equiv 0 \tag{2.6}$$

holds, where $\equiv$ means that the equality holds identically. Differentiating (2.6) by x_m, we have

$$\begin{pmatrix} (\lambda_1)_{x_1} & \cdots & (\lambda_1)_{x_{n-m+1}} \\ \vdots & \vdots & \vdots \\ (\lambda_{n-m+1})_{x_1} & \cdots & (\lambda_{n-m+1})_{x_{n-m+1}} \end{pmatrix}\begin{pmatrix} x'_1 \\ \vdots \\ x'_{n-m+1} \end{pmatrix} + \begin{pmatrix} \lambda'_1 \\ \vdots \\ \lambda'_{n-m+1} \end{pmatrix} \equiv 0. \tag{2.7}$$

On the other hand, $g=f|_{S(f)}$ is parametrized by

$$\Big(x_1(x_{n-m+2},\dots,x_m),\dots,x_{n-m+1}(x_{n-m+2},\dots,x_m),x_{n-m+2},\dots,x_{m-1},$$
$$f_m\big(x_1(x_{n-m+2},\dots,x_m),\dots,x_{n-m+1}(x_{n-m+2},\dots,x_m),x_{n-m+2},\dots,x_m\big),\dots,$$
$$f_n\big(x_1(x_{n-m+2},\dots,x_m),\dots,x_{n-m+1}(x_{n-m+2},\dots,x_m),x_{n-m+2},\dots,x_m\big)\Big).$$

Since $f'_m = \cdots = f'_n = 0$ on $S(f)$, the transposition of the matrix representation of dg is

$$
(2.8)\qquad \left.\left(\begin{array}{ccc|c|ccc}
(x_1)_{x_{n-m+2}} & \cdots & (x_{n-m+1})_{x_{n-m+2}} & & * & \cdots & * \\
\vdots & \vdots & \vdots & E & \vdots & \vdots & \vdots \\
(x_1)_{x_{m-1}} & \cdots & (x_{n-m+1})_{x_{m-1}} & & * & \cdots & * \\
\hline
x'_1 & \cdots & x'_{n-m+1} & 0 & \displaystyle\sum_{i=1}^{n-m+1} (f_m)_{x_i} x'_i & \cdots & \displaystyle\sum_{i=1}^{n-m+1} (f_n)_{x_i} x'_i
\end{array}\right)\right|_{S(f)},
$$

where E stands for the identity matrix. Thus the matrix (2.8) is not full-rank if and only if $x'_1 = \cdots = x'_{n-m+1} = 0$. By (2.7), the condition $x'_1 = \cdots = x'_{n-m+1} = 0$ is equivalent to $(\eta\Lambda)|_{S(f)} = 0$. This implies that $S(g) = S_2(f)$. $\square$

Let 0 be a 2-singular point of f. We say that 0 is 2-*non-degenerate* if

$$d(\eta\Lambda)_0(T_0 S(f)) = T_0 \boldsymbol{R}^{n-m+1}$$

holds. This condition does not depend on the choice of coordinate systems and the choice of η. Moreover, 2-non-degeneracy implies that $S_2(f)$ is a manifold. In fact, it holds that $S_2(f) = \{p \in S(f) \,|\, \eta_p \in T_p S(f)\} = \{p \in S(f) \,|\, \eta\Lambda(p) = 0\}$, and that $d(\eta\Lambda)_0(T_0 S(f)) = T_0 \boldsymbol{R}^{n-m+1}$ implies that 0 is a regular value of $\eta\Lambda : S(f) \to \boldsymbol{R}^{n-m+1}$. Since $\dim S(f) = m - (n-m+1)$ holds, $m - (n-m+1) \geq n-m+1$ is needed for 2-non-degeneracy.

Lemma 2.3. *Let $f : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ be a map-germ with* $\operatorname{rank} df_0 = m-1$. *A singular point* 0 *is* 2*-non-degenerate if and only if* $(\Lambda, \eta\Lambda) = 0$, *and*

$$\operatorname{rank} d(\Lambda, \eta\Lambda)_0 = 2(n-m+1).$$

Proof. Let us assume $(\Lambda, \eta\Lambda) = 0$ and $\operatorname{rank} d(\Lambda, \eta\Lambda)_0 = 2(n-m+1)$. Then we see that $\operatorname{rank} d\Lambda_0 = n-m+1$, and we see non-degeneracy. Furthermore, $\eta\Lambda(0) = 0$, so we also see the 2-singularity. Thus it is enough to show that 2-non-degeneracy is equivalent to rank $d(\Lambda, \eta\Lambda)_0 = 2(n-m+1)$ at a 2-singular point.

Let us assume that 0 is 2-singular. Since the dimension of $S(f)$ is $2m-n+1$, and by the 2-singularity, it holds that $\eta_0 \in T_0 S(f)$, we take vector fields $\xi_2, \ldots, \xi_m$ satisfying that $\eta, \xi_2, \ldots, \xi_{2m-n-1}$ at 0 form a basis of $T_0 S(f)$. By $S(f) = \{\Lambda = 0\}$, it holds that $\eta\Lambda = \xi_2\Lambda = \cdots = \xi_{2m-n-1}\Lambda = 0$. Thus the transposition of the matrix representation of $d(\Lambda, \eta\Lambda)_0$ is

$$\left(\begin{array}{ccc|ccc}
\eta\lambda_1 & \cdots & \eta\lambda_{n-m+1} & \eta^2\lambda_1 & \cdots & \eta^2\lambda_{n-m+1} \\
\xi_2\lambda_1 & \cdots & \xi_2\lambda_{n-m+1} & \xi_2\eta\lambda_1 & \cdots & \xi_2\eta\lambda_{n-m+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\xi_{2m-n-1}\lambda_1 & \cdots & \xi_{2m-n-1}\lambda_{n-m+1} & \xi_{2m-n-1}\eta\lambda_1 & \cdots & \xi_{2m-n-1}\eta\lambda_{n-m+1} \\
\hline
\xi_{2m-n}\lambda_1 & \cdots & \xi_{2m-n}\lambda_{n-m+1} & \xi_{2m-n}\eta\lambda_1 & \cdots & \xi_{2m-n}\eta\lambda_{n-m+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\xi_m\lambda_1 & \cdots & \xi_m\lambda_{n-m+1} & \xi_m\eta\lambda_1 & \cdots & \xi_m\eta\lambda_{n-m+1}
\end{array}\right)(0)$$

$$=: \left(\begin{array}{c|c} O & J_2 \\ \hline J_1 & * \end{array}\right)(0).$$

By the non-degeneracy, $\operatorname{rank} J_1(0) = n - m + 1$ holds. Hence $\operatorname{rank} d(\Lambda, \eta\Lambda)_0 = 2(n - m + 1)$ is equivalent to $\operatorname{rank} J_2(0) = n - m + 1$. Since $\eta, \xi_2, \ldots, \xi_{2m-n-1}$ at 0 form a basis of $T_0S(f)$, we see that $\operatorname{rank} J_2(0) = n - m + 1$ is equivalent to 2-non-degeneracy. $\square$

Let 0 be a 2-non-degenerate singular point of f. We say that 0 is 3-*singular* if $\eta_0 \in T_0S_2(f)$ holds, namely, $\eta^2\Lambda(0) = 0$, where $\eta^j\Lambda$ stands for $\eta\cdots\eta\Lambda$ (j times). If 0 is 3-singular, we set $S_3(f) = \{p \in S_2(f) \mid \eta_p \in T_pS_2(f)\}$. This does not depend on the choice of η, and it holds that $S_3(f) = \{p \in S_2(f) \mid \eta^2\Lambda(p) = 0\} = \{p \in (\boldsymbol{R}^m, 0) \mid \Lambda(p) = \eta\Lambda(p) = \eta^2\Lambda(p) = 0\}$.

Accordingly, we define higher order singularities and non-degeneracies inductively. For a fixed $1 \le i \le m/(n - m + 1)$, and for $j \le i - 1$, assume that j-singularity and j-non-degeneracy of a singular point 0 of f are defined, and $S_j(f) = \{p \in S_{j-1}(f) \mid \eta_p \in T_pS_{j-1}\} = \{p \in S_{j-1}(f) \mid \eta^{j-1}\Lambda(p) = 0\}$ and $d(\eta^{j-1}\Lambda)_0(T_0S_{j-1}(f)) = T_0\boldsymbol{R}^{n-m+1}$ holds. This condition implies that $S_j(f)$ is a manifold.

Let 0 be an $(i-1)$-non-degenerate singular point of f. We say that 0 is i-*singular* if $\eta_0 \in T_0S_{i-1}$ holds. We define $S_i = \{p \in S_{i-1} \mid \eta_p \in T_pS_{i-1}\}$. Then since $S_{i-1}(f) = \{p \in S_{i-2}(f) \mid \eta^{i-2}\Lambda(p) = 0\}$, we see that $S_i = \{p \in S_{i-1} \mid \eta^{i-1}\Lambda = 0\}$.

Let 0 be an i-singular point of f. We call 0 is i-*non-degenerate* if

$$d(\eta^{i-1}\Lambda)_0(T_0S_{i-1}(f)) = T_0\boldsymbol{R}^{n-m+1}$$

holds. We show the following lemma.

Lemma 2.4. *For an i-singular point, the i-non-degeneracy does not depend on the choice of η.*

Proof. Let $\tilde{\eta} = \alpha\eta + \beta$, where α is a non-zero function and β is a vector field satisfying $\beta = 0$ on $S(f)$. It is enough to show that

$$\tilde{\eta}^{i-1}\Lambda = \alpha^{i-1}\eta^{i-1}\Lambda \quad (\text{on } S_{i-1}(f)).$$

We show this by induction. If $i = 2$, it is obvious. We assume that $\tilde{\eta}^{i-2}\Lambda = \alpha^{i-2}\eta^{i-2}\Lambda$ holds on $S_{i-2}(f)$. Then

$$\begin{aligned}\tilde{\eta}^{i-1}\Lambda - \alpha^{i-1}\eta^{i-1}\Lambda &= (\alpha\eta + \beta)\tilde{\eta}^{i-2}\Lambda - \alpha^{i-1}\eta^{i-1}\Lambda \\ &= \alpha\Big(\eta\underline{\big(\tilde{\eta}^{i-2}\Lambda - \alpha^{i-2}\eta^{i-2}\Lambda\big)} + \eta(\alpha^{i-2})\eta^{i-2}\Lambda\Big) + \beta\tilde{\eta}^{i-2}\Lambda\end{aligned} \tag{2.9}$$

holds. Since the underlined part of (2.9) vanishes on $S_{i-2}(f)$, and $S_{i-1}(f) = \{\eta \in TS_{i-2}\}$, and $\eta^{i-2}\Lambda = 0$ on S_{i-1}, the right hand side of (2.9) vanishes on $S_{i-1}(f)$. $\square$

This procedure can be continued when $i \leq m/(n-m+1)$. We see that $S_i(f) = (\Lambda, \eta\Lambda, \dots, \eta^{i-1}\Lambda)^{-1}(0)$. More precisely, we have the following lemma.

Lemma 2.5. *Let $f : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ be a map-germ satisfying $\operatorname{rank} df_0 = m-1$. The $(i+1)$-non-degeneracy of a singular point 0 is equivalent to*

$$(\Lambda, \eta\Lambda, \dots, \eta^i\Lambda) = 0 \quad \textit{and} \quad \operatorname{rank} d(\Lambda, \eta\Lambda, \dots, \eta^i\Lambda)_0 = (i+1)(n-m+1).$$

Proof. We show the necessity by induction. By Lemma 2.3, we have 2-non-degeneracy. Let us assume that j-non-degeneracy ($j \leq i$) is proven. The $(j+1)$-singularity of 0 follows immediately from $\eta^j\Lambda(0) = 0$ for $j \leq i$. We show $(j+1)$-non-degeneracy. By j-non-degeneracy, we have submanifolds

$$S_j \subset S_{j-1} \subset \cdots \subset S_1 \subset (\boldsymbol{R}^m, 0).$$

We take a basis of each tangent space at 0 as follows: $\Xi_j = \{\xi_{j,1}, \dots, \xi_{j,m-j(n-m+1)}\}$ is a basis of T_0S_j, $\Xi_k = \{\xi_{k,1}, \dots, \xi_{k,n-m+1}\} \cup \Xi_{k+1}$ is a basis of T_0S_k ($k = j-1, \dots, 1$), and $\Xi_0 = \{\xi_{0,1}, \dots, \xi_{0,n-m+1}\} \cup \Xi_1$ is a basis of $T_0\boldsymbol{R}^m$. Since $S_k(f) = \{\Lambda = \eta\Lambda = \cdots = \eta^{k-1}\Lambda = 0\}$ ($1 \leq k \leq j$), if $\xi \in T_0S_k(f)$, then $\xi\Lambda = \cdots \xi\eta^{k-1}\Lambda = 0$ holds at 0. Thus the transposition of the matrix representation of $d(\Lambda, \eta\Lambda, \dots, \eta^j\Lambda)_0$ is

$$\begin{array}{cc}
 & \begin{array}{ccccc} \Lambda & \eta\Lambda & \cdots & \eta^{j-1}\Lambda & \eta^j\Lambda \\ \vdots & \vdots & \cdots & \vdots & \vdots \end{array} \\
\begin{array}{c} \Xi_j \ \cdots \\ \Xi_{j-1} \cdots \\ \vdots \\ \Xi_1 \ \cdots \\ \Xi_0 \ \cdots \end{array} & \left(\begin{array}{c|c|c|c|c} O & O & \cdots & O & J_j \\ \hline O & O & \cdots & J_{j-1} & \\ \hline \vdots & \vdots & \iddots & & \\ \hline O & J_2 & & & \\ \hline J_1 & & & & \end{array}\right)
\end{array},$$

where

$$\begin{array}{cc} & \eta^k\Lambda \\ & \vdots \\ \Xi_l \cdots & A \end{array}$$

means that A is a matrix formed by differentials of $\eta^k\Lambda = \eta^k(\lambda_1, \ldots, \lambda_{n-m+1})$ by $\Xi_l = (\xi_{l,1}, \ldots, \xi_{l,L})$. Then we see that $\operatorname{rank} J_j = n - m + 1$, and this implies $(j+1)$-non-degeneracy. $\square$

Theorem 2.6. *The map-germ $f : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ is an r-Morin singularity if and only if 0 is r-non-degenerate but not r-singular.*

To prove this theorem, the assumption does not depend on the choice of coordinate system and choice of null vector field, we may assume that f is of the form

$$f(x_1, \ldots, x_m) = \bigl(x_1, \ldots, x_{m-1}, f_m(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)\bigr), \tag{2.10}$$

and $\eta = \partial x_m$. Then $\Lambda = (f'_m, \ldots, f'_n)(x_1, \ldots, x_m)$ holds, where $' = \partial/\partial x_m$. Then the theorem follows directly from the following lemma due to Morin.

Lemma 2.7. (Morin, [14, p 5663, Lemme]) *Let $f : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ is a map-germ written in the form (2.10). Then f at 0 is an r-Morin singularity if and only if $(f_m^{(j)}, \ldots, f_n^{(j)})(0) = 0$ $(1 \le j \le r)$ and $(f_m^{(r+1)}, \ldots, f_n^{(r+1)})(0) \neq 0$ hold, and $\operatorname{rank} d(F, F', \ldots, F^{(r-1)})_0 = r(n - m + 1)$ holds, where $F = (f'_m, \ldots, f'_n)$, and $f_i^{(j)}$ stands for $(\partial^j/\partial x_m^j) f_i$ $(m \le i \le n)$.*

We give a proof of Theorem 2.6 here for the sake of those readers who are not familiar with singularity theory. The proof is based on [14, p 5664-5665]. The proof is a little complicated, thus we would like to state a short sketch of it previously. By the usual usage of the Malgrange preparation theorem and by Tschirnhaus transformation, one may assume that f has the form $\bigl(x_1, \ldots, x_{m-1}, g_1(x), \ldots, g_{n-m+1}(x)\bigr)$, where $x = (x_1, \ldots, x_m)$ and

$$g_i(x) = \sum_{j=1}^{r} \tilde{g}_{ij}(x) x_m^j \ (i = 1, \ldots, n-m), \quad g_{n-m+1}(x) = \sum_{j=1}^{r-1} \tilde{g}_{n-m+1,j}(x) x_m^j + x_m^{r+1}. \tag{2.11}$$

If the coordinate change on the source $(x_1, \ldots, x_m) \mapsto (\tilde{x}_1, \ldots, \tilde{x}_m)$ defined by

$$\begin{aligned} &\tilde{x}_1 = \tilde{g}_{11}(x), \ldots, \tilde{x}_r = \tilde{g}_{1r}(x), \tilde{x}_{r+1} = \tilde{g}_{21}(x), \ldots, \tilde{x}_{2r} = \tilde{g}_{2r}(x), \ldots, \\ &\qquad \tilde{x}_{r(n-m)+1} = \tilde{g}_{n-m+1,1}(x), \ldots, \tilde{x}_{r(n-m)+r-1} = \tilde{g}_{n-m+1,r-1}(x), \\ &\qquad\qquad \tilde{x}_{r(n-m)+r} = x_{r(n-m)+r}, \ldots, \tilde{x}_m = x_m \end{aligned}$$

is allowed, then (2.11) can be written in the form

$$g_i(x) = \sum_{j=1}^{r} \tilde{x}_{i-1+j}\tilde{x}_m^j \ (i = 1, \dots, n-m), \quad g_{n-m+1}(x) = \sum_{j=1}^{r-1} \tilde{x}_{n-m+j}(x)\tilde{x}_m^j + \tilde{x}_m^{r+1}.$$

Then by a suitable coordinate change on the target, the claim is proven. Most of the proof is occupied to show that these coordinate changes are regular.

Proof of Theorem 2.6. By $(r-1)$-singularity and non r-singularity,

$$(f_m^{(j)}, \dots, f_n^{(j)})(0) = 0 \quad (1 \le j \le r), \quad (f_m^{(r+1)}, \dots, f_n^{(r+1)})(0) \neq 0$$

holds. By a linear transformation, we may assume $f_i^{(r+1)}(0) \neq 0$ for all $m \le i \le n$. We consider a quotient space

$$\mathcal{M}_m / \big\langle x_1, \dots, x_{m-1}, f_i(x) \big\rangle_{\mathcal{M}_m} = \langle x_m^{r+1} \rangle_{\mathcal{M}_m} \quad (m \le i \le n), \tag{2.12}$$

where $\mathcal{M}_m = \{ f : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}, 0) \}$ is a ring of function-germs. Then by the Malgrange preparation theorem, there exist functions $\alpha_{n,k}$ $(0 \le k \le r)$ such that

$$x_m^{r+1} = \alpha_{n,0}\big(x_1, \dots, x_{m-1}, f_n(x)\big) - \sum_{k=1}^{r} \alpha_{n,k}\big(x_1, \dots, x_{m-1}, f_n(x)\big) x_m^k \tag{2.13}$$

holds. We consider a diffeomorphism-germ

$$\psi(x_1, \dots, x_m) = \left(x_1, \dots, x_{m-1}, x_m + \frac{1}{r}\alpha_{n,r}\big(x_1, \dots, x_{m-1}, f_n(x)\big) \right),$$

and set $\tilde{x} = \psi(x)$, where $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_m)$ and $x = (x_1, \dots, x_m)$. We remark that ψ^{-1} has the form $\psi^{-1}(\tilde{x}) = \big(\tilde{x}_1, \dots, \tilde{x}_{m-1}, \psi_m^{-1}(\tilde{x})\big)$. Then by a calculation, we see that there exist functions $\beta_{n,k}$ $(0 \le k \le r-1)$ such that

$$\tilde{x}_m^{r+1} = \beta_{n,0}\Big(\tilde{x}_1, \dots, \tilde{x}_{m-1}, f_n\big(\psi^{-1}(\tilde{x})\big)\Big) - \sum_{k=1}^{r-1} \beta_{n,k}\Big(\tilde{x}_1, \dots, \tilde{x}_{m-1}, f_n\big(\psi^{-1}(\tilde{x})\big)\Big)\tilde{x}_m^k \tag{2.14}$$

holds. Again by (2.12), there exist functions $\beta_{i,k}$ $(0 \le k \le r,\ m \le i \le n-1)$ such that

$$\tilde{x}_m^{r+1} = \beta_{i,0}\Big(\tilde{x}_1, \dots, \tilde{x}_{m-1}, f_i\big(\psi^{-1}(\tilde{x})\big)\Big) - \sum_{k=1}^{r} \beta_{i,k}\Big(\tilde{x}_1, \dots, \tilde{x}_{m-1}, f_i\big(\psi^{-1}(\tilde{x})\big)\Big)\tilde{x}_m^k \tag{2.15}$$

holds. Differentiating (2.14) and (2.15) $r+1$ times by $\tilde{x}_m$, we see that

$$\frac{\partial}{\partial y}\beta_{n,0}(x_1, \dots, x_{m-1}, y) \neq 0, \quad \frac{\partial}{\partial y}\beta_{i,0}(x_1, \dots, x_{m-1}, y) \neq 0 \quad (m \le i \le n-1)$$

at 0. Moreover, we have the following lemma.

Lemma 2.8. *Vectors*

$$\begin{aligned}
&d\beta_{m,1}(x_1,\dots,x_{m-1},0),\dots,d\beta_{m,r}(x_1,\dots,x_{m-1},0),\\
&d\beta_{m+1,1}(x_1,\dots,x_{m-1},0),\dots,d\beta_{m+1,r}(x_1,\dots,x_{m-1},0),\\
&\qquad\dots,\\
&d\beta_{n-1,1}(x_1,\dots,x_{m-1},0),\dots,d\beta_{n-1,r}(x_1,\dots,x_{m-1},0),\\
&d\beta_{n,1}(x_1,\dots,x_{m-1},0),\dots,d\beta_{n,r-1}(x_1,\dots,x_{m-1},0)
\end{aligned}$$

are linearly independent at 0.

Proof. Differentiating (2.14) and (2.15) by $\tilde{x}_m$ and $\tilde{x}_l$ $(1 \le l \le m-1)$, we see that

$$0 = (\beta_{i,0})_y (f_i)'_{x_l} (\psi_m^{-1})' - (\beta_{i,1})_{x_l} \quad (m \le i \le n)$$

holds at 0. This implies $d\beta_{i,1}(x_1,\dots,x_{m-1},0) = a_{i,11} df_i'(x_1,\dots,x_{m-1},0)$ at 0, where $a_{i,11} \in \boldsymbol{R}$ is non-zero. Again differentiating (2.14) and (2.15) twice by $\tilde{x}_m$ and $\tilde{x}_l$ $(1 \le l \le m-1)$, we see that

$$0 = (\beta_{i,0})_y (f_i)''_{x_l} ((\psi_m^{-1})')^2 - (\beta_{i,1})_y (f_i)'_{x_l} (\psi_m^{-1})' - 2(\beta_{i,2})_{x_l} \quad (m \le i \le n)$$

holds at 0. Thus it holds that $d\beta_{i,2}(x_1,\dots,x_{m-1},0) = a_{i,21} d(f_i')(x_1,\dots,x_{m-1},0) + a_{i,22} d(f_i'')(x_1,\dots,x_{m-1},0)$ at 0, where $a_{i,22}, a_{i,21} \in \boldsymbol{R}$ and $a_{i,22} \neq 0$. By the same arguments, we see that

$$\begin{aligned}
d\beta_{i,k}(x_1,\dots,x_{m-1},0) &= \sum_{j=1}^{k} a_{i,kj} d\big(f_i^{(j)}\big)(x_1,\dots,x_{m-1},0)\ (1 \le k \le r,\ m \le i \le n-1),\\
d\beta_{n,k}(x_1,\dots,x_{m-1},0) &= \sum_{j=1}^{k} a_{n,kj} d\big(f_n^{(j)}\big)(x_1,\dots,x_{m-1},0)\ (1 \le k \le r)
\end{aligned}$$

at 0, where $a_{i,kk} \neq 0, a_{n,kk} \neq 0$. This implies that

$$\begin{aligned}
\operatorname{rank}\big(&d\beta_{m,1}(x_1,\dots,x_{m-1},0),\dots,d\beta_{m,r}(x_1,\dots,x_{m-1},0),\\
&d\beta_{m+1,1}(x_1,\dots,x_{m-1},0),\dots,d\beta_{m+1,r}(x_1,\dots,x_{m-1},0),\\
&\qquad\dots,\\
&d\beta_{n-1,1}(x_1,\dots,x_{m-1},0),\dots,d\beta_{n-1,r}(x_1,\dots,x_{m-1},0),\\
&d\beta_{n,1}(x_1,\dots,x_{m-1},0),\dots,d\beta_{n,r-1}(x_1,\dots,x_{m-1},0)\big)
\end{aligned}$$

is the same as

$$\begin{aligned}
\operatorname{rank}\big(&df_m'(x_1,\dots,x_{m-1},0),\dots,df_m^{(r)}(x_1,\dots,x_{m-1},0),\\
&df_{m+1}'(x_1,\dots,x_{m-1},0),\dots,df_{m+1}^{(r)}(x_1,\dots,x_{m-1},0),\\
&\qquad\dots,\\
&df_{n-1}'(x_1,\dots,x_{m-1},0),\dots,df_{n-1}^{(r)}(x_1,\dots,x_{m-1},0),\\
&df_n'(x_1,\dots,x_{m-1},0),\dots,df_n^{(r-1)}(x_1,\dots,x_{m-1},0)\big),
\end{aligned}$$

and this is full-rank by assumption. $\square$

Assume that

$$\begin{aligned}
&\operatorname{rank}\bigl(df'_m(x_1,\dots,x_{r(n-m+1)-1},0,\dots,0),\dots,df^{(r)}_m(x_1,\dots,x_{r(n-m+1)-1},0,\dots,0),\\
&\quad df'_{m+1}(x_1,\dots,x_{r(n-m+1)-1},0,\dots,0),\dots,df^{(r)}_{m+1}(x_1,\dots,x_{r(n-m+1)-1},0,\dots,0),\\
&\qquad \dots,\\
&\quad df'_{n-1}(x_1,\dots,x_{r(n-m+1)-1},0,\dots,0),\dots,df^{(r)}_{n-1}(x_1,\dots,x_{r(n-m+1)-1},0,\dots,0),\\
&\quad df'_{n}(x_1,\dots,x_{r(n-m+1)-1},0,\dots,0),\dots,df^{(r-1)}_{n}(x_1,\dots,x_{r(n-m+1)-1},0,\dots,0)\bigr)\\
&= r(n-m+1)-1.
\end{aligned}$$

Then the map θ defined by

(2.16)

$$\begin{aligned}
\tilde{x}\mapsto\Bigl(&\beta_{m,1}\Bigl(\tilde{x}_1,\dots,\tilde{x}_{m-1},f_m\bigl(\psi^{-1}(\tilde{x})\bigr)\Bigr),\dots,\beta_{m,r}\Bigl(\tilde{x}_1,\dots,\tilde{x}_{m-1},f_m\bigl(\psi^{-1}(\tilde{x})\bigr)\Bigr),\\
&\beta_{m+1,1}\Bigl(\tilde{x}_1,\dots,\tilde{x}_{m-1},f_{m+1}\bigl(\psi^{-1}(\tilde{x})\bigr)\Bigr),\dots,\beta_{m+1,r}\Bigl(\tilde{x}_1,\dots,\tilde{x}_{m-1},f_{m+1}\bigl(\psi^{-1}(\tilde{x})\bigr)\Bigr),\\
&\quad ,\dots,\\
&\beta_{n-1,1}\Bigl(\tilde{x}_1,\dots,\tilde{x}_{m-1},f_{n-1}\bigl(\psi^{-1}(\tilde{x})\bigr)\Bigr),\dots,\beta_{n-1,r}\Bigl(\tilde{x}_1,\dots,\tilde{x}_{m-1},f_{n-1}\bigl(\psi^{-1}(\tilde{x})\bigr)\Bigr),\\
&\beta_{n,1}\Bigl(\tilde{x}_1,\dots,\tilde{x}_{m-1},f_{n}\bigl(\psi^{-1}(\tilde{x})\bigr)\Bigr),\dots,\beta_{n,r-1}\Bigl(\tilde{x}_1,\dots,\tilde{x}_{m-1},f_{n}\bigl(\psi^{-1}(\tilde{x})\bigr)\Bigr),\\
&\ \tilde{x}_{r(n-m+1)},\dots,\tilde{x}_m\Bigr)
\end{aligned}$$

is a diffeomorphism-germ on the source, and Θ defined by

(2.17)

$$\begin{aligned}
X\mapsto\bigl(&\beta_{m,1}(X_1,\dots,X_{m-1},X_m),\dots,\beta_{m,r}(X_1,\dots,X_{m-1},X_m),\\
&\beta_{m+1,1}(X_1,\dots,X_{m-1},X_{m+1}),\dots,\beta_{m+1,r}(X_1,\dots,X_{m-1},X_{m+1}),\\
&\quad \dots,\\
&\beta_{n-1,1}(X_1,\dots,X_{m-1},X_{n-1}),\dots,\beta_{n-1,r}(X_1,\dots,X_{m-1},X_{n-1}),\\
&\beta_{n,1}(X_1,\dots,X_{m-1},X_{n}),\dots,\beta_{n,r-1}(X_1,\dots,X_{m-1},X_{n}),\\
&X_{r(n-m+1)},\dots,X_{m-1},\beta_{m,0}(X_1,\dots,X_{m-1},X_m),\dots,\beta_{n,0}(X_1,\dots,X_{m-1},X_n)\bigr),
\end{aligned}$$

where $X=(X_1,\dots,X_n)$, is also a diffeomorphism-germ on the target. We set $\theta(x)=\bar{x}=(\bar{x}_1,\dots,\bar{x}_m)$. Then we see that $\Theta\circ f\circ\psi^{-1}\circ\theta^{-1}$ has the following expression:

$$\begin{aligned}
\beta_{i,j}\bigl(f\circ\psi^{-1}\circ\theta^{-1}(\bar{x})\bigr)&=\beta_{i,j}\Bigl(\bar{x}_1,\dots,\bar{x}_{m-1},f_i\bigl(\psi^{-1}\circ\theta^{-1}(\bar{x})\bigr)\Bigr)\\
&=\beta_{i,j}\Bigl(\tilde{x}_1,\dots,\tilde{x}_{m-1},f_i\bigl(\psi^{-1}(\tilde{x})\bigr)\Bigr)\\
&=\bar{x}_{r(i-m)+j}
\end{aligned}$$

when $m \le i \le n-1,\ 1 \le j \le r$ or $i = n,\ 1 \le j \le r-1$, and

$$\begin{aligned}\beta_{i,0}\big(f \circ \psi^{-1} \circ \theta^{-1}(\bar{x})\big) &= \beta_{i,0}\Big(\bar{x}_1, \dots, \bar{x}_{m-1}, f_i\big(\psi^{-1} \circ \theta^{-1}(\bar{x})\big)\Big)\\ &= \beta_{i,0}\Big(\tilde{x}_1, \dots, \tilde{x}_{m-1}, f_i\big(\psi^{-1}(\tilde{x})\big)\Big)\\ &= \tilde{x}_m^{r+1} + \sum_{j=1}^{R} \beta_{i,j}\Big(\tilde{x}_1, \dots, \tilde{x}_{m-1}, f_i\big(\psi^{-1}(\tilde{x})\big)\Big)\tilde{x}_m^j\\ &= \bar{x}_m^{r+1} + \sum_{j=1}^{R} \bar{x}_{r(i-m)+j}\bar{x}_m^j,\end{aligned}$$

where $R = r$ for $m \le i \le n-1$ and $R = r-1$ for $i = n$. Therefore f is $\mathcal{A}$-equivalent to

$$x \mapsto \big(x_1, \dots, x_{m-1}, \hat{h}_m(x), \dots, \hat{h}_{m-1}(x), h_n(x)\big),$$

where $\hat{h}_i(x) = h_i(x) + x_m^{r+1}$, and $h_i(x)$ $(i = m, \dots, n)$ are as in (1.2). By suitable linear translations on the source and target, we see that f is $\mathcal{A}$-equivalent to the form as in (1.1). This completes the proof. $\square$

By Theorem 2.6 and Lemma 2.5, we have the following criteria.

Corollary 2.9. *Let $f : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ be a map-germ satisfying* $\operatorname{rank} df_0 = m-1$. *Then f at 0 is an r-Morin singularity if and only if*

- *$\eta\Lambda = \cdots = \eta^{r-1}\Lambda = 0$ and $\eta^r\Lambda \ne 0$ hold at 0, and*
- $\operatorname{rank} d(\Lambda, \eta\Lambda, \dots, \eta^{r-1}\Lambda)_0 = r(n-m+1)$ *holds.*

Here, $f = (f_1, \dots, f_m)$ satisfies $d(f_1, \dots, f_{m-1}) = m-1$, $\Lambda = (\lambda_1, \dots, \lambda_{n-m+1})$, $\lambda_i =$ $\det(f_1, \dots, f_{m-1}, f_{m-1+i})$ *and η is the null vector field.*

Applying Lemma 2.7 for a given map-germ f, it needs that f is written in the normalized form (2.10), and to obtain this form, the implicit function theorem is applied. On the other hand, since our criteria uses only coordinate free data of f, the author believes that our criteria (Theorem 2.6 and Corollary 2.9) is convenient to Lemma 2.7 and indispensable in certain cases. In fact, applications [9, 10, 12, 17, 30] of this kind of criteria might be difficult by using only of the criteria which needs the normalization. We remark that our characterization can be interpreted as a vector field representation of the intrinsic derivative. See [18] about the intrinsic derivative, and see also [1, 6]. In fact, the image of $v \in T_p\boldsymbol{R}^m$ by $D(df)_p : T_p\boldsymbol{R}^m \to \operatorname{Hom}(K_p, L_{f(p)})$ coincides with $d\Lambda_p(v) : \boldsymbol{R} \to \boldsymbol{R}^{n-m+1}$, where $K_p = \ker df_p$, $L_{f(p)} = \operatorname{coker} df_p$, and $T_p\boldsymbol{R}^k$ (resp. $T_p \operatorname{Hom}(K_p, L_{f(p)})$) is canonically identified with $\boldsymbol{R}^k$ $(k = 1, n-m+1)$ (resp. $\operatorname{Hom}(K_p, L_{f(p)})$).

§ 3. Application to singularities of ruling maps

A *one-parameter family of n-planes in $\boldsymbol{R}^{2n}$* is a map defined by

$$F_{(\gamma,\delta)}(t,u_1,\dots,u_n)=\gamma(t)+\sum_{i=1}^{n}u_i\delta_i(t)$$

where $\gamma : J \to \boldsymbol{R}^{2n}$ is a curve and $\delta(t)=(\delta_1(t),\dots,\delta_n(t)) : J \to (\boldsymbol{R}^{2n})^n$ satisfies $\delta_i\cdot\delta_j=1$ if $i=j$ and $\delta_i\cdot\delta_j=0$ if $i\neq j$, where J is an open interval, and $\cdot$ stands for the canonical inner product. We call γ the *base curve*, and δ the *director frame* of $F_{(\gamma,\delta)}$. This is a generalization of ruled surfaces in $\boldsymbol{R}^3$. Ruled surfaces are classical objects in differential geometry. However, it has again paid attention in several areas [19, 29, 31]. In general, ruled surfaces and their generalizations have singularities, and they have been investigated in several articles [8, 11, 15]. To study the geometry and singularities of this kind of map, the striction curve plays a crucial role (See [7, 11], for example). One can always choose a director frame satisfying $\delta_i\cdot\delta_j'=0$ for any i,j. A curve $\sigma(t)=\gamma(t)+\sum_{i=1}^{n}u_i(t)\delta_i(t)$ is a *striction curve* if $\sigma'\cdot\delta_i'\equiv 0$ $(1\le i\le n)$ holds, where $\equiv$ means that the equality holds identically. If $(\delta(t),\delta'(t))=(\delta_1(t),\dots,\delta_n(t),\delta_1'(t),\dots,\delta_n'(t))$ are linearly independent, then we obtain a striction curve $\sigma(t)=\gamma(t)+\sum_{i=1}^{n}u_i(t)\delta_i(t)$ by setting

$$\begin{pmatrix}u_1(t)\\ \vdots\\ u_n(t)\end{pmatrix}=-\Big(\big(\delta_i'(t)\cdot\delta_j'(t)\big)_{i,j=1,\dots,n}\Big)^{-1}\begin{pmatrix}\gamma'\cdot\delta_1'\\ \vdots\\ \gamma'\cdot\delta_n'\end{pmatrix}(t).$$

One can easily show that the image of the striction curve coincides with the set of singular points of $F_{(\gamma,\delta)}$. Moreover, $p=(t,u_1,\dots,u_n)$ is a 1-Morin singularity if and only if the striction curve is an immersion at p ([22, Theorem 2.5] and [23, Theorem 4]). We give an alternative proof of this fact by using our criteria.

Proof. Let $F_{(\gamma,\delta)}$ be a one-parameter family of n-planes in $\boldsymbol{R}^{2n}$. We assume that for any t, $(\delta(t),\delta'(t))$ are linearly independent, $\delta_i\cdot\delta_j'=0$ $(i,j=1,\dots,n)$, and γ is a striction curve. Then $S(F_{(\gamma,\delta)})=\{u_1=\cdots=u_n=0\}$. By the definition of striction curve, there exist $\alpha_i(t)$ $(1\le i\le n)$ such that $\gamma'(t)=\sum_{i=1}^{n}\alpha_i(t)\delta_i(t)$ holds. Hence we see that the null vector field η can be taken as a function of t and $\eta(t)=-\partial t+\sum_{i=1}^{n}\alpha_i(t)\partial u_i$. Moreover, since $(t,u_1,\dots,u_n)$ and the coordinate system generated by $(\delta,\delta')(0)$ satisfies the condition (2.1), $\Lambda=(\lambda_1,\dots,\lambda_n)$ is

$$\lambda_j=\det\left(\gamma'+\sum_{i=1}^{n}u_i\delta_i',\delta,\delta_1',\dots,\widehat{\delta_j'},\dots,\delta_n'\right)=\det\left(\gamma'+u_j\delta_j',\delta,\delta_1',\dots,\widehat{\delta_j'},\dots,\delta_n'\right),$$

where $\delta=(\delta_1,\dots,\delta_n)$ and $(\delta_1',\dots,\widehat{\delta_j'},\dots,\delta_n')=(\delta_1',\dots,\delta_{j-1}',\delta_{j+1}',\dots,\delta_n')$. Then by Corollary 2.9, $F_{(\gamma,\delta)}$ at $p=(t,0,\dots,0)$ is a 1-Morin singularity if and only if $\eta\Lambda\neq 0$.

By a direct calculation,

$$\begin{aligned}
\eta\lambda_j(p) =& -\det\Big(\gamma' + u_j\delta_j', \delta, \delta_1', \dots, \widehat{\delta_j'}, \dots, \delta_n'\Big)'\Big|_{u_j=0}\\
&+\det\Big(\alpha_j\delta_j', \delta, \delta_1', \dots, \widehat{\delta_j'}, \dots, \delta_n'\Big)(t)\\
=& -\det\Big(\gamma'', \delta, \delta_1', \dots, \widehat{\delta_j'}, \dots, \delta_n'\Big)(t)\\
&-\det\Big(\gamma', \delta_1, \dots, \delta_j', \dots, \delta_n, \delta_1', \dots, \widehat{\delta_j'}, \dots, \delta_n'\Big)(t) + (-1)^{n+j-1}\alpha_j\Delta\\
=& -\det\Big(\alpha_j\delta_j', \delta, \delta_1', \dots, \widehat{\delta_j'}, \dots, \delta_n'\Big)(t)\\
&-\det\Big(\alpha_j\delta_j, \delta_1, \dots, \delta_j', \dots, \delta_n, \delta_1', \dots, \widehat{\delta_j'}, \dots, \delta_n'\Big)(t) + (-1)^{n+j-1}\alpha_j\Delta\\
=& (-1)^{n+j}\alpha_j\Delta + (-1)^{n+j-1}\alpha_j\Delta + (-1)^{n+j-1}\alpha_j\Delta\\
=& (-1)^{n+j-1}\alpha_j\Delta,
\end{aligned}$$

where $\Delta = \det(\delta, \delta')$. Hence $\eta\Lambda \neq 0$ is equivalent to $(\alpha_1, \dots, \alpha_n) \neq 0$, and it is equivalent to $\gamma' \neq 0$. □

§ 4. $\mathcal{A}$-isotopy of map-germs

We define an equivalence relation called $\mathcal{A}$-*isotopy*, which is a strengthened version of $\mathcal{A}$-equivalence. Let d be a natural number. A map-germ $f \in C^\infty(m,n)$ is said to be d-*determined* if any $g \in C^\infty(m,n)$ satisfying $j^d f(0) = j^d g(0)$ is $\mathcal{A}$-equivalent to f, where $j^d f(0)$ is the d-jet of f at 0. Let $\mathrm{Diff}^d(k)$ be the set of d-jets of diffeomorphism-germs $(\boldsymbol{R}^k, 0) \to (\boldsymbol{R}^k, 0)$ equipped with the relative topology as a subset $\mathrm{Diff}^d(k) \subset J^d(k,k)$, where $J^d(k,k)$ is canonically identified with a Euclidean space.

Definition 4.1. Let $f, g \in C^\infty(m,n)$ be $\mathcal{A}$-equivalent map-germs that are d-determined. Then f and g are $\mathcal{A}$-*isotopic* if there exist continuous curves $\sigma : [0,1] \to \mathrm{Diff}^d(m)$ and $\tau : [0,1] \to \mathrm{Diff}^d(n)$ such that $\sigma(0)$, $\tau(0)$ are both d-jets of the identity, and

$$j^d(g)(0) = j^d\big(\tau(1) \circ f \circ \sigma(1)\big)(0)$$

holds.

Namely, f and g are $\mathcal{A}$-isotopic if and only if $j^d f(0)$ and $j^d g(0)$ are located on the same arc-wise connected component of the d-jet of the $\mathcal{A}^d$-orbit of $j^d f(0)$. Since the set $\mathrm{Diff}^{d,+}(m)$ of d-jets of orientation-preserving diffeomorphism-germs is arc-wise connected, f and g are $\mathcal{A}$-isotopic if and only if there exist orientation preserving diffeomorphism-germs $\sigma^+ : (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^m, 0)$ and $\tau^+ : (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ such that $j^d g(0) = j^d(\tau^+ \circ f \circ \sigma^+)(0)$ holds. This notion of $\mathcal{A}$-isotopic is a slightly strengthened version of $\mathcal{A}$-equivalence. By the above arguments, there are at most four $\mathcal{A}$-isotopy classes in an $\mathcal{A}$-equivalent class. However, the number of $\mathcal{A}$-isotopy classes of an $\mathcal{A}$-equivalent class of a given map-germ f may represent a property of f. In this section, we study the number of $\mathcal{A}$-isotopy classes of each Morin singularity as an application of our criteria (Corollary 2.9). We remark that this type of problem is asked by Takashi Nishimura [16, p.226]. We also remark that homotopy types of the $\mathcal{A}^2$-orbit of the fold are considered by Yoshifumi Ando [2, 3], thus we are mainly interested in the case $r \geq 2$.

It is easy to see that any corank 1 germ is $\mathcal{A}$-isotopic to the form (2.10). Furthermore, since we only used the diffeomorphisms (2.16) and (2.17) to obtain the normal form (1.1) from (2.10), any r-Morin singularity is $\mathcal{A}$-isotopic to

(4.1)
$$h_{r,(\varepsilon_1,\varepsilon_2)}(x) = \left(\varepsilon_1 x_1, x_2, \ldots, x_{m-1}, \varepsilon_1 x_1 x_m + \sum_{j=2}^{r} x_j x_m^j, h_2(x), \ldots, h_{n-m}(x), \varepsilon_2 h_{n-m+1}(x)\right),$$

where $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$, and $h_2, \ldots, h_{n-m+1}$ are as in (1.1). We remark that the final linear translations are orientation-preserving. We have the following.

Proposition 4.2. (I) *If r is even, then $h_{r,(\varepsilon_1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to $h_{r,(\varepsilon_1,1)}$. Moreover, if $m > r(m-n+1)$ holds, then $h_{r,(\varepsilon_1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to $h_{0,r}$.* (II) *If r is odd, then $h_{r,(\varepsilon_1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to $h_{r,(1,\varepsilon_2)}$. Moreover, if $m > r(m-n+1)$ holds, then $h_{r,(\varepsilon_1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to $h_{0,r}$.*

The proof of this proposition is not difficult, but rather long. We postpone it to Section 5. By Proposition 4.2, the $\mathcal{A}$-isotopic condition for r-Morin singularities of suspensions ($m > r(n-m+1)$) is the same as $\mathcal{A}$-equivalence, so we stick to the non-suspension case ($m = r(n-m+1)$). In this case, by Corollary 2.9, a necessary condition that f is $\mathcal{A}$-equivalent to an r-Morin singularity is

$$\det d(\Lambda, \Lambda', \ldots, \Lambda^{(r-1)})(0) \neq 0. \tag{4.2}$$

Set $D = \operatorname{sgn} \det d(\Lambda, \Lambda', \ldots, \Lambda^{(r-1)})(0)$, and $a = n - m$. Calculating D for (4.1), we obtain $D = \varepsilon_1^{(a+1)r+1}\varepsilon_2^r$. Furthermore, the sign of D may depend on the choice of oriented frame $\{\xi_1, \ldots, \xi_{m-1}, \eta\}$, and an orientation-preserving diffeomorphism on the target. Let $\{\tilde{\xi}_1, \ldots, \tilde{\xi}_{m-1}, \tilde{\eta}\}$ be another frame, and let $\tilde{D}$ stand for the sign of (4.2) with respect to this frame. Then $\tilde{\eta}(0) = \alpha\eta(0)$ holds. If $\alpha > 0$ then $\tilde{D} = D$, and if $\alpha < 0$, then $\tilde{D} = (-1)^{(r-1)r(a+1)/2}D$ holds. On the other hand, let $\Phi = (\Phi_1, \ldots, \Phi_n)$ be an orientation-preserving diffeomorphism on the target, and let $\bar{D}$ stand for the sign of (4.2) of $\Phi \circ f$. By (2.4), if $(\Phi_1, \ldots, \Phi_{m-1})|_{\{x_m = \cdots = x_n = 0\}}$ is orientation-preserving, then $\bar{D} = D$, and if $(\Phi_1, \ldots, \Phi_{m-1})|_{\{x_m = \cdots = x_n = 0\}}$ is orientation-reversing, then $\bar{D} = (-1)^{ar}D$ holds. We divide r into four cases via modulo four. Let l be an integer.

Case 1: $r = 4l$ In this case, $h_{r,(\varepsilon_1,\varepsilon_2)}$ and $h_{r,(\varepsilon'_1,\varepsilon'_2)}$ are $\mathcal{A}$-isotopic if and only if $\varepsilon_1 = \varepsilon'_1$.

Proof. By Proposition 4.2, ε_2 may be deleted. By the above arguments, $D = \varepsilon_1$ is an invariant of the $\mathcal{A}$-isotopic condition. □

Case 2: $r = 4l + 1$ In this case, if a is even, then $h_{r,(\varepsilon_1,\varepsilon_2)}$ and $h_{r,(\varepsilon'_1,\varepsilon'_2)}$ are $\mathcal{A}$-isotopic if and only if $\varepsilon_2 = \varepsilon'_2$. If a is odd, then $h_{r,(\varepsilon_1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to $h_{0,r}$.

Proof. By Proposition 4.2, ε_1 may be deleted. By the above arguments again, $D = \varepsilon_2$ is an invariant of the $\mathcal{A}$-isotopic condition, and we have the first conclusion. For a proof of the second conclusion, see Section 5. □

In particular, the $\mathcal{A}$-class and the $\mathcal{A}$-isotopy class coincides for the Whitney umbrella $(m = 2, n = 3, r = 1)$.

Case 3: $r = 4l + 2$ In this case, if a is odd, then $h_{r,(\varepsilon_1,\varepsilon_2)}$ and $h_{r,(\varepsilon'_1,\varepsilon'_2)}$ are $\mathcal{A}$-isotopic if and only if $\varepsilon_1 = \varepsilon'_1$. If a is even, then $h_{r,(\varepsilon_1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to $h_{0,r}$.

Proof. By Proposition 4.2, ε_2 may be deleted. By the above arguments again, $D = \varepsilon_1$ is an invariant of the $\mathcal{A}$-isotopic condition, and we have the first conclusion. For a proof of the second conclusion, see Section 5. □

Case 4: $r = 4l + 3$ In this case, $h_{r,(\varepsilon_1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to $h_{0,r}$. See Section 5 for a proof.

Summarizing up the above arguments, we can summarize the number of $\mathcal{A}$-isotopy classes of $\mathcal{A}$-classes for each Morin singularity. We summarize it in the following table.

	$m = r(n-m+1)$		$m > r(n-m+1)$
	a : odd (invariant)	a : even (invariant)	
$r = 4l$	2 (ε_1)	2 (ε_1)	1
$r = 4l+1$	1	2 (ε_2)	1
$r = 4l+2$	2 (ε_1)	1	1
$r = 4l+3$	1	1	1

Table 1. Number of $\mathcal{A}$-isotopy classes in the $\mathcal{A}$-classes.

§ 5. Proofs

Here, we use the following terminology: Let I be a set of indices such that $\#I$ is even. Then the *π-rotations of I* are diffeomorphisms $(x_1, \dots, x_k) \mapsto (\tilde{x}_1, \dots, \tilde{x}_k)$, where $\tilde{x}_j = \varepsilon x_j$ if $j \in I$, and $\tilde{x}_j = x_j$ if $j \notin I$, with $\varepsilon = -1$. We see that applying π-rotations both on the source and the target does not change the $\mathcal{A}$-isotopy class.

Proof of Proposition 4.2. First we show (I). Set $\varepsilon_2 = -1$. By a π-rotation of $\{m, n\}$ on the target, $h_{r,(\varepsilon_1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to

$$\left(\varepsilon_1 x_1, \dots, x_{m-1}, \varepsilon_2\left(\varepsilon_1 x_1 x_m + \sum_{j=2}^{r} x_j x_m^j\right), h_2(x), \dots, h_{n-m}(x), h_{n-m+1}(x)\right). \tag{5.1}$$

Considering π-rotations of $\{1, \dots, r\}$ on the source, (5.1) is $\mathcal{A}$-isotopic to

$$\left(\varepsilon_1\varepsilon_2 x_1, \varepsilon_2 x_2, \dots, \varepsilon_2 x_r, x_{r+1}, \dots, x_{m-1}, \varepsilon_1 x_1 x_m + \sum_{j=2}^{r} x_j x_m^j, h_2(x), \dots, h_{n-m+1}(x)\right). \tag{5.2}$$

Considering π-rotations of $\{1, \dots, r\}$ on the target, (5.2) is $\mathcal{A}$-isotopic to

$$\left(\varepsilon_1 x_1, \dots, x_{m-1}, \varepsilon_1 x_1 x_m + \sum_{j=2}^{r} x_j x_m^j, h_2(x), \dots, h_{n-m+1}(x)\right), \tag{5.3}$$

which proves the first part of (I). We assume $m > r(m-n+1)$ and set $\varepsilon_1 = -1$. Then x_{m-1} is not contained in any terms of $h_1, \dots, h_{n-m+1}$. Considering π-rotations of $\{2, \dots, r, m-1\}$ on the source, (5.3) is $\mathcal{A}$-isotopic to

$$\big(\varepsilon_1 x_1, \dots, \varepsilon_1 x_r, x_{r+1}, \dots, x_{m-2}, \varepsilon_1 x_{m-1}, \varepsilon_1 h_1(x), h_2(x), \dots\big), \tag{5.4}$$

where h_1 is as in (1.1). Considering π-rotations of $\{1, \dots, r, m-1, m\}$ on the target, (5.4) is $\mathcal{A}$-isotopic to $h_{0,r}$, which proves the second part of (I).

Secondly, we show (II). Set $\varepsilon_1 = -1$. Considering π-rotations of $\{2, \dots, r\}$ on the source, (5.1) is $\mathcal{A}$-isotopic to

$$\big(\varepsilon_1 x_1, \dots, \varepsilon_1 x_r, x_{r+1}, \dots, x_{m-1}, \varepsilon_1 h_1(x), h_2(x), \dots, h_{n-m}(x), \varepsilon_2 h_{n-m+1}(x)\big). \tag{5.5}$$

Then by π-rotations on the target, we see that (5.5) is $\mathcal{A}$-isotopic to $h_{r,(1,\varepsilon_2)}$, which proves the first part of (II). We assume $m > r(m-n+1)$ and set $\varepsilon_2 = -1$. Then by π-rotations on the target, $h_{r,(1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to

$$\big(x_1, \dots, x_{m-1}, \varepsilon_2 h_1(x), h_2(x), \dots, h_{n-m+1}(x)\big). \tag{5.6}$$

Considering π-rotations of $\{1, \dots, r, m-1\}$ of the source, (5.6) is $\mathcal{A}$-isotopic to

$$\big(\varepsilon_2 x_1, \dots, \varepsilon_2 x_r, x_{r+1}, \dots, x_{m-2}, \varepsilon_2 x_{m-1}, h_1(x), h_2(x), \dots, h_{n-m+1}(x)\big). \tag{5.7}$$

Then by π-rotations on the target, we see that (5.7) is $\mathcal{A}$-isotopic to $h_{0,r}$, which proves the second part of (II). $\square$

Proof of the claim of the second part of Case 2. Let us assume $r = 4l + 1$ and a is odd. By Proposition 4.2, $h_{r,(\varepsilon_1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to $h_{r,(1,\varepsilon_2)}$. We show $h_{r,(1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to $h_{0,r}$. Set $\varepsilon_2 = -1$. Considering π-rotations of

$$\Big\{ \underbrace{\underbrace{1, 3, \dots, r}_{\text{odd}},\ \underbrace{r+1, r+3, \dots, 2r}_{\text{odd}},\ \dots,\ \underbrace{(a-1)r+1, (a-1)r+3, \dots, ar}_{\text{odd}}}_{\text{odd}}, \\ \underbrace{ar+1, ar+3, \dots, ar+r-2}_{\text{even}},\ m \Big\}$$

on the source, we see that $h_{r,(1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to

$$\big(\varepsilon_2 x_1, x_2, \varepsilon_2 x_3, x_4, \dots, \varepsilon_2 x_{m-2}, x_{m-1}, h_1(x), \dots, h_{n-m}(x), \varepsilon_2 h_{n-m+1}(x)\big),$$

noticing $(a+1)r = m$. By π-rotations on the target, we have the result. $\square$

Proof of the claim of the second part of Case 3. Let us assume $r = 4l + 2$ and a is even. By Proposition 4.2, $h_{r,(\varepsilon_1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to $h_{r,(\varepsilon_1,1)}$. We show $h_{r,(\varepsilon_1,1)}$ is $\mathcal{A}$-isotopic to $h_{0,r}$. Set $\varepsilon_1 = -1$. Considering π-rotations of

$$\Big\{ 1, \underbrace{2, 4, \dots, r}_{\text{odd}}, \\ \underbrace{\underbrace{r+1, r+3, \dots, 2r-1}_{\text{odd}},\ \dots,\ \underbrace{r(a-1)+1, r(a-1)+3, \dots, r(a-1)+r-1}_{\text{odd}}}_{\text{odd}}, \\ \underbrace{ar+2, ar+4, \dots, ar+r-2}_{\text{even}}, m \Big\}$$

we see that $h_{r,(\varepsilon_1,1)}$ is $\mathcal{A}$-isotopic to

$$\big(x_1, \varepsilon_1 x_2, x_3, \varepsilon_1 x_4, \dots, \varepsilon_1 x_{m-2}, x_{m-1}, \varepsilon_1 h_1(x), h_2(x), \dots, h_{n-m}(x), \varepsilon_1 h_{n-m+1}(x)\big).$$

By π-rotations on the target, we have the result. $\square$

Proof of the claim of Case 4. Let us assume $r = 4l + 3$. By Proposition 4.2, $h_{r,(\varepsilon_1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to $h_{r,(1,\varepsilon_2)}$. We show $h_{r,(1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to $h_{0,r}$. Set $\varepsilon_2 = -1$. Considering π-rotations of

$$\Big\{ \underbrace{1, 3, \dots, r}_{\text{even}},\ \dots,\ \underbrace{a(r-1)+1, a(r-1)+3, \dots, a(r-1)+r}_{\text{even}}, \\ \underbrace{ar+1, ar+3, \dots, ar+r-2}_{\text{odd}},\ m \Big\},$$

we see that $h_{r,(1,\varepsilon_2)}$ is $\mathcal{A}$-isotopic to

$$(\varepsilon_2 x_1, x_2, \varepsilon_2 x_3, \ldots, \varepsilon_2 x_{m-2}, x_{m-1}, h_1(x), \ldots, \varepsilon_2 h_{n-m+1}(x)).$$

By π-rotations on the target, we have the result. $\square$

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