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# Singular controls for port-Hamiltonian systems

By

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**Abstract**

The port-Hamiltonian system is a generalized Hamiltonian system and is regarded as an especial input-output control system. We see the port-Hamiltonian systems from the viewpoint of geometric control theory. Controllability and observability are basic concepts in control theory and important for system design. Singular controls play an important role in a control system, especially in the sense of optimal control problem. We see the following properties are equivalent for linear port-Hamiltonian systems: controllability, observability and nonexistence of singular control.

## § 1. Introduction

Let $(M, \omega)$ be a symplectic manifold, that is, $\omega$ is a nondegenerate closed differential two-form on $M$. We may define a Hamiltonian vector field $X_f$ associated to a function $f : M \to \mathbb{R}$ by the equation $\omega(X_f, \cdot) = -df$.

The port-Hamiltonian system with function $f$ on a symplectic manifold $M$ is given by a family of differential equations with a parameter $u \in \mathbb{R}^r$ and functions $y_1, \cdots, y_r$;

$$(1.1) \qquad \begin{cases} \dot{x} = X_f(x) + \displaystyle\sum_{i=1}^{r} u_i g_i(x), \\ y_j = g_j(x) f, \quad (1 \leq j \leq r) \end{cases}$$

where $x$ is a point of $M$ and $g_1, \cdots, g_r$ are vector fields on $M$. We call the parameter $u$ and the functions $y_j$ the control parameter and outputs respectively. The origin of this notion is the bond graph which is a method of design in engineering science [6].

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The concept of the port-Hamiltonian system is regarded as a generalized Hamiltonian system. The meaning of "generalized" is divided into two meanings: the one is that the underlying structure which defining Hamiltonian vector fields is generalized, the other is that differential equations defined by using Hamiltonian vector fields are generalized.

On the former, we may give a definition of Hamiltonian vector fields by using just a bi-derivation. The way to define a Hamiltonian vector field is the same as the way by using a Poisson tensor, formally (See Definition 3.1). We remark that the formulation of a Hamiltonian system is given by a Poisson tensor usually. We may consider, moreover, the notion of Dirac structures which is a generalization of both symplectic structures and Poisson structures. The port-Hamiltonian system can be treated as an implicit Hamiltonian system on Dirac structures [9]. A Dirac structure on a manifold $M$ is a subbundle $L$ of the bundle $TM \oplus T^*M$ which is maximally isotropic under the pairing $\langle , \rangle_+$, where $\langle (X, \eta), (Y, \mu) \rangle_+ = \frac{1}{2}(\eta(Y) + \mu(X))$. If $L$ satisfies a certain integrable condition, we call $L$ an integrable Dirac structure. For example, integrable Dirac structure induced by a symplectic structure $\omega$ is a graph of an isotropic map $\omega^\sharp : TM \to T^*M, X \mapsto \iota_X\omega$, where $\iota_X\omega$ is interior product; for more details, see [3]. In this paper, we concentrate on the port-Hamiltonian systems which are defined on pseudo-Poisson manifolds (see Definition3.1).

On the latter, we may find out from the first equation of (1.1) that the Hamiltonian vector field $X_f$ is affected by a curve $u(t)$, which is called a control, along the frame fields $g_1, \cdots, g_r$. By considering the second equation of (1.1) as the result of the effect of a control $u(t)$ on the system, the port-Hamiltonian system is regarded as one of input-output systems (see Definition 2.8).

*Remark.* If the vector fields $g_1, \cdots, g_r$ are equal to zero, then the system (1.1) is just a Hamiltonian system since each outputs $y_j$ will be zero.

In control theory, the concept of controllability and observability are basic and important. The system (1.1) is said to be controllable if a point on $M$ reach to any point along a curve $x(t)$ defined by the first equation of (1.1) with a suitable control $u(t)$. The system (1.1) is said to be observable if we can recognize the solution $x(t)$ determined by a control $u(t)$ and an initial condition from the outputs $y_j$ and the control $u(t)$. For exact definitions, see Definitions 2.5 and Definition 2.9.

For a control system, we may consider singular controls which are defined by singular points of an endpoint map (see Definition 2.3). Singular controls are interesting and important objects, particularly in the optimal control problem. From the standpoint of singularity theory, we consider a characterization of port-Hamiltonian systems by singular controls.

In Section 2, we obtain Proposition 2.7 which connects controllability to existence of singular controls on a linear input-output system. In Section 3, we obtain Theorem 3.4 which relating controllability, observability and existence of singular controls each other on a linear port-Hamiltonian system. This is very simple observation, but interesting because it relates the three important concepts together. In Section 4, we see some physical examples of linear-port-Hamiltonian systems and a nonlinear port-Hamiltonian system.

## § 2. Control systems and singular controls

Let $M$ be an $n$-dimensional smooth manifold, $\{E, M, \alpha\}$ a fibre bundle with $r$-dimensional fibre on $M$, $\{TM, M, \pi_{TM}\}$ a tangent bundle on $M$.

**Definition 2.1.** A **control system** on $M$ is a pair $(\alpha, F)$ of the fibration $\alpha : E \to M$ and a smooth map $F : E \to TM$ such that $\alpha = \pi_{TM} \circ F$.

**Definition 2.2.** For $T > 0$, we say a map $c : [0, T] \to E$ is an **admissible control** if the $c$ is an $L^\infty$ measurable bounded map such that $x(t) := \alpha \circ c(t)$ satisfies Lipschitz condition and the differential equation $\dot{x}(t) = F(c(t))$.

We denote by $\mathcal{U}_{x_0,T}$ the set of admissible controls for which the initial value of the differential equation $\dot{x}(t) = F(c(t))$ is $x_0$. The $\mathcal{U}_{x_0,T}$ is an open submanifold of Banach manifold $L^\infty([0, T], E)$.

**Definition 2.3.** We define the **endpoint map** $End_{x_0} : \mathcal{U}_{x_0,T} \to M$ which assigns an admissible control $c$ to the endpoint of the solution $x(T)$. A singular point of the endpoint map is called a **singular control**, namely a singular control $c(t)$ is a point on which the differential map $d_c(End_{x_0}) : T_c(\mathcal{U}_{x_0,T}) \to T_{x(T)}M$ is not surjective.

When the local trivialization $\alpha^{-1}(V) \cong V \times U$ (where $V \subset M$) of the fibre $\{E, M, \alpha\}$ is given, the map $F : E \to TM$ is parametrized by $(x, u) \in V \times U$ as $F(x, u) = f_u(x)$. We may regard $f_u(x)$ as a family of vector fields with the parameter $u$ which is called a **control parameter**. The control system is called an **affine control system** if a parametrization $f_u(x)$ of the map $F : E \to TM$ is written by $f_u(x) = g_0(x) + \sum_{i=1}^{r} u_i g_i(x)$ with vector fields $g_0, \cdots g_r$ on $M$, namely the control system is expressed by

$$\dot{x} = g_0(x) + \sum_{i=1}^{r} u_i g_i(x).$$

The vector fields $g_1, \cdots, g_r$ are called **input vector fields**.

Define a function $H : T^*V \times U \to \mathbb{R}$ for a parametrized control system by $H(x, p, u) = \langle p, g_0(x) + \sum_{i=1}^{r} u_i g_i(x) \rangle$, where $\langle , \rangle$ is a natural pairing of a covector and a vector. Singular controls of a parametrized control system are characterized by the constrained Hamiltonian system as follows.

**Proposition 2.4** ([1]). *An admissible control $c(t) = (x(t), u(t))$ is a singular control if and only if there exists $p(t)$ such that the curve $(x(t), p(t), u(t))$ on $T^*V\backslash\{0\}\times U$ satisfies followings*

$$\begin{cases} \dot{x}(t) = \dfrac{\partial H}{\partial p}(x(t), p(t), u(t)), \\ \dot{p}(t) = -\dfrac{\partial H}{\partial x}(x(t), p(t), u(t)), \\ \dfrac{\partial H}{\partial u_i}(x(t), p(t), u(t)) = \langle p, g_i(x)\rangle = 0 \ (1 \leq i \leq r). \end{cases}$$

We find that singular controls are related to the optimal control problem because Proposition 2.4 comes from Pontryagin maximum principle. In detail see [1].

We now introduce the important notions in control theory; controllability and observability.

**Definition 2.5.** A control system is **controllable** if for any points $x_0$ and $x_1$ of $M$, there exists $T > 0$ and an admissible control $c \in \mathcal{U}_{x_0,T}$ such that $End_{x_0}(c) = x_1$.

We say a control system $(\alpha, F)$ is linear if $M = \mathbb{R}^n$ and a parametrization $f_u(x)$ of $F$ is written by $f_u(x) = Ax + Bu$ with matrices $A \in M_{n\times n}(\mathbb{R})$ and $B \in M_{n\times r}(\mathbb{R})$. Let $(A, B)$ denotes this linear control system $\dot{x} = Ax + Bu$. There is a well known condition which is equivalent to controllability on linear control systems. We define an $n \times nr$ matrix $(B \mid AB \mid \cdots \mid A^{n-1}B)$ called a **controllability matrix** for linear control system $(A, B)$ to explain the condition.

**Proposition 2.6** ([7]). *On a linear control system $(A, B)$, the following statements are equivalent.*

- *The linear system $(A, B)$ is controllable.*
- *The controllability matrix $(B \mid AB \mid \cdots \mid A^{n-1}B)$ has rank $n$.*

We shall state the following proposition which relates the controllability to existence of singular controls on linear control systems.

**Proposition 2.7.** *The following statements are equivalent on a linear control system $(A, B)$.*

*i). The linear system $(A, B)$ is controllable.*

*ii). There are no singular controls for the linear system $(A, B)$.*

*Proof.* *i)* $\Longrightarrow$ *ii)* : Assume that there exists a singular control $u^*(t)$. We have the constrained Hamiltonian system on $T^*M$ for the control $u^*(t)$ and the Hamiltonian $\hat{H}(x,p,u)$ defined by

$$\hat{H}(x,p,u) = \langle p, Ax + Bu \rangle.$$

By Proposition 2.4, the constrained Hamiltonian system

$$\begin{cases} \dot{x} = \dfrac{\partial \hat{H}}{\partial p} = Ax + Bu^*, \\ \dot{p} = -\dfrac{\partial \hat{H}}{\partial x} = {}^t\!Ap, \\ {}^t\!Bp = 0, \end{cases}$$

has a solution with $p \neq 0$. Since $B$ is a constant matrix, it follows from the constraint ${}^t\!Bp = 0$ that the equalities

$${}^t\!B\dot{p} = 0, {}^t\!B\ddot{p} = 0, {}^t\!Bp^{(3)} = 0, \cdots, {}^t\!Bp^{(n-1)} = 0,$$

where $p^{(k)} := \dfrac{d^k p}{dt^k}$. Furthermore, we get

$$p^{(k)} = {}^t\!A^k p$$

from the second equation of the Hamiltonian system. Then for the controllability matrix we have

$${}^t\!p \left( B \mid AB \mid \cdots \mid A^{n-1}B \right) = \begin{pmatrix} {}^t\!B \\ {}^t\!B{}^t\!A \\ {}^t\!B{}^t\!A^2 \\ \vdots \\ {}^t\!B{}^t\!A^{n-1} \end{pmatrix} p = \begin{pmatrix} {}^t\!Bp \\ {}^t\!B\dot{p} \\ {}^t\!B\ddot{p} \\ \vdots \\ {}^t\!Bp^{(n-1)} \end{pmatrix} = 0.$$

Thus the condition $p \neq 0$ implies that

$$\operatorname{rank} \left( B \mid AB \mid \cdots \mid A^{n-1}B \right) < n.$$

By virtue of Proposition 2.6, the system is not controllable.
*ii)* $\Longrightarrow$ *i)* : Conversely, assume

$$\operatorname{rank} \left( B \mid AB \mid \cdots \mid A^{n-1}B \right) < n.$$

Then there exists $p_0 \neq 0$ such that

$$\begin{pmatrix} {}^t\!B \\ {}^t\!B{}^t\!A \\ {}^t\!B{}^t\!A^2 \\ \vdots \\ {}^t\!B{}^t\!A^{n-1} \end{pmatrix} p_0 = 0.$$

Put $p(t) = \exp(t\,{}^tA)p_0$ and let us now see $p(t)$ satisfies the constrained Hamiltonian system. We see the second equation holds:

$$\dot{p} = {}^tA \exp(t\,{}^tA)p_0 = {}^tAp(t).$$

By using Cayley-Hamilton theorem and by the assumption, we have

$${}^tB{}^tA^n p_0 = {}^tB \sum_{i=0}^{n-1} \alpha_i {}^tA^i p_0 = 0.$$

As a consequence of the induction relative to $n$ implies

$$\begin{aligned} {}^tBp(t) &= {}^tB\left(E + t{}^tA + \frac{t^2}{2}{}^tA^2 + \cdots\right)p_0 \\ &= 0. \end{aligned}$$

□

We remark that for nonlinear systems, Proposition 2.7 does not hold in general. For example, if a control system is given by a contact structure then the control system is controllable but does not have any singular controls [1].

We introduce input-output systems in order to define the observability. Recall that a control system $(\alpha, F)$ is given by a fibration $\alpha : E \to M$ and a smooth map $F : E \to TM$.

**Definition 2.8.** An **input-output system** on $M$ is a triple $(\alpha, F, h)$ of a control system $(\alpha, F)$ and a function $h : E \to \mathbb{R}^r$ called an **output**. An input-output system is expressed by

$$\begin{cases} \dot{x} = f_u(x), \\ y = h(x, u), \end{cases}$$

with a local trivialization.

Consider the case that an input-output system is trivialized, that is, admissible controls $c$ are parametrized; $c(t) = (x(t), u(t))$. Let $x(u, x_i, t)$ denotes the solution of the differential equation $\dot{x} = f_u(x)$ with initial values $x_i$ and the control $u$.

**Definition 2.9.** Two points $x_1$ and $x_2$ of $M$ are said to be **indistinguishable**, which is denoted by $x_1 I x_2$, if for arbitrary $T > 0$ and a control $u : [0, T] \to U$, the outputs $h(x(u, x_1, t), u)$ and $h(x(u, x_2, t), u)$ along the solutions $x(u, x_1, t)$ and $x(u, x_2, t)$ respectively coincide together on the closed interval $[0, T]$. We say an input-output system is **observable** if $x_1 I x_2$ implies $x_1 = x_2$.

We say an input-output system $(\alpha, F, h)$ is linear if the control system $(\alpha, F)$ is linear and $h : E \to \mathbb{R}^r$ is given by linear map $h(x,u) = Cx + Du$ with a parametrization $(x,u)$, where $C \in M_{r\times n}(\mathbb{R})$ and $D \in M_{r\times r}(\mathbb{R})$. That is, the linear input-output system is given by

$$\begin{cases} \dot{x} = Ax + Bu, \\ h(x,u) = Cx + Du. \end{cases}$$

Let $(A, B, C, D)$ denotes the linear input-output system above.

Observability on a linear input-output system is characterized by the following proposition.

**Proposition 2.10** ([7]). *The following statements are equivalent on linear-input output systems.*

- *A linear input-output system $(A, B, C, D)$ is observable.*
- *The $rn \times n$ matrix* $\begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$ *has rank $n$.*

The matrix appeared above is called the **observability matrix**. We remark that there is the duality on Proposition 2.6 and Proposition 2.10 at least for linear system.

## § 3. Port-Hamiltonian systems

We introduce pseudo-Poisson structures which port-Hamiltonian systems are defined on.

*Remark.* More generally, port-Hamiltonian systems may be treated as implicit Hamiltonian system defined on Dirac manifolds. In details, see [9].

**Definition 3.1.** A **pseudo-Poisson structure** or a **pseudo-Poisson tensor** on a smooth manifold $M$ is a bivector field $\Pi \in \wedge^2(TM)$ where $\wedge^2(TM)$ denotes the set of alternative $(2,0)$-tensor fields on $M$. We call $(M, \Pi)$ a **pseudo-Poisson manifold**.

A bi-vector field $\Pi$ on $M$ induces a pseudo-Poisson bracket $\{,\}_\Pi : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ by

$$\{f, g\}_\Pi := \langle \Pi, df \wedge dg \rangle = \sum_{i,j} \Pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j},$$

where $(x_1, \cdots, x_n)$ is a local coordinate on $M$ and $\Pi_{ij}$ is the local expression of $\Pi$. The bracket is said to be a **Poisson bracket**, if the bracket satisfies the Jacobian identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$$

*Remark.* If the local expression of the pseudo-Poisson tensor $\Pi_{ij}$ is a constant matrix, then the bracket induced by the pseudo-Poisson tensor $\{,\}_\Pi$ satisfies the Jacobian identity, that is, the bracket will be a Poisson bracket.

**Definition 3.2.** Let $(M,\Pi)$ be a pseudo-Poisson manifold. A **Hamiltonian vector field** $X_f$ on $M$ associated to $f : M \to \mathbb{R}$ is defined by the equation

$$X_f g = \{f,g\}_\Pi.$$

We consider an input-output affine control system which its first term is a Hamiltonian vector field.

**Definition 3.3.** A **port-Hamiltonian system** on a pseudo-Poisson manifold $(M,\Pi)$ is defined by an input-output system

$$\begin{cases} \dot{x} = X_f(x) + \displaystyle\sum_{i=1}^{r} u_i g_i(x), \\ y_j = g_j(x) f, \quad (j = 1, \cdots, r) \end{cases}$$

with vector fields $g_1, \cdots, g_r$ and a $C^\infty$ function $f$.

*Remark.*

Consider the case that the port-Hamiltonian system is a linear input-output system. That is, a local expression of the pseudo-Poisson structure $\Pi_{ij}$ is a constant matrix $J$, the function $f$ is quadratic form given by ${}^t x F x$ with an $n \times n$ matrix $F$ and input vector fields $g_i$ are constant vectors. In this way the linear port-Hamiltonian system has a representation of an input-output system $(JF, G, {}^t GF, O)$ with $G = (g_1, \cdots, g_r)$, namely the port-Hamiltonian system is expressed by

$$\begin{cases} \dot{x} = JFx + Gu, \\ y = {}^t GFx. \end{cases}$$

Thus the controllability matrix and observability matrix which appeared in Proposition 2.6 and Proposition 2.10 are now

$$\left(B \mid AB \mid \cdots \mid A^{n-1}B\right) = \left(G \mid JFG \mid \cdots \mid (JF)^{n-1}G\right), \quad \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} = \begin{pmatrix} {}^t GF \\ {}^t GFJF \\ \vdots \\ {}^t GF(JF)^{n-1} \end{pmatrix},$$

for linear port-Hamiltonian system, respectively. Hence we have the relation between the observability matrix and controllability matrix;

$$\begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix} = \begin{pmatrix} {}^tGF \\ ({}^tGF)JF \\ \vdots \\ ({}^tGF)(JF)^{n-1} \end{pmatrix} = \left({}^tFG \mid {}^t(JF){}^tFG \mid \cdots \mid {}^t(JF)^{n-1}{}^tFG\right)$$

$$= F\left(G \mid -JFG \mid \cdots \mid (-1)^{n-1}(JF)^{n-1}G\right)$$
$$= F\left(B \mid -AB \mid \cdots \mid (-1)^{n-1}A^{n-1}B\right).$$

Thus we have the following theorem from Proposition 2.7.

**Theorem 3.4.** *Consider a linear port-Hamiltonian system* $\Sigma_{lp} = (JF, G, {}^tGF, O)$. *If* $\det F \neq 0$, *then the following statements are equivalent:*

*i).* $\Sigma_{lp}$ *is not controllable.*

*ii).* $\Sigma_{lp}$ *is not observable.*

*iii).* $\Sigma_{lp}$ *has a singular control.*

The interesting point of Theorem 3.4 is the relation between the observability and the existence of singular controls. This motivates us to study relations between observability and singular controls on a nonlinear port-Hamiltonian system.

## § 4. Examples

We shall give several examples related to Theorem 3.4.

**Example 4.1** ([5][10])**.** Consider the LC-circuit given in the Figure 1 and we shall give an its port-Hamiltonian expression.

For the components in an electrical circuit, the following basic laws hold: the magnetic flux $\varphi_L$ and the current $I_L$ across a capacitor with inductance $L$ and capacitance $C$ is related via

$$\varphi_L(t) = L\frac{I_L}{C}$$

whereas the electric charge $Q_C$ and the voltage $V_C$ of the capacitor with capacitance $C$ are related via

$$L_C(t) = C\frac{dV_C}{dt}(t).$$

The conservation of charge and energy in electrical circuits are described by Kirchhoff's circuit laws. Kirchhoff's first law states that at any node in an electrical circuit, the

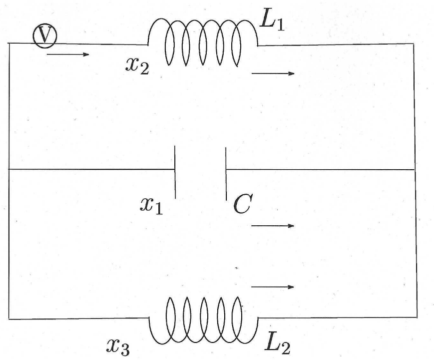


Figure 1. LC-circuit

sum of currents flowing into the node is equal to the sum of currents flowing out of the node. Moreover, Kirchhoff's second law says that the directed sum of the electrical potential differences around any closed circuit must be zero.
Applying these laws to the example, we obtain the following differential equations:

$$\begin{aligned}
\frac{dQ}{dt}(t) &= I_C \overset{\text{Kirchhoff's 1st}}{=} I_{L_1}(t) - I_{L_2}(t) = \frac{\varphi_{L_1}}{L_1} - \frac{\varphi_{L_2}}{L_2}, \\
\frac{d\varphi_{L_1}}{dt}(t) &= -V_{L_1}(t) \overset{\text{Kirchhoff's 2nd}}{=} -V_C(t) + V(t) = -\frac{Q}{C}(t) + V(t), \\
\frac{d\varphi_{L_2}}{dt}(t) &= -V_{L_2}(t) \overset{\text{Kirchhoff's 2nd}}{=} \frac{Q}{C}(t).
\end{aligned}$$

We assume that we can only measure the current $I_{L_1}$, that is, we define the output as

$$y(t) = I_{L_1}(t).$$

We may regard the system as a port-Hamiltonian system in conformity with the preceding. Replace the variables $(Q, \varphi_{L_1}, \varphi_{L_2})$ to $(x_1, x_2, x_3)$ as a coordinate of the Poisson manifold $(M, \pi)$ with structure represented by

$$\pi = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Take the function $f$ on $M$ as the energy function

$$f(x_1, x_2, x_3) = \frac{1}{2C}x_1^2 + \frac{1}{2L_1}x_2^2 + \frac{1}{2L_2}x_3^2$$

of the system and we get the port-Hamiltonian expression of the above system:

$$\begin{cases} \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \frac{\partial f}{\partial x_2}(x) \\ \frac{\partial f}{\partial x_3}(x) \end{pmatrix} + V \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ y = g(x)f = \dfrac{\partial f}{\partial x_2}(x) = I_{L_1}, \end{cases}$$

where the input vector field $g$ is ${}^t(0,1,0)$. The control parameter $V$ is voltage source here.

Since this is a linear port-Hamiltonian system, we can apply Theorem 3.4 to the system. Since the controllability matrix is written by

$$(B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B) = \left( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mid \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \mid \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \mid 2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mid 2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \mid \cdots \right),$$

the controllability matrix has full rank and the system is controllable. Hence from Theorem 3.4, the system is observable and has no singular controls.

The following example is very alike but contrasting.

**Example 4.2.** Consider the LC-circuit given in the Figure 2 which is very alike

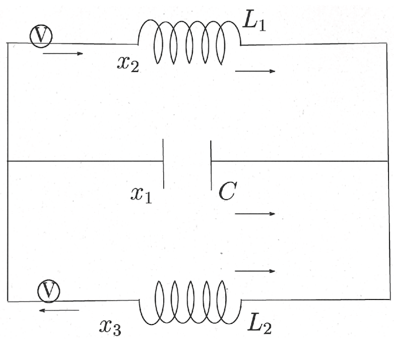


Figure 2. LC-circuit 2

to Example 4.1. The same way as Example 4.1, we get the following port-Hamiltonian expression;

$$\begin{cases} \begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \frac{\partial f}{\partial x_2}(x) \\ \frac{\partial f}{\partial x_3}(x) \end{pmatrix} + V \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \\ y = g(x)f = \dfrac{\partial f}{\partial x_2}(x) = I_{L_1} + I_{L_2}, \end{cases}$$

where the input vector field $g$ is ${}^t(0,1,1)$. Since the controllability matrix of the system is

$$(B \mid AB \mid A^2B \mid \cdots \mid A^{n-1}B) = \left( \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \mid \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \mid \cdots \mid \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right),$$

the rank of the matrix is 1. Thus the system is not controllable. From Theorem 3.4, the system is not observable and there exists singular controls for the system.

The port-Hamiltonian system provides a way to study nonholonomic systems. We present an important example of a nonlinear port-Hamiltonian system which modeling the motion with nonholonomic constraint.

**Example 4.3.** [4] We introduce a port-Hamiltonian system which models a motion of a rolling coin. Let $(M, \Pi)$ be a five dimensional pseudo-Poisson manifold and $x = (q_1, q_2, q_3, p_1, p_2)$ a local coordinate of pseudo-Poisson manifold $M$. The physical meanings of the coordinate is as follows; $q_1$ is the angle between the direction of coin and $x$-axis , $(q_2, q_3)$ is a point on a plain, $p_1$ is a inertia momentum of rolling and $p_2$ is a inertia momentum of spinning.

Let $H$ be a smooth function on $M$ representing energy of the system;

$$H(q_1, q_2, q_3, p_1, p_2) = \frac{1}{2}\left(p_1^2 + p_2^2\right).$$

With the input vector fields ${}^tg_1 = (0,0,0,1,0)$ and ${}^tg_2 = (0,0,0,0,1)$, we have a port-Hamiltonian system;

$$\begin{cases} \begin{pmatrix} \dot{q_1} \\ \dot{q_2} \\ \dot{q_3} \\ \dot{p_1} \\ \dot{p_2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cos q_1 \\ 0 & 0 & 0 & 0 & \sin q_2 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\cos q_1 & -\sin q_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial q_1}(x) \\ \frac{\partial H}{\partial q_2}(x) \\ \frac{\partial H}{\partial q_3}(x) \\ \frac{\partial H}{\partial p_1}(x) \\ \frac{\partial H}{\partial p_2}(x) \end{pmatrix} + u_1 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \\ y_j = g_j(x)H = p_j \quad , (j = 1, 2). \end{cases}$$

*Remark.* The structure $\Pi$ is a genuine pseudo-Poisson, that is, the pseudo-Poisson tensor $\Pi$ is not a Poisson tensor. Moreover, since the dimension of the manifold is five, the $\Pi$ will be degenerate.

For this port-Hamiltonian system, we cannot apply Theorem 3.4 because this system is nonlinear. But we can find a singular control by Proposition 2.4 as following.

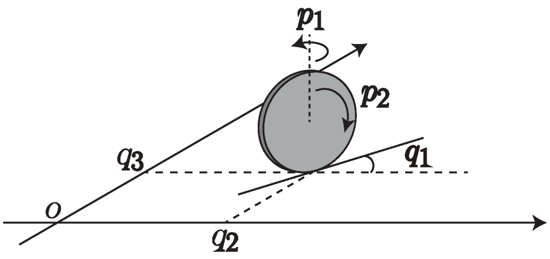


Figure 3. A rolling coin

The Hamiltonian $\hat{H}$ of the port-Hamiltonian system is given by

$$\hat{H}(x, \Psi, u) = \langle \Psi, X_f + ug \rangle,$$

where $\Psi = (\psi_1, \psi_2, \psi_3, \pi_1, \pi_2)$ is the local coordinate of $T_x^*M$. For a singular control, we get a constrained Hamiltonian system of $\hat{H}$ as follows;

$$\begin{cases} \dot{q_1}(t) = p_1, & \dot{\psi_1}(t) = p_2 \sin q_1 \psi_2 - p_2 \cos q_1 \psi_3, \\ \dot{q_2}(t) = p_2 \cos q_1, & \dot{\psi_2}(t) = 0, \\ \dot{q_3}(t) = p_2 \sin q_1, & \dot{\psi_3}(t) = 0, \\ \dot{p_1}(t) = u_1(t), & \dot{\pi_1}(t) = \psi_1, \\ \dot{p_2}(t) = u_2(t), & \dot{\pi_2}(t) = -\psi_2 \cos q_1 - \psi_3 \sin q_1, \\ \langle \Psi(t), g_i \rangle = \pi_i = 0 \ (i = 1, 2), & \Psi(t) \neq 0. \end{cases}$$

From the equations of $\dot{\psi}_1$ and $\dot{\pi}_2$, we have

$$\begin{pmatrix} -p_2 \sin q_1 & p_2 \cos q_1 \\ \cos q_1 & \sin q_1 \end{pmatrix} \begin{pmatrix} \psi_2 \\ \psi_3 \end{pmatrix} = 0.$$

Because the existence of singular controls implies the simultaneous equations have a nontrivial solution, we have

$$\det \begin{pmatrix} -p_2 \sin q_1 & p_2 \cos q_1 \\ \cos q_1 & \sin q_1 \end{pmatrix} = 0.$$

Since this is equal to $-p_2(\sin^2 q_1 + \cos^2 q_1) = 0$, we get $p_2 = 0$. Consequently, we have the forms of singular controls on this system;

$$\begin{cases} u_1(t) : \text{any arbitrary function}, \\ u_2(t) = 0. \end{cases}$$

We get the solution corresponds to a singular control with an initial condition;

$$\begin{aligned}
q_1(t) &= \int p_1(t)dt, & \psi_1(t) &= 0, \\
q_2(t) &= Q_2 : \text{const}, & \psi_2(t) &= C_2 : \text{const}, \\
q_3(t) &= Q_3 : \text{const}, & \psi_3(t) &= C_3 : \text{const}, \\
p_1(t) &= \int u_1(t)dt, & \pi_1(t) &= 0, \\
p_2(t) &= 0, & \pi_2(t) &= 0.
\end{aligned}$$

The meaning of the singular trajectory of the example is that singular controls do not control rolling component and the coin spins on a fixed point in $\mathbb{R}^2$ under the singular controls.

In this example, there exists a singular control and we found out the physical meaning of singular controls. But it is not well known about the existence of singular controls of port-Hamiltonian systems and its meanings. Theorem 3.4 does not holds on nonlinear systems, but it is interesting to study its generalization to nonlinear systems under some conditions.

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