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Singular controls with non-trivial independent trajectory of generic driftless control-affine systems

By

Wataru Yukuno *

Abstract

We consider singular controls with non-trivial independent trajectory and corresponding bi-extremals for generic smooth driftless control-affine systems on a finite dimensional smooth manifold in the sense of Whitney topology. Then we have the following theorem: for generic driftless control-affine systems of two or more smooth vector fields on a finite dimensional manifold, any bi-extremal with the non-trivial independent trajectory is of minimal order and any singular control with the non-trivial independent trajectory is of corank one.

§ 1. Introduction

Bonnard and Kupka studied in [2] singular controls of generic control-affine systems with one drift and one control. After that, Chitour, Jean and Trélat generalized the result of [2] in the paper [4]. Chitour, Jean and Trélat also treated the driftless control-affine system case in [3],[4]. They studied, in those papers, the properties of singular controls for generic driftless control-affine systems.

In [3], Chitour, Jean and Trélat introduce and describe the notions called "of minimal order" and "of corank one" related to singular controls. For the exact definitions, see Definition 4.2 and Definition 5.1 of our paper. Their results imply that, for a generic linearly-independent driftless control-affine system, any singular bi-extremal with the non-trivial singular trajectory is of minimal order and any singular control with the non-trivial singular trajectory is of corank one.

In the important paper [4], the results in [3] are widely generalized to general driftless control-affine systems including possibly linearly-dependent systems of vector

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*Department of Mathematics, Hokkaido University, Japan.
e-mail: `yukuwata@math.sci.hokudai.ac.jp`

fields. In particular, as a generalization of result in [3], by using a claim(Theorem 2.13 of [4]), they showed in [4] the following result: Let $(X_1, \cdots, X_m), 2 \leqq m \leqq n$ be a generic system of smooth vector fields over an n-dimensional manifold M, regarded as a driftless control-affine system. Then any singular X-bi-extremal with the non-trivial singular X-trajectory is of minimal order. Any singular X-control with the non-trivial singular X-trajectory is of corank one.

Unfortunately, Theorem 2.13 of [4] seems to be not correct. In fact, it was claimed there that, for a generic system $X = (X_1, \cdots, X_m), 2 \leqq m \leqq n$ of smooth vector fields over an n-dimensional manifold M, any X-trajectory $x : [0,T] \to M$ satisfies that

$$\dot{x}(t) = 0, \text{ for a.e. } t \in I_{\text{dep}}(x).$$

Here

$$I_{\text{dep}}(x) = \{t \in [0,T] \mid X_1(x(t)), \cdots, X_m(x(t)) \text{ are linearly dependent}\},$$

(see also Definition 6.1 of our paper). However the statement is not correct, because clearly there exist a counterexample: For instance, consider a generic system $X = (X_1, X_2, X_3)$ on $\mathbb{R}^3$. Then X_1, X_2, X_3 can be linearly dependent on a surface $\Sigma \subset \mathbb{R}^3$. Moreover setting $D_x = \langle X_1(x), X_2(x), X_3(x)\rangle_{\mathbb{R}}$ for $x \in \mathbb{R}^3$, we see $T_x\Sigma$ is transverse to D_x in $T_x\mathbb{R}^3$ and $\dim(D_x) = 2$ for any $x \in U$, on an open dense subset $U \subset \Sigma$. Then we have a line field $T\Sigma \cap D$ on U and any immersive integral curve $x : [0,T] \to U$ of the line field satisfies $I_{\text{dep}}(x) = [0,T]$ and

$$\dot{x}(t) \neq 0 \text{ for any } t \in [0,T].$$

Therefore we need to modify the formulation of theorems in [4]. In fact we adopt the different definition on minimal order property from that in [4].

Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields over an n-dimensional manifold M, and Ω be an open subset of $\mathbb{R}^m$. Consider the driftless control-affine systems

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x)$$

with the control parameter $(u_1, \cdots, u_m) \in \Omega$.

To formulate our main theorems, we introduce the new concept of an independent X-trajectory (see Definition 6.1) for a system $X = (X_1, \cdots, X_m)$. An X-trajectory $x : [0,T] \to M$ is called *independent* if the set of $t \in [0,T]$ such that $X_1(x(t)), \cdots, X_m(x(t))$ are linearly dependent over $\mathbb{R}$ has measure zero. An X-trajectory $x : [0,T] \to M$ is called *non-trivial* if for any sub-interval $J \subset [0,T]$, the restriction of x to $J, x|_J : J \to M$ is not constant. We need also the notiton of singular X-controls (see §2) and that of X-bi-extremals (see §3).

By using the ideas due to Chitour, Jean and Trélat in [3] and Thom's transversality theorem (for instance see [5]), we have the following theorems (see Theorem 6.2, Theorem 6.12):
Let $\mathrm{VF}(M)^m$ denote the set of systems of smooth vector fields $X = (X_1, \cdots, X_m)$ over M. We endow $\mathrm{VF}(M)^m$ with the Whitney smooth topology.

Main theorem 1 *Suppose* $2 \leqq m \leqq n$. *Then there exists an open dense subset* G_0 *of* $\mathrm{VF}(M)^m$ *such that, if* $X \in G_0$, *if* $z : [0,T] \to T^*M$ *is an* X*-bi-extremal and if the singular trajectory* $x = \pi \circ z : [0,T] \to M$ *is non-trivial and independent, then the bi-extremal* z *is of minimal order.*

Here $\pi : T^*M \to M$ is the canonical projection from the cotangent bundle of M.

Main theorem 2 *Suppose* $2 \leqq m \leqq n$. *Then there exists an open dense subset* G_1 *of* $\mathrm{VF}(M)^m$ *such that, if* $X \in G_1$, *if* $u : [0,T] \to \Omega$ *is a singular* X*-control for a given initial point* $x_0 \in M$ *and if the corresponding singular trajectory* $x : [0,T] \to M, x(0) = x_0$ *is non-trivial and independent, then the control* u *is of corank one.*

We put a remark on the case $m = 1$ and on the case $m > n$:
Let $m = 1$. A generic vector field on a manifold has isolated singularities. Then the endpoint mapping has singularities everywhere of corank $\geqq n - 1$. On the other hand, by definition, no bi-extremal is of minimal order if $m = 1$. Therefore main theorems 1 and 2 never hold in the case.
Let $m > n$. Then the dependent locus turns to be the whole space, and there is no independent trajectory. Therefore in this case main theorems become void.

In general, singular trajectories are very difficult to treat in control problems. Then main theorems of this paper guarantee that, for a generic system, the singularities of endpoint mappings are not so bad. Under additional conditions, any singularity is of corank one. Moreover, in the Hamiltonian formalism, if we choose a bi-extremal for a given singular trajectory, then the singular control can be recovered by some differentiations of minimal order (see [3],[4]).

In §2 we recall a singular control as a singular point of an end-point mapping on a driftless control-affine system. In §3 we introduce an equivalent condition of a singular control and describe a singular extremal and a singular bi-extremal. In §4 we define the concept that a singular bi-extremal corresponding to a singular control is of minimal order. Then note that the singular control is obtained as a solution of a homogeneous linear equation defined by the Goh matrix. In §5 we describe a singular control of corank one. In §6 we introduce the notion of independent trajectories and show the main results of this paper.

§ 2. Driftless control-affine systems and singular X-controls

We recall the basic definitions in a driftless control-affine system needed in this paper: Let M be an n-dimensional C^∞ manifold and $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields on M. We consider the control system

$$\dot{x} = \sum_{i=1}^{m} u_i X_i(x)$$

defined by X with the control parameters $u_1, \cdots, u_m$, which is called a *driftless control-affine system.*

Let $\Omega \subset \mathbb{R}^m$ be an open set. Let $x_0 \in M$ and $T > 0$. For an L^∞ (i.e. essentially bounded) curve $u : [0,T] \to \Omega$, we consider the *Cauchy problem* $(*)_{x_0,u}$, namely we consider the differential equation with the initial condition:

$$(*)_{x_0,u} \begin{cases} \dot{x}(t) = \sum_{i=1}^{m} u_i(t) X_i(x(t)) \text{ for a.e. } t \in [0,T], \\ x(0) = x_0 \end{cases}$$

By using the classical Carathéodory theory ([1] 2.4.1), we have that there exists a locally unique Lipschitz solution. A curve $u : [0,T] \to \Omega$ is called an X-*control* and the solution of $(*)_{x_0,u}$ is called an X-*trajectory*.

In particular, an X-control $u : [0,T] \to \Omega$ is called *admissible* if the global solution $x = x_u$ of $(*)_{x_0,u}$ exists. We call $x(T)$ the endpoint of the trajectory x. We use $\mathcal{U}_{x_0,T}$ to denote the set of admissible X-controls. Then $\mathcal{U}_{x_0,T} \subset L^\infty([0,T],\Omega)$ is a Banach open manifold ([1]).

The endpoint mapping $\mathrm{End}_{x_0}^T : \mathcal{U}_{x_0,T} \to M$ is defined by taking endpoint, namely, $\mathrm{End}_{x_0}^T(u) := x_u(T)$ for the trajectory x_u for $u \in \mathcal{U}_{x_0,T}$.

An X-control $u : [0,T] \to \Omega$ is called a *singular* or an *abnormal* if u is a singular point of $\mathrm{End}_{x_0}^T$, namely if the differential $(\mathrm{End}_{x_0}^T)_* : T_u\mathcal{U}_{x_0,T} \to T_{x_u(T)}M$ is not surjective. When u is a singular X-control, the corresponding trajectory x_u is called a *singular* X-*trajectory* or an *abnormal* X-*extremal.*

Unless otherwise stated, after this we consider only a driftless control-affine case.

§ 3. Equivalent condition of singular X-controls

We explain the equivalent condition of singular X-controls. In general case, it is difficult to study the properties of singular points of a functional from a Banach manifold to a finite dimensional manifold. However, in case of the endpoint mapping, as a part of Pontryagin maximal principle, an admissible X-control of the endpoint mapping is singular if and only if there exists an X-bi-extremal that satisfies the constrained Hamiltonian equations (Proposition 3.1).

Let M be an n-dimensional manifold and $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields on M. Let $\Omega \subset \mathbb{R}^m$ be an open set. Then the local description of the equivalent condition of singular X-controls can be given:

Proposition 3.1. *Let $x_0 \in M$. Let $u : [0,T] \to \Omega$ be an admissible X-control and $x : [0,T] \to M$ be the X-trajectory with $x(0) = x_0$. Then u is a singular X-control if and only if there exists an absolutely continuous curve $z : [0,T] \to T^*M$ such that, $x = \pi \circ z$, and that the following equations hold for any local coordinates $(x,p;u) = (x_1, \cdots, x_n, p_1, \cdots, p_n; u_1, \cdots, u_m)$ of $T^*M \times \Omega$, with a canonical coordinates (x,p) of T^*M :*

$$(\sharp): \begin{cases} (1)\ \dot{x}_i(t) = \frac{\partial H}{\partial p_i}(x(t), p(t); u(t))\ (1 \leqq i \leqq n) & \textit{for a.e. } t \in [0,T] \\ (2)\ \dot{p}_i(t) = -\frac{\partial H}{\partial x_i}(x(t), p(t); u(t))\ (1 \leqq i \leqq n) & \textit{for a.e. } t \in [0,T] \\ (3)\ \frac{\partial H}{\partial u_j}(x(t), p(t); u(t)) = 0\ (1 \leqq j \leqq m) & \textit{for a.e. } t \in [0,T] \\ (4)\ p(t) \neq 0 & \textit{for every } t \in [0,T] \end{cases},$$

where we define $H : T^*M \times \Omega \to \mathbb{R}$ by $H(x,p;u) = \langle p, \sum_{i=1}^m u_i X_i(x) \rangle$.

If a point $x_0 \in M$ is fixed and a singular X-control u is given, then the curve z is called a singular X-bi-extremal or an abnormal X-bi-extremal corresponding to u. The equations ($\sharp$) is called the *constrained Hamiltonian equations.* Note that for a singular X-control $u : [0,T] \to \Omega$, a corresponding X-bi-extremal z_u is not a unique solution of ($\sharp$) because the initial condition of (2) is not given.

§ 4. Singular controls of minimal order

Let M be an n-dimensional manifold. Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields on M. In order to define singular X-controls of minimal order, we prepare the (generalized) Goh matrix ([4], Definition 2.15.).

We describe the (generalized) Goh matrix. For integers i, j $(1 \leqq i, j \leqq m)$, we define the Hamiltonians $H_i : T^*M \to \mathbb{R}$ and $H_{ij} : T^*M \to \mathbb{R}$ by

$$\begin{cases} H_i(z) = \langle z, X_i(x) \rangle, \\ H_{ij}(z) = \langle z, [X_i, X_j](x) \rangle. \end{cases}, z \in T^*M.$$

where $x = \pi(z)$, $\pi : T^*M \to M$ is the canonical projection. Note that

$$\langle z, [X_i, X_j](x) \rangle = \{H_i, H_j\}(x),\ z \in T^*M,$$

where $\{H_i, H_j\}$ is the Poisson bracket of H_i and H_j. Then we define the *Goh matrix* $G : T^*M \to \mathcal{S}_m(\mathbb{R})$ associated to the system of vector fields $X = (X_1, \cdots, X_m)$ on M by :

$$G(z) := (H_{i,j}(z))_{1 \leqq i,j \leqq m}\,,\; z \in T^*M,$$

where $\mathcal{S}_m(\mathbb{R})$ is the set of $m \times m$ skew-symmetric matrices. Note that since $G(z)$ is a skew-symmetric matrix, rank $G(z)$ is even. If m is even, then there exists a polynomial function $P : \mathcal{S}_m(\mathbb{R}) \cong \mathbb{R}^{\frac{m(m-1)}{2}} \to \mathbb{R}$ of degree $\frac{m}{2}$ in the variables $(H_{ij})_{1 \leqq i<j \leqq m}$ such that for any $G \in S_m(\mathbb{R})$, $\det G$ is the square of $P(G)$, which is called the Pfaffian. We define the Hamiltonian $\hat{P} : T^*M \to \mathbb{R}$ associated to the system of vector fields $X = (X_1, \cdots, X_m)$ on M by

$$\hat{P}(z) := P((H_{ij}(z))_{1 \leqq i<j \leqq m}),\; z \in T^*M.$$

Note that the following holds:

$$\det(G(z)) = (\hat{P}(z))^2, z \in T^*M.$$

Then we define the *generalized Goh matrix* $\hat{G} : T^*M \to \mathrm{M}_{m+1,m}(\mathbb{R})$ associated to the system of vector fields $X = (X_1, \cdots, X_m)$ on M by an $(m+1) \times m$ -matrix

$$\hat{G}(z) := \begin{pmatrix} (H_{ij}(z))_{1 \leqq i,j \leqq m} \\ \left(\left\{\hat{P}, H_j\right\}(z)\right)_{1 \leqq j \leqq m} \end{pmatrix}, z \in T^*M.$$

Now, we show that any X-singular control is obtain as a solution of a homogeneous linear equation by Goh matirices or generized Goh matrices:
Let $\Omega \subset \mathbb{R}^m$ be an open subset.

Proposition 4.1. *Fix $x_0 \in M$. Let $u : [0,T] \to \Omega$ be a singular X-control and $z : [0,T] \to T^*M$ be a singular X-bi-extremal corresponding to u. Then, the followings hold:*

(1) *u is a solution of the equation $G(z(t))u(t) = 0$ for almost every $t \in [0,T]$.*

(2) *If m is even, then u is a solution of the equation $\hat{G}(z(t))u(t) = 0$ for almost every*

$t \in [0,T]$.

Proof. (1) Since u is a singular X-control, by Proposition 3.1, for each integer $i\,(1 \leqq i \leqq m)$ and for every $t \in [0,T]$,

$$H_i(z(t)) = 0.$$

By differentiating both sides, for each integer $i\,(1 \leqq i \leqq m)$ and for almost every $t \in [0,T]$,

$$\sum_{j=1}^{m} H_{ij}(z(t))\,u_j(t) = 0.$$

This is equivalent to the following equation: for almost every $t \in [0,T]$,

$$G(z(t))u(t) = 0.$$

(2) If m is even, then $\operatorname{rank} G(z(t)) < m-1$ for $t \in [0,T]$. Therefore for every $t \in [0,T]$,

$$\hat{P}(z(t)) = 0.$$

By differentiating both sides, for almost every $t \in [0,T]$,

$$\sum_{j=1}^{m} \left\{\hat{P}, H_j\right\}(w(t))u_j(t) = 0.$$

This is equivalent to the following equation: for almost every $t \in [0,T]$,

$$\hat{G}(z(t))\,u(t) = 0.$$

□

Definition 4.2. A singular X-bi-extremal $z : [0,T] \to T^*M$ is called of *minimal order* if it satisfies the following condition: If m is odd (resp. even), then $\operatorname{rank} G(z(t)) = m-1$ (resp. $\operatorname{rank} \hat{G}(z(t)) = m-1$) for almost every $t \in [0,T]$.

Remark By Proposition 4.1, if a singular X-bi-extremal $z : [0,T] \to T^*M$ corresponding to singular X-control $u : [0,T] \to \Omega$ is of minimal order, then we can deduce an expression for $u(t)$, up to time reparameterization.

The definition of a singular bi-extremal of minimal order (Definition 4.2 of our paper) is different from Definition 2.16 of [4]. Definition 2.16 of [4] is that, a singular X-bi-extremal $z : [0,T] \to T^*M$ is called of *minimal order* if it satisfies the following conditions:

$$\begin{cases} \text{(i) } \dot{x}(t) = 0, \text{ for almost every } t \in I_{\text{dep}}(x). \\ \text{(ii) If } m \text{ is odd (resp. even), then } \operatorname{rank} G(z(t)) = m-1 \text{ (resp. } \operatorname{rank} \hat{G}(z(t)) = m-1) \\ \qquad \text{for almost every } t \in [0,T] \setminus I_{\text{dep}}(x). \end{cases}$$

Here

$$I_{\text{dep}}(x) = \{t \in [0,T] \mid X_1(x(t)), \cdots, X_m(x(t)) \text{ are linearly dependent}\}.$$

Theorem 2.13 of [4] claims that any X-trajectory satisfies the above condition (i) for generic X, which is not correct as is mentioned in §1. Therefore in our Definition

4.2, we do not suppose the above condition (i). Instead we adopt the condition (ii) as the minimal order condition by replacing $I_{\rm dep}(x)$ with $[0,T]$, in order to get a natural definition (Definition 4.2).

§ 5. Singular controls of corank one

Let M be an n-dimensional manifold. Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields on M. Let $\Omega \subset \mathbb{R}^m$ be an open subset and let $x_0 \in M$.

Definition 5.1. A singular X-control $u : [0,T] \to \Omega$ is called of *corank one* if the codimension of $(End_{x_0}^T)_*$ is one, namely if $\dim M - \dim \mathrm{Im}\left((End_{x_0}^T)_*\right) = 1$ at u.

In particular, if the control system is driftless control-affine system, then it is well-known that the following theorem:

Proposition 5.2. ([3]) *A singular X-control $u : [0,T] \to \Omega$ is of corank one if and only if for any corresponding two singular X-bi-extremals $z_1, z_2 : [0,T] \to T^*M$ to u, there exists a non-zero real number λ such that $z_1 = \lambda\, z_2$ on $[0,T]$.*

§ 6. Singular control with non-trivial independent trajectory

In this section, we prove the main theorems (Main theorem 1 and Main theorem 2). In order to describe the main theorems, we define the independent trajectory on a driftless control-affine system. Let M be an n-dimensional manifold. Let $X = (X_1, \cdots, X_m)$ be a system of smooth vector fields on M.

Definition 6.1. A singular X-trajectory $x : [0,T] \to M$ on a driftless control-affine system $\dot{x} = \sum_{i=1}^m u_i X_i$ is called *independent* if the set

$$I_{\rm dep}(x) := \{t \in [0,T] \,|\, X_1(x(t)), \cdots, X_m(x(t)) \text{ are linearly dependent over } \mathbb{R}\}$$

has measure zero.

Recall that $\mathrm{VF}(M)$ denotes the space of smooth vector fields over M and $\mathrm{VF}(M)^m$ the m-tuple product of $\mathrm{VF}(M)$ with the Whitney smooth topology. Then, as is stated in Introduction, the following theorem holds:

Theorem 6.2. (Main theorem 1) *Suppose $2 \leqq m \leqq n$. Then there exists an open dense subset G_0 of $\mathrm{VF}(M)^m$ such that, if $X \in G_0$, if $z : [0,T] \to T^*M$ is an X-bi-extremal and if the singular trajectory $x = \pi \circ z : [0,T] \to M$ is non-trivial and independent, then the bi-extremal z is of minimal order. Here $\pi : T^*M \to M$ is the canonical projection from the cotangent bundle of M.*

The proof goes in parallel with the proof of Theorem 2.4 in [3]. However note that Theorem 2.4 in [3] implies the results only for driftless control-affine systems which are independent everywhere. Since we treat general systems of vector fields which may have non-void locus of dependence, we need appropriate modifications of the proof given in [3].

Outline of proof : Let $d \geqq 1$ be an integer. Put $N = 2d$. We denote the space of all N-jets of vector fields $X \in \mathrm{VF}(M)$ by $J^N(\mathrm{VF}(M))$, and the fibre product over M of m-tuple spaces of $J^N(\mathrm{VF}(M))$, by $J^N(\mathrm{VF}(M))^m$. Then, we will show the main theorem 1 by the following procedures:

[Step1] (See Definition 6.5) Construct the "bad set" with respect to minimal order, $B_{mo}(d) \subset J^N(\mathrm{VF}(M))^m$. Note that, $B_{mo}(d)$ is semi-algebraic and in particular, dimensions of $B_{mo}(d)$ and its closure $\overline{B_{mo}(d)}$ are well-defined.

[Step2] (See Lemma 6.10) Show that, if $X \in \mathrm{VF}(M)^m$ satisfies the condition that, for any $x \in M$, $j^N_x X \notin B_{mo}(d)$, if $z : [0,T] \to T^*M$ is an X-bi-extremal, and if the singular trajectory $\pi \circ z : [0,T] \to M$ is non-trivial and independent, then z is of minimal order.

[Step3] (See Lemma 6.11) Compute the codimension of $\overline{B_{mo}(d)}$ in $J^N(\mathrm{VF}(M))^m$.

[Step4] (See the subsection §6.4) For $N > 4n$ $(d > 2n)$, let G_0 be the set of $X \in \mathrm{VF}(M)^m$ such that the jet $j^N_x X$ is not included in $\overline{B_{mo}(d)}$ in $J^N(\mathrm{VF}(M))^m$. Then, show that, G_0 is an open dense subset of $\mathrm{VF}(M)^m$ in the sense of Whitney smooth topology by Thom transversality theorem (for instance see [5]).

In order to show the main theorem 1, we prepare the subsections: §6.1 to §6.4. In §6.1, after preparing the notations, permutated Hamiltonians, Goh matrices and elementary determinants in Definition 6.3, 6.4, we define the bad set $B_{mo}(d)$ of $J^N(\mathrm{VF}(M))^m$ in Definition 6.5. In §6.2, in Lemma 6.8, 6.9, we describe some properties of rank of Goh matrix for any singular X-bi-extremals if $X \in \mathrm{VF}(M)^m$ satisfies the condition that for any $x \in M$, $j^N_x X$ is included in complement of $B_{mo}(d)$ in $J^N(\mathrm{VF}(M))^m$. In particular, the important Lemma 6.10 prepared for the proof of main theorem 1 can be immediately derived from the Lemmata 6.8, 6.9 : if $X \in \mathrm{VF}(M)^m$ satisfies the condition that , $j^N_x X \notin B_{mo}(d)$ for any $x \in M$, then any singular X-bi-extremal with the non-trivial independent singular X-trajectory is of minimal order. In §6.3, we compute the codimension of the closure of $B_{mo}(d)$ in $J^N(\mathrm{VF}(M))^m$ in Lemma 6.11. In §6.4, by using these Lemmata 6.10, 6.11, we show the main theorem 1.

§ 6.1. Construction of bad set.

In this section, we construct the semi-algebric set $B_{mo}(d)$, which is called the bad set with respect to minimal order for an integer d :

Let $\mathfrak{S}_m$ be the set of permutations with m elements, and $X = (X_1, \cdots, X_m) \in \mathrm{VF}(M)^m$. Then, we prepare some notations, permutated Hamiltonians, Goh matrices and elementary determinants, in order to define $B_{mo}(d)$ for an integer d :

Definition 6.3. Let $i\,(1 \leqq i \leqq m), j\,(1 \leqq j \leqq m)$ and $r\,(1 \leqq r \leqq m-1)$ be integers. Then, we define *permutated Hamiltonians* $H_i, H_{i,j} : \mathfrak{S}_m \times T^*M \to \mathbb{R}$ and *permutated Goh matrices* $G : \mathfrak{S}_m \times T^*M \to M_m(\mathbb{R})$, $G^r : \mathfrak{S}_m \times T^*M \to M_r(\mathbb{R})$ by

$$\begin{cases} H_i(\sigma, z) := \{z, X_{\sigma(i)}(\pi(z))\}, \quad H_{ij}(\sigma, z) := \{z, [X_{\sigma(i)}, X_{\sigma(j)}](\pi(z))\}, \\ G(\sigma, z) := (H_{i,j}(\sigma, z))_{1 \leqq i,j \leqq m}, \; G^r(\sigma, z) := (H_{i,j}(\sigma, z))_{1 \leqq i,j \leqq r}, \end{cases}$$

where $\pi : T^*M \to M$ is the canonical projection.

Definition 6.4. We inductively define the real valued functions on $\mathfrak{S}_m \times T^*M$, which are called *elementary determinants*: Let $\sigma \in \mathfrak{S}_m, z \in T^*M$. Then,
(I) For an integer $r\,(1 \leqq r \leqq m-1)$,

$$\Delta_0^r(\sigma, z) := \det(G^r(\sigma, z)), \quad \Delta_0^0(\sigma, z) := 1.$$

(II) For integers $r\,(1 \leqq r \leqq m-1)$, $k\,(r+1 \leqq k \leqq m)$, (with the convention that the index $m+1$ stands for $r+1$),

$$\begin{cases} \Delta_{0,s+1}^{r,k}(\sigma, z) := \det\left(\begin{array}{c|c} G^r(\sigma, z) & (H_{ij}(\sigma, z))_{1 \leqq i \leqq r} \\ \hline \multicolumn{2}{c}{\left(\{\Delta_{0,s}^{r,k}, H_j\}(\sigma, z)\right)_{j=1,\cdots,r,k}} \end{array}\right) \quad (s = 0, 1, \cdots), \\ \Delta_{0,0}^{r,k}(\sigma, z) := \det\left(\begin{array}{c|c} G^r(\sigma, z) & (H_{ij}(\sigma, z))_{1 \leqq i \leqq r} \\ \hline (H_{(k+1)j}(\sigma, z))_{1 \leqq j \leqq r} & H_{(k+1)k}(\sigma, z) \end{array}\right). \end{cases}$$

(III) For $r\,(1 \leqq r \leqq m-1)$, $p\,(1 \leqq p \leqq m-r-1)$, and $s_1, \cdots, s_p \geqq 1$,

$$\begin{cases} \Delta_{0,s_1,\cdots,s_p,s+1}^{r,r+1,\cdots,r+p,k}(\sigma, z) := \det\begin{pmatrix} (H_{ij}(\sigma, z))_{\substack{1 \leqq i \leqq r \\ j=1,\cdots,r+p,k}} \\ \hline \left(\{\Delta_{0,s_1-1}^{r,r+1}, H_j\}(\sigma, z)\right)_{j=1,\cdots,r+p,k} \\ \vdots \\ \left(\{\Delta_{0,s_1,\cdots,s_p-1}^{r,r+1,\cdots,r+p}, H_j\}(\sigma, z)\right)_{j=1,\cdots,r+p,k} \\ \left(\{\Delta_{0,s_1,\cdots,s_p,s}^{r,r+1,\cdots,r+p,k}, H_j\}(\sigma, z)\right)_{j=1,\cdots,r+p,k} \end{pmatrix} \quad (s = 0, 1, \cdots), \\ \Delta_{0,s_1,\cdots,s_p,0}^{r,r+1,\cdots,r+p,k}(\sigma, z) := \Delta_{0,s_1,\cdots,s_p-1}^{r,r+1,\cdots,r+p-1,k}(\sigma, z). \end{cases}$$

(IV) Let m be an even integer. We denote the Pfaffian polynomial of G, by $p : \mathbb{R}^{\frac{m(m-1)}{2}} \to \mathbb{R}$. Then, $P(\sigma, z) := p((H_{ij}(\sigma, z))_{1 \leqq i < j \leqq m})$.

(i) for every $k \in \{m-1, m\}$,

$$\begin{cases} \delta^k_{s+1}(\sigma, z) := \det \left(\begin{array}{c|c} G^{m-2}(\sigma, z) & (H_{ik}(\sigma, z))_{1 \leqq i \leqq m-2} \\ \hline (\{\delta^k_s, H_j\}(\sigma, z))_{1 \leqq j \leqq m-2} & \{\delta^k_s, H_k\}(\sigma, z) \end{array} \right) \ (s = 0, 1 \cdots), \\ \delta^k_0(\sigma, z) := P(\sigma, z). \end{cases}$$

(ii) for every integer $s_1 \geqq 1$,

$$\begin{cases} \delta_{s_1, s+1}(\sigma, z) := \begin{pmatrix} (H_{ij}(\sigma, z))_{\substack{1 \leqq i \leqq m-2 \\ 1 \leqq j \leqq m}} \\ \hline \left(\{\delta^{m-1}_{s_1 - 1}, H_j\}(\sigma, z)\right)_{1 \leqq j \leqq m} \\ (\{\delta_{s_1, s}, H_j\}(\sigma, z))_{1 \leqq j \leqq m} \end{pmatrix} \ (s = 0, 1, \cdots) \\ \delta_{s_1, 0}(\sigma, z) := \delta^m_{s_1}(\sigma, z). \end{cases}$$

By using the elementary determinants, we define the bad set $B_{mo}(d)$ for an integer d :

Definition 6.5. Let d be an integer and $N = 2d$. For an integer p, let $N_{p,d}$ be the set of $(p+1)$-tuples $\overline{s} = (0, s_1, \cdots, s_p)$ in $\{0\} \times (\mathbb{N})^p$ with $s_1 + \cdots + s_p < d + p$. We define the "bad set" with respect to minimal order, $B_{mo}(d) \subset J^N(\mathrm{VF}(M))^m$ by the image of $\hat{B}_{mo}(d)$ by the canonical projection $J^N(\mathrm{VF}(M))^m \times_M T^*M \to J^N(\mathrm{VF}(M))^m$:

$$\hat{B}_{mo}(d) = \{(j^N_x X, z) \,|\, x = \pi(z) \in M, z \in T^*M, \, (j^N_x X, z) \in \hat{B}^0_{mo}(d) \cup \hat{B}^1_{mo}(d)\},$$

Then, $B_{mo}(d)$ is defined by the following:

$$B_{mo}(d) = \{j^N_x X \,|\, x = \pi(z) \in M, \, (j^N_x X, z) \in \hat{B}^0_{mo}(d) \cup \hat{B}^1_{mo}(d) \, \text{for some} \, z \in T^*M\},$$

where $\hat{B}^0_{mo}(d), \hat{B}^1_{mo}(d) \subset J^N(\mathrm{VF}(M))^m \times_M T^*M$ are written in Definition 6.6, 6.7 respectively:

Definition 6.6. If $m = 2$, then $\hat{B}^0_{mo}(d) = \emptyset$. On the other hand, if $m \geqq 3$, then we define $\hat{B}^0_{mo}(d) \subset J^N(\mathrm{VF}(M))^m \times_M T^*M$ by the union of the sets $\hat{B}^0_{mo}(d, \sigma, r, \overline{s}, z)$ with $\sigma \in \mathfrak{S}_m$, even integers $r \, (0 \leqq r \leqq m-3)$, and $\overline{s} \in N_{p,d}$ with $0 \leqq p < m$. Here the definition of $\hat{B}^0_{mo}(d, \sigma, r, \overline{s}, z)$ is below:

For $z \in T^*M$ with $\pi(z) = x \in M$, $\sigma \in \mathfrak{S}_m$, an even integer $r (0 \leqq r \leqq m-3)$, and $\overline{s} \in N_{p,d}$ with $p \, (0 \leqq p < m)$, let $\hat{B}^0_{mo}(d, \sigma, r, \overline{s}, z)$ be the set of $(j^N_x X, z) \in J^N(\mathrm{VF}(M))^m \times_M T^*M$ such that:

1). $X_1(x), \cdots, X_m(x)$ are linearly independent;

2). $\Delta^r_0(\sigma, z) \neq 0$;

3). for every integer $i \, (0 \leqq i \leqq p)$,

(a) $\Delta^{r, r+1, \cdots, r+i}_{0, s_1, \cdots, s_i}(\sigma, z) \neq 0$;

(b) for every integer $k\,(r+i \leqq k \leqq m)$ and $s\,(1 \leqq s \leqq s_i - 1)$,

$$\Delta^{r,r+1,\cdots,r+i-1,k}_{0,s_1,\cdots,s_{i-1},s}(\sigma, z) = 0;$$

4). for every $k\,(r+p+1 \leqq k \leqq m)$ and $s\,(1 \leqq s \leqq d+p-(s_1+\cdots+s_p)$,

$$\Delta^{r,r+1,\cdots,r+p,k}_{s_1,\cdots,s_p,s}(\sigma, z) = 0.$$

Definition 6.7. If m is an odd integer, then $\hat{B}^1_{mo}(d) = \emptyset$. On the other hand, if $m \geqq 2$ is an even integer, then we define $\hat{B}^1_{mo}(d) \subset J^N(\mathrm{VF}(M))^m \times_M T^*M$ by the union of the sets $\hat{B}^1_{mo}(d, \sigma, s_1, z)$ with $\sigma \in \mathfrak{S}_m$ and integers $s_1\,(1 \leqq s_1 \leqq d)$. Here the definition of $\hat{B}^1_{mo}(d, \sigma, s_1, z)$ is below:
Let $m \geqq 2$ be an even integer. For $\sigma \in \mathfrak{S}_m$ and an integer $s_1 (1 \leqq s_1 \leqq d)$, we define $\hat{B}^1_{mo}(d, \sigma, s_1, z)$ by the set of elements $(j^N_x X, z) \in J^N_x(\mathrm{VF}(M))^m \times_M T^*M$ such that:
1). $X_1(x), \cdots, X_m(x)$ are linearly independent;
2). $\Delta^{m-2}_0(\sigma, z) \neq 0$;
3). (a) if $s_1 < d$, then $\delta^{m-1}_{s_1}(\sigma, z) \neq 0$;
(b) for $k \in \{m-1, m\}$ and $s\,(0 \leqq s \leqq s_1 - 1), \delta^k_s(\sigma, z) = 0$;
4). for $s\,(1 \leqq s \leqq d - s_1), \delta_{s_1,s}(\sigma, z) = 0$.

§ 6.2. The property of singular bi-extremals avoiding bad set.

We describe some properties of rank of Goh matrix for any singular X-bi-extremals with the non-trivial and independent singular X-trajectory if $X \in \mathrm{VF}(M)^m$ satisfies the condition that for any $x \in M$, $j^N_x X$ is included in complement of $B_{mo}(d)$ in $J^N(\mathrm{VF}(M))^m$. In particular, we will show the important Lemma 6.10 prepared for the proof of the main theorem 1 by using the Lemmata 6.8, 6.9.

Lemma 6.8. *Suppose that, $2 \leqq m \leqq n$. Let d be a positive integer and $N = 2d$. Let $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j^N_x X \notin B_{mo}(d)$. Then, if $z : [0, T] \to T^*M$ is an X-bi-extremal and if the singular X-trajectory $\pi \circ z : [0, T] \to M$ is non-trivial and independent, then*

$$m - 2 \leqq \mathrm{rank}\ G(z(t)) \leqq m - 1, \text{ for } a.e.\, t \in [0, T].$$

Here $\pi : T^*M \to M$ is the canonical projection from cotangent bundle of M.

Proof. By the assumption that $\pi \circ z$ is non-trivial and by Proposition 4.1 (1), we have the inequality rank $G(z(t)) \leqq m - 1$ for a.e. $t \in [0, T]$. Therefore it suffices to show the inequality $m - 2 \leqq \mathrm{rank}\ G(z(t))$ for a.e. $t \in [0, T]$.

In $m=2$ case, Lemma 6.8 clearly holds. We consider $m \geqq 3$. In order to prove by contradiction, assume that, $z:[0,T]\to T^*M$ is an X-bi-extremal and the singular X-trajectory $\pi\circ z:[0,T]\to M$ is non-trivial and independent but there exists a measurable subset of positive measure $K\subset[0,T]$ such that

$$\operatorname{rank} G(z(t)) \leqq m-3 \ \text{ for every } t\in K.$$

Since $\pi\circ z$ is independent, the compliment of $I_{\mathrm{dep}}(\pi\circ z)$, $I_{\mathrm{indep}}(\pi\circ z)$ has positive measure. Then, let $J := I_1\cap I_{\mathrm{indep}}(\pi\circ z)$. J has positive measure and for every $t\in J$, $X_1((\pi\circ z)(t)),\cdots,X_m((\pi\circ z)(t))$ are linearly independent over $\mathbb{R}$.

After this, in the same way as the proof of Lemma 3.8 in [3], we can prove this Lemma 6.8 :
In fact, for an integer p $(1\leqq p\leqq m)$, we denote by I_p a subset of $\{1,\cdots,m\}$ with cardinality p. Let r be the maximum of even integers p $(1\leqq p\leqq m-3)$ such that there exists a subset $I_p\subset\{1,\cdots,m\}$ satisfying $\det\left(H_{i,j}(z(t))\right)_{(i,j)\in I_p^2}\not\equiv 0$ on J. Let J_r be the set of $t\in J$ such that there exists $I_r\subset\{1,\cdots,m\}$ satisfynig $(H_{i,j}(z(t))_{(i,j)\in I_r^2}\neq 0$. Note J_r has positive measure. Therefore there exists $\sigma\in\mathfrak{S}$ such that

$$\Delta_0^r(\sigma,z(t))\neq 0 \text{ for every } t\in J_r.$$

On the other hand, for any subset $I_{r+2}\subset\{1,\cdots,m\}$, $\det\left(H_{i,j}(z(t)\right)_{(i,j)\in I_{r+2}^2}\equiv 0$ on J_r and $\operatorname{rank}\left(H_{i,j}(z(t)\right)_{(i,j)\in I_{r+2}^2}\leqq r$ on J_r. Therefore for any subset $I_{r+1}\subset I_{r+2}$, $\det\left(H_{i,j}(z(t)\right)_{(i,j)\in I_{r+1}^2}=0$ on J_r. In particular, for $k=r+1,\cdots,m$,

$$\Delta_{0,0}^{r,k}(\sigma,z(t))=0 \text{ for every } t\in J_r.$$

By differentiating both sides, for $k=r+1,\cdots,m$,

$$\sum_{i=1}^{m} u_i(t)\,\{\Delta_{0,0}^{r,k},H_i\}(\sigma,z(t))=0 \ \text{ for a.e. } t\in J_r.$$

Here, we denote by $G_0(\sigma,z(t))$ the following $m\times m$- matrix:

$$\begin{pmatrix} H_{11}(\sigma,z(t)) & \cdots & H_{1m}(\sigma,z(t)) \\ \vdots & & \vdots \\ H_{r1}(\sigma,z(t)) & \cdots & H_{rm}(\sigma,z(t)) \\ \{\Delta_{0,0}^{r,r+1},H_1\}(\sigma,z(t)) & \cdots & \{\Delta_{0,0}^{r,r+1},H_m\}(\sigma,z(t)) \\ \vdots & & \vdots \\ \{\Delta_{0,0}^{r,m},H_1\}(\sigma,z(t)) & \cdots & \{\Delta_{0,0}^{r,m},H_m\}(\sigma,z(t)) \end{pmatrix}.$$

Then $G_0(\sigma,z(t))\not\equiv 0$ on J_r. Note that, the first diagonal minors of order r of $G_0(\sigma,z(t))$ is $\Delta_0^r(\sigma,z(t))$,which never vanishes on J_r. and by definition, the diagonal minors of order $r+1$ containing $\Delta_0^r(\sigma,z(t))$ are $\Delta_{0,1}^{r,k}(\sigma,z(t))$, $k=r+1,\cdots,m$.

Claim. There exists $k_1 \in \{1, \cdots, m\}$, an integer $s_1 (1 \leqq s_1 < d+1)$, and a subset $J_{r+1} \subset J_r$ of positive measure such that

$$\begin{cases} \Delta^{r,k}_{0,\ell}(\sigma, z(t)) \equiv 0 \text{ on } J_{r+1}, \text{ for any integers } k, \ell \ (r+1 \leqq k \leqq m,\ 0 \leqq \ell \leqq s_1 - 1); \\ \Delta^{r,k_1}_{0,s_1}(\sigma, z(t)) \neq 0 \,\text{for every } t \in J_{r+1}. \end{cases}$$

In fact, assume the claim is false. Then for every $k\,(r+1 \leqq k \leqq m)$ $\Delta^{r,k}_{0,1}(\sigma, z(t)) = 0$ on J_r. We consider the matrix $G_1(\sigma, z(t))$ obtained by replacing the last $m - r$ rows of $G_0(\sigma, z(t))$ with rows

$$(\{\Delta^{r,k}_{0,0}, H_\ell\}(\sigma, z(t)))_{1 \leqq \ell \leqq m}, \text{ for } k\ (1 \leqq k \leqq m).$$

By construction, $\det G_1(\sigma, z(t)) \equiv 0$ on J_r. The contradiction assumption implies that, for $k\,(r+1 \leqq k \leqq m)$, $\Delta^{r,k}_{0,2}(\sigma, z(t))) \equiv 0$ of J_r. Proceeding similarly, there exists $t \in J_r$ such that $j_{\pi \circ z(t)} X$ belongs to $B(d, \sigma, r, 0, z(t))$. This contradicts the assumption that, for any $x \in M$, $j_x X \in B_{mo}(d)$. Thus the claim is proved.

Up to a permutation, assume $k_1 = r+1$. We define a non-invertible matrix by replacing in G_0 :

$$\begin{cases} \text{the } (r+1)\text{-th line by } (\{\Delta^{r,r+1}_{0,s_1-1}, H_\ell\}(\sigma, z(t)))_{1 \leqq \ell \leqq m}; \\ \text{for } j\,(r+2 \leqq j \leqq m), \text{the } j\text{-th line by } (\{\Delta^{r,j}_{0,s_1-1}, H_\ell\}(\sigma, z(t)))_{1 \leqq \ell \leqq m}. \end{cases}$$

To this matrix is applied the previous reasoning on G_0. Thus, by a finite number of steps, we obtain that there exists subset $J_{m-1} \subset J$ of positive measure, and $\overline{s} = (0, s_1, \cdots, s_m)$ in $N_{m-1,d}$, such that

$$\begin{cases} \Delta^{r,\cdots,r+m-1}_{0,s_1,\cdots,s_{m-1}}(\sigma, z(t))) \neq 0 \text{ for every } t \in J_{m-1}; \\ \Delta^{r,\cdots,r+m-1,r+m}_{0,s_1,\cdots,s_{m-1},\ell}(\sigma, z(t))) \equiv 0 \text{ on } J_{m-1}, \ell \geqq 0. \end{cases}$$

As a consequence, for every $t \in J_{m-1}, j_{\pi \circ z(t)} X$ belongs to $B(d, \sigma, r, \overline{s}, z(t))$. This contradicts the assumption that, for any $x \in M$, $j_x X \in B_{mo}(d)$. □

Lemma 6.9. *Suppose that, m is even and $2 \leqq m \leqq n$. Let d be a positive integer and $N = 2d$. Let $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j^N_x X \notin B_{mo}(d)$. Then, if $z : [0, T] \to T^*M$ is a singular X-bi-extremal and if the singular X-trajectory $\pi \circ z : [0, T] \to M$ is non-trivial and independent, then*

$$\text{rank } \hat{G}(z(t)) = m - 1 \text{ for } a.e.\, t \in [0, T].$$

Proof. By the assumption that $\pi \circ z$ is non-trivial and by Proposition 4.1 (2), we have the inequality rank $G(z(t)) \leqq m - 1$ for a.e. $t \in [0, T]$. Therefore we will show the inequality rank $G(z(t)) \geqq m - 1$ for a.e. $t \in [0, T]$.

In order to prove by contradiction, assume that, $z : [0,T] \to T^*M$ is an X-bi-extremal and the singular X-trajectory $\pi\circ z : [0,T] \to M$ is non-trivial and independent but there exists a measurable subset of positive measure $K \subset [0,T]$ such that

$$\operatorname{rank}\hat{G}(z(t)) \leqq m-2 \text{ for every } t \in K.$$

Since $\pi \circ z$ is independent, $I_{\text{indep}}(\pi \circ z)$ has positive measure, where $I_{\text{indep}}(\pi \circ z)$ is the compliment of $I_{\text{dep}}(\pi\circ z)$ in $[0,T]$. When let $J := I_1 \cap I_{\text{indep}}(\pi\circ z)$, $J \subset [0,T]$ has positive measure and for any $t \in J$, $X_1((\pi \circ z)(t)), \cdots, X_m((\pi \circ z)(t))$ are linearly independent over M. After this, in the same way as the proof of Lemma 3.8 in [3], we can prove this Lemma 6.9:
In fact, By the previous proof, we may assume that there exists $\sigma \in \mathfrak{S}_m$ such that

$$\Delta_0^{m-2}(\sigma, z(t)) \neq 0 \text{ for every } t \in J.$$

In particular, rank $\mathrm{G}(z(t)) = m-2$. Moreover, for $k = m-1$ and $k = m$,

$$\delta_0^k(\sigma, z(t)) = 0 \text{ and } \delta_1^k(\sigma, z(t)) = 0 \text{ for every } t \in J_r.$$

Similarly to the argument of Lemma 3.8, we claim that there exists an integer $s_1\,(1 \leqq s_1 < d)$, and $k_1 \in \{m-1, m\}$, such that

$$\delta_{s_1}^{k_1}(\sigma, z(t)) \neq 0 \text{ for every } t \in J.$$

In fact, otherwise, for $s\,(0 \leqq s \leqq d)$ and for $k \in \{m-1, m\}$, $\delta_s^k(\sigma, z(t)) = 0$. In that case, for every $t \in J, j^N_{\pi\circ z(t)}X$ belongs to $B^1(d, \sigma, d, z(t))$. This contradicts for any $x \in M$, $j_x X \in B_{mo}(d)$.

Up to a permutation, we may assume $k_1 = m-1$. Let $J_1 \subset J$ be a subset of positive measure such that

$$\delta_{s_1}^{m-1}(\sigma, z(t)) \neq 0 \text{ for every}\, t \in J_1.$$

Similarly to the argument of Lemma 3.8, for every $s \geqq 0$, we have

$$\delta_{s_1,s}(\sigma, z(t)) = 0 \text{ for } t \in J_r.$$

Then, for every $t \in J_1$, $j^N_{\pi\circ z(t)}X$ belongs to $B^1(d, \sigma, s_1, z(t))$. This contradicts for any $x \in M$, $j_x X \in B_{mo}(d)$. □

Lemma 6.10. *Suppose that, $2 \leqq m \leqq n$. Let d be a positive integer and $N = 2d$. Let $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j_x^N X \notin B_{mo}(d)$. Then, if $z : [0,T] \to T^*M$ is a singular X-bi-extremal and if the singular X-trajectory $\pi \circ z : [0,T] \to M$ is non-trivial and independent, then $z : [0,T] \to T^*M$ is of minimal order.*

Proof. Since rank $G(z(t))$ is even, by Lemma 6.8 if m is odd, then $m-2$ can be replace with $m-1$ in the statement of Lemma 6.8:
If $m\,(2 \leqq m \leqq n)$ is odd, if $z : [0,T] \to T^*M$ is a singular X-bi-extremal and if $\pi \circ z : [0,T] \to M$ is non-trivial and independent, then

$$\text{rank } G(z(t)) = m-1 \ \text{ for a.e.}\, t \in [0,T].$$

On the other hand, by Lemma 6.9, the following immediately holds:
If $m\,(2 \leqq m \leqq n)$ is even, if $z : [0,T] \to T^*M$ is a singular X-bi-extremal and if $\pi \circ z : [0,T] \to M$ is non-trivial and independent, then

$$\text{rank } \hat{G}(z(t)) = m-1 \ \text{ for a.e.}\, t \in [0,T].$$

Therefore, these imply that for every integer $m\,(2 \leqq m \leqq n)$, z is of minimal order (see Definition 4.2). □

§ 6.3. Codimension of bad set.

Let d be an integer and $N = 2d$. Let $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$ be the m-tuple product space of polynomial vector fields of degree $\leqq N$ over $\mathbb{R}^n$.

In this section, we compute the codimension of the bad set of the closure $\overline{B_{mo}(d)}$ in $J^N(\mathrm{VF}(M))^m$. In order to prove this Lemma 6.11, we constuct the typical fiber $G(d) \subset \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$ of $B_{mo}(d)$. $G(d)$ and its closure $\overline{G(d)}$ are semi-algebraic for d. In particular, dimensions of $G(d), \overline{G(d)}$ are well-defined. By using the codimension of $\overline{G(d)}$ in $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$, we show Lemma 6.11:

Lemma 6.11. $\operatorname{codim}(\overline{B_{mo}(d)}, J^N(\mathrm{VF}(M))^m) \geqq d - n.$

Proof. We describe only the outline of the proof of Lemma 6.11 because this bad set $B_{mo}(d)$ is the completely same bad set $B_{mo}(d)$ defined as 3.1.2 in [3]:

Step 1: Construct the typical fiber $G(d)$ of $B_{mo}(d)$: The typical fiber $G(d) \subset \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$ is constructed by the union of $G^0(d)$ and $G^1(d)$. Note that $G^0(d), G^1(d)$ are semi-algebraic sets for d. Therefore $G(d) = G^0(d) \cup G^1(d)$ and its closure $\overline{G(d)}$ are semi-algebraic also. In particular dimensions of $G^0(d), G^1(d), G(d), \overline{G(d)}$ are well-defined and we have that

$$\dim(\overline{G(d)}) = \dim(G(d)) = \max\{\dim(G^0(d)), \dim(G^1(d))\}.$$

Moreover we have that the dimensions of $B_{mo}(d)$ and $\overline{B_{mo}(d)}$ are well-defined and that they are equal.

The definition of $G^0(d), G^1(d) \subset \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$ is below:

(1) Construction of the trivial fiber $G^0(d)$ of $\hat{B}^0_{mo}(d)$: If $m = 2$, then $G^0(d) = \emptyset$. If $m \geqq 3$, then $G^0(d)$ is the canonical projection of $G^0(d; T^*_0\mathbb{R}^n)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. $G^0(d; T^*_0\mathbb{R}^n)$ is defined by the set of $(Q, p) = (Q_1, \cdots, Q_m, p) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^m$ such that there exist $\sigma \in \mathfrak{S}_m$, an even integer $r\,(r \leqq m-3)$, and $\overline{s} \in N_{q,d}$ with $q(0 \leqq q < m)$ such that (Q, p) satisfies the conditions 1) to 4):
1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;
2). $\Delta^r_0(\sigma, z_0) \neq 0$;
3). for every integer $i\,(0 \leqq i \leqq q)$, $\Delta^{r,r+1,\cdots,r+i}_{0,s_1,\cdots,s_i}(\sigma, z_0) \neq 0$.
4). (a) for every integers $i\,(0 \leqq i \leqq q)$, $k\,(r+i \leqq k \leqq m)$, and $s\,(1 \leqq s \leqq s_i - 1)$,

$$\Delta^{r,r+1,\cdots,r+i-1,k}_{0,s_1,\cdots,s_i-1,s}(\sigma, z_0) = 0;$$

(b) for every integer $k\,(r+p+1 \leqq k \leqq m)$ and $s\,(1 \leqq s \leqq d + q - (s_1 + \cdots + s_p))$,

$$\Delta^{r,r+1,\cdots,r+q,k}_{0,s_1,\cdots,s_q,s}(\sigma, z_0) = 0,$$

where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0, p)$.
(2) Construction of the trivial fiber $G^1(d)$ of $\hat{B}^1_{mo}(d)$: In m is an odd integer, then $G^1(d) = \emptyset$. If m is an even integer, then $G^1(d)$ is the canonical projection of $G^1(d; T^*_0\mathbb{R}^n)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. $G^1(d; T^*_0\mathbb{R}^n)$ is defined by the set of $(Q, p) = (Q_1, \cdots, Q_m, p) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^m$ such that there exists a positive integer $s_1\,(1 \leqq s_1 \leqq d)$ and $\sigma \in \mathfrak{S}_m$ such that (Q, p) satisfies the conditions 1) to 4):
1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;
2). $\Delta^{m-2}_0(\sigma, z_0) \neq 0$;
3). if $s_1 < d$, then $\delta^{m-1}_{s_1}(\sigma, z_0) \neq 0$;
4). (a) for every integer $k \in \{m - 1.m\}$ and $s\,(0 \leqq s \leqq s_1 - 1)$, $\delta^k_s(\sigma, z_0) = 0$;
(b) for every integer $s\,(1 \leqq s \leqq d - s_1), \delta_{s_1,s}(\sigma, z_0) = 0$,
where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0, p)$.

Step 2: Construct the two mappings $\phi^0(\sigma, r, \overline{s}), \phi^1(\sigma, s_1) : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n \to \mathbb{R}^d$: **(1) Construction of $\phi^0(\sigma, r, \overline{s})$** : Let $\sigma \in \mathfrak{S}_m$, $r\,(0 \leqq r \leqq m-3)$ be an even integer, and $\overline{s} = (0, s_1, \cdots, s_q) \in N_{q,d}$ with $0 \leqq q < m$. Then, we define the mapping $\phi^0(\sigma, r, \overline{s}) : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q, p) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$,

$$\phi^0(\sigma, r, \overline{s})(Q, p) := \begin{cases} \Delta^{r,r+1,\cdots,r+i-1,r+i}_{0,s_1,\cdots,s_{i-1},\overline{s}}(Q)(\sigma, z_0) & i = 1, \cdots, q, \text{and } \overline{s} = 1, \cdots, s_i - 1, \\ \Delta^{r,r+1,\cdots,r+q,r+q+1}_{0,s_1,\cdots,s_q,\overline{s}}(Q)(\sigma, z_0) & \overline{s} = 1, 2, \cdots, d + q - (s_1 + \cdots + s_q), \end{cases}$$

where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0, p)$, and $\Delta^{r,r+1,\cdots,r+i-1,r+i}_{0,s_1,\cdots,s_{i-1},\overline{s}}(Q)$, $\Delta^{r,r+1,\cdots,r+q,r+q+1}_{0,s_1,\cdots,s_q,\overline{s}}(Q)$ are the elementary determinants associated to Q.
(2) Construction of $\phi^1(\sigma, s_1)$: Let $\sigma \in \mathfrak{S}_m$ and $s_1\,(1 \leqq s_1 \leqq d)$. Then, we define the

mapping $\phi^1(\sigma, s_1) : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q,p) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$,

$$\phi^1(\sigma, s_1)(Q,p) := \begin{cases} \delta^m_s(Q)(\sigma, z_0), \ s = 0, 1 \cdots, s_1 - 1; \\ \delta_{s_1,s}(Q)(\sigma, z_0), \ s = 1, 2 \cdots, d - s_1, \end{cases}$$

where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0,p)$, and $\delta^m_s(Q), \delta_{s_1,s}(Q)$ are the elementary determinants associated to Q.

Step 3: Construct the two open sets $T^0_{\sigma,r,\overline{s}}, T^1_{\sigma,s_1} \subset \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$:

(1) Construction of $T^0_{\sigma,r,\overline{s}}$: Let $\sigma \in \mathfrak{S}_m$, $r\,(0 \leqq r \leqq m-3)$ be an even integer, and $\overline{s} = (0, s_1, \cdots, s_q) \in N_{q,d}$ with $0 \leqq q < m$. Then, $T^0_{\sigma,r,\overline{s}}$ is defined by the set of $(Q,p) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$ such that (Q,p) satisfies the conditions 1) and 2):

1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;

2). for every integer $i\,(0 \leqq i \leqq p)$, $\Delta^{r,r+1,\cdots,r+i}_{0,s_1,\cdots,s_i}(\sigma, z_0) \neq 0$,

where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0,p)$.

(2) Construction of T^1_{σ,s_1} : Let $\sigma \in \mathfrak{S}_m$ and $s_1\,(1 \leqq s_1 \leqq d)$. Then, T^1_{σ,s_1} is defined by the set of $(Q,p) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$ such that (Q,p) satisfies the conditions 1) to 3):

1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;

2). $\Delta^{m-2}_0(\sigma, z_0) \neq 0$;

3). if s_1 is $s_1 < d$, then $\delta^{m-1}_{s_1}(\sigma, z_0) \neq 0$,

where z_0 is the elements of $T^*_0\mathbb{R}^n$ given in coordinates by $(0,p)$.

Then, $T^0_{\sigma,r,\overline{s}}$ and T^1_{σ,s_1} are open subsets of $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \times \mathbb{R}^n$.

Step 4: The followings immediately hold:

(1) If $m \geqq 3$, then $G_0(d; T^*_0\mathbb{R}^n)$ is the union of the kernels of the restriction to $T^0_{\sigma,r,\overline{s}}$ of the mapping $\phi^0(\sigma, r, \overline{s})$ with $\sigma \in \mathfrak{S}_m$, even integers $r\,(0 \leqq r \leqq m-3)$, and $\overline{s} \in N_{q,d}$ with $0 \leqq q < m$.

(2) If $m \geqq 2$ is even, then $G^1(d; T^*_0\mathbb{R}^n)$ is the union of the kernels of the restriction to T^1_{σ,s_1} of the mapping $\phi^1(\sigma, s_1)$ with $\sigma \in \mathfrak{S}_m$ and $s_1(1 \leqq s_1 \leqq d)$.

Step 5: Let Ω_0 be the set of $Q \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that $Q_1(0), \cdots, Q_m(0)$ are linearly independent. Then, the followings hold (see Lemma 3.5, 3.6 in [3]):

(1) If $m \geqq 3, \sigma \in \mathfrak{S}_m$, and $r\,(0 \leqq r \leqq m-3)$ is an even integer, then the restriction to the intersection $T^0_{\sigma,r,\overline{s}} \cap \hat{V}$ of the mapping $\phi^0(\sigma, r, \overline{s})$ is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0 \times \mathbb{R}^n$.

(2) If $m \geqq 2$ is even, $\sigma \in \mathfrak{S}_m$, and an integer s_1 satisfies $1 \leqq s_1 \leqq d$, then the restriction to the intersection $T^1_{\sigma,s_1} \cap \hat{V}$ of the mapping $\phi^1(\sigma, s_1)$ is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0 \times \mathbb{R}^n$.

Step 6: $\mathrm{codim}(\overline{B_{mo}(d)}, J^N(\mathrm{VF}(M))^m) \geqq d - n$:

Let $k \in \{0,1\}$. By step 4, 5, $\mathrm{codim}(G^k(d; T^*_0\mathbb{R}^n), \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^m) = d$. On the other hand, $G^k(d)$ is the canonical projection of $G^k(d;T_0^*\mathbb{R}^n)$ by $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)\times\mathbb{R}^n\to\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$. Therefore, $\mathrm{codim}(G^k(d),\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n))\geqq d-n$. Since $G(d)=G^0(d)\cup G^1(d)$ is the typical fiber of $B_{mo}(d)$,

$$\mathrm{codim}(B_{mo}(d),J^N(\mathrm{VF}(M))^m)=\mathrm{codim}(G(d),\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n))\geqq d-n.$$

Since the dimensions of $B_{mo}(d)$ and $\overline{B_{mo}(d)}$ are equal,

$$\mathrm{codim}(\overline{B_{mo}(d)},J^N(\mathrm{VF}(M))^m)\geqq d-n.$$

□

§ 6.4. Proof of main theorem 1

Proof of Theorem 6.2 (Main theorem 1) :
Let $d>2n$ be a positive integer. Let $N=2d(>4n)$. Let G_0 be the set of $X\in\mathrm{VF}(M)^m$ such that for any $x\in M$, j_x^NX is not included in the closure of $B_{mo}(d)$ in $J^N(\mathrm{VF}(M))^m$:

$$G_0:=\left\{X\in\mathrm{VF}(M))^m\ \middle|\ j_x^NX\notin\overline{B_{mo}(d)}\text{ for any }x\in M.\right\}.$$

By Lemma 6.11,

$$\mathrm{codim}(\overline{B_{mo}(d)},J^N(\mathrm{VF}(M))^m)\geqq d-n>n.$$

Then G_0 is an open dense subset of $\mathrm{VF}(M)^m$ by using the transversality theorem (see [5]).

Let $X=(X_1,\cdots,X_m)\in G_0$. Then, for any $x\in M$, $j_x^NX\notin B_{mo}(d)$. Therefore, by using Lemma 6.10, if $z:[0,T]\to T^*M$ is a singular X-bi-extremal and if the singular X-trajectory $x=\pi\circ z:[0,T]\to M$ is non-trivial and independent, then $z:[0,T]\to T^*M$ is of minimal order. □

Next we prove the following theorem:

Theorem 6.12. (Main theorem 2) *Suppose $2\leqq m\leqq n$. Then there exists an open dense subset G_1 of $\mathrm{VF}(M)^m$ such that, if $X\in G_1$, if $u:[0,T]\to\Omega$ is a singular X-control for a given initial point $x_0\in M$ and if the corresponding singular trajectory $x:[0,T]\to M,x(0)=x_0$ is non-trivial and independent, then the control u is of corank one.*

Outline of proof : Let $d'\geqq 1$ be an integer. We set $d=2d'-1$ and $N=d+1=2d'$.
[Step1] Construct the "bad set" with respect to corank one, $B_C(d)\subset J^N(\mathrm{VF}(M))^m$. Note that $B_C(d)$ is semi-algebraic and in particular, dimensions of $B_C(d)$ and its closure $\overline{B_C(d)}$ are well-defined.

[Step2] (See Lemma 6.18) Show that, if $X \in \mathrm{VF}(M)^m$ satisfies for any $x \in M, j_x^N X \notin B_{mo}(d') \cup B_C(d)$, if $u : [0,T] \to \Omega$ is a singular X-control for a given initial point $x_0 \in M$ and if the corresponding X-trajectory $x : [0,T] \to M, x(0) = x_0$ is non-trivial and independent, then the control u is of corank one.
[Step3] (See Lemma 6.19) Compute the codimension of $\overline{B_C(d)}$ in $J^N(\mathrm{VF}(M))^m$.
[Step4] (See the subsection §6.8) Let $d' > 2n$. Then $N = 2d' > 4n$ and $d = 2d' - 1 > 3n$. Let G_1 be the set of $X \in \mathrm{VF}(M)^m$ such that, for any $x \in M$, the jets $j_x^N X$ is not included in the closure of $B_{mo}(d') \cup B_C(d)$ in $J^N(\mathrm{VF}(M))^m$. Then, show that G_1 is an open dense subset of $\mathrm{VF}(M)^m$ in the sense of Whitney smooth topology by Thom transversality theorem (for instance see [5]).

In order to show the main theorem 2, we prepare the subsections: §6.5 to §6.8. In §6.5, after preparing some notations, namely, permutated Hamiltonians, extended Lie derivatives and elementary determinants in Definition 6.13, 6.14, 6.15, we define the bad set $B_C(d)$ in Definition 6.5. In §6.6, we show the important Lemma 6.18 prepared for the proof of main theorem 2: if $X \in \mathrm{VF}(M)^m$ satisfies the condition that, for any $x \in M$, $j_x^N X \notin B_{mo}(d)$, then any singular X-control with the non-trivial independent singular X-trajectory is of corank one. In §6.7, we compute the codimension of $B_C(d)$ in $J^N(\mathrm{VF}(M))^m$ in Lemma 6.19. In §6.8, by using these Lemmata 6.18, 6.19, we show the main theorem 2.

§ 6.5. Construction of bad set.

In this section, we construct the semi-algebraic set $B_C(d)$, which is called the bad set with respect to corank one for an integer d.

Let $\mathfrak{S}_m$ be the set of permutations with m elements, and $X = (X_1, \cdots, X_m) \in \mathrm{VF}(M)^m$. Then, we prepare some notations, permutated Hamiltonians, extended Lie derivatives and elementary determinants, in order to define $B_C(d) \subset J^N(\mathrm{VF}(M))^m$ for an integer d.

Definition 6.13. For $k \in \{1, 2\}$, we define the real valued functions $H_{ij}^{[k]}, \Delta_0^{[k],r}, P^{[k]}, \delta_s^{[k],i}$ on $\mathfrak{S}_m \times T^*M \times_M T^*M$, which are called *permutated Hamiltonians*:
Let $\sigma \in \mathfrak{S}_m$ and $(z^{[1]}, z^{[2]}) \in T^*M \times_M T^*M$. Then,

$$
\begin{aligned}
&H_{ij}^{[k]}(\sigma, z^{[1]}, z^{[2]}) = H_{ij}(\sigma, z^{[k]}) \text{ for } i, j \ (1 \leqq i, j \leqq m) \\
&\Delta_0^{[k],r}(\sigma, z^{[1]}, z^{[2]}) = \Delta_0^r(\sigma, z^{[k]}) \text{ for every integer } r \ (0 \leqq r \leqq m-1) \\
&P^{[k]}(\sigma, z^{[1]}, z^{[2]}) := P(z^{[k]}), \text{and } P^{[k],m-2}(\sigma, z^{[1]}, z^{[2]}) := P^{m-2}(z^{[k]}) \\
&\delta_s^{[k],i}(\sigma, z^{[1]}, z^{[2]}) := \delta_s^i(\sigma, z), \text{ for every integer } s \geqq 0 \text{ and } i \in \{m-1, m\},
\end{aligned}
$$

where P (resp. P^{m-2}) is defined by

$$
P(\sigma, z) := P\left((H_{ij}(\sigma, z))_{1 \leqq i < j \leqq m}\right) \ \left(\text{resp. } P^{m-2}(\sigma, z) := P\left((H_{ij}(\sigma, z))_{1 \leqq i < j \leqq m-2}\right)\right),
$$

for $\sigma \in \mathfrak{S}, z \in T^*M$ by using the Pfaffian polynomial of G (resp. G^{m-2}).

Definition 6.14. Let $\mathcal{F}_\bullet$ be the set of $F \in C^\infty(T^*M \times_M T^*M)$ such that there exists $F_1, F_2 \in C^\infty(T^*M)$ such that $F(z^{[1]}, z^{[2]}) = F_1(z^{[1]})F_2(z^{[2]})$ for $(z^{[1]}, z^{[2]}) \in T^*M \times_M T^*M$. Let $\mathcal{F}$ be the set of $F \in C^\infty(T^*M \times_M T^*M)$ such that F is a linear combination of a finite number of elements of $\mathcal{F}_\bullet$ on $\mathbb{R}$. Then, for the Hamiltonian vector field $\overrightarrow{H}$ of $X \in \mathrm{VF}(M)$, we define *an extended Lie derivative* $\mathcal{L}_{\overrightarrow{H}} : \mathcal{F} \to \mathcal{F}$ by the following: For $F \in \mathcal{F}_\bullet$ and $(z^{[1]}, z^{[2]}) \in T^*M \times_M T^*M$,

$$\mathcal{L}_{\overrightarrow{H}}(F)(z^{[1]}, z^{[2]}) := \mathcal{L}_{\overrightarrow{H}}(F_1)(z^{[1]})\, F_2(z^{[2]}) + F_1(z^{[1]})\, \mathcal{L}_{\overrightarrow{H}}(F_2)(z^{[2]}),$$

and extend it by linearly to $\mathcal{F}$. Here, $\mathcal{L}_{\overrightarrow{H}}(F_k)$ is the Lie derivative of F_k with respect to $\overrightarrow{H}$ for $k \in \{1, 2\}$.

Definition 6.15. We inductively define the real valued functions on $\mathfrak{S}_m \times T^*M \times_M T^*M$, which are called elementary determinants: Let $\sigma \in \mathfrak{S}_m$ and $(z^{[1]}, z^{[2]}) \in T^*M \times_M T^*M$. Then

$$\begin{cases} \Theta_{s+1}(\sigma, z^{[1]}, z^{[2]}) := \det \begin{pmatrix} \left(H^{[1]}_{ij}(\sigma, z^{[1]}, z^{[2]})\right)_{\substack{1 \leqq i \leqq m-1,\\ 1 \leqq j \leqq m}} \\ \hline \left(\mathcal{L}_{\overrightarrow{H_j}} \Theta_s(\sigma, z^{[1]}, z^{[2]})\right)_{1 \leqq j \leqq m} \end{pmatrix} \quad (s = 0, 1, \cdots), \\ \Theta_0(\sigma, z^{[1]}, z^{[2]}) := \det \begin{pmatrix} \left(H^{[1]}_{ij}(\sigma, z^{[1]}, z^{[2]})\right)_{\substack{1 \leqq i \leqq m-1,\\ 1 \leqq j \leqq m}} \\ \hline \left(H^{[2]}_{ij}(\sigma, z^{[1]}, z^{[2]})\right)_{1 \leqq j \leqq m} \end{pmatrix}, \end{cases}$$

and

$$\begin{cases} \theta_s(\sigma, z^{[1]}, z^{[2]}) := \det \begin{pmatrix} \left(H^{[1]}_{ij}(\sigma, z^{[1]}, z^{[2]})\right)_{\substack{1 \leqq i \leqq m-1,\\ 1 \leqq j \leqq m}} \\ \hline \left(\{P^{[1]}, H^{[1]}_j\}(\sigma, z^{[1]}, z^{[2]})\right)_{1 \leqq j \leqq m} \\ \hline \left(\mathcal{L}_{\overrightarrow{H_j}} \theta_s(\sigma, z^{[1]}, z^{[2]})\right)_{1 \leqq j \leqq m} \end{pmatrix} \quad (s = 0, 1, \cdots) \\ \theta_0(\sigma, z^{[1]}, z^{[2]}) := \det \begin{pmatrix} \left(H^{[1]}_{ij}(\sigma, z^{[1]}, z^{[2]})\right)_{\substack{1 \leqq i \leqq m-1,\\ 1 \leqq j \leqq m}} \\ \hline \left(\{P^{[1]}, H^{[1]}_j\}(\sigma, z^{[1]}, z^{[2]})\right)_{1 \leqq j \leqq m} \\ \hline \left(\{P^{[12]}, H^{[2]}_j\}(\sigma, z^{[1]}, z^{[2]})\right)_{1 \leqq j \leqq m} \end{pmatrix}, \end{cases}$$

Definition 6.16. Let $N = d+1$. We define the "bad set" with respect to corank one, $B_C(d)$ by the canonical projection of $\hat{B}_C(d)$ by $J^N(\mathrm{VF}(M))^m \times_M T^*M \times_M T^*M \to J^N(\mathrm{VF}(M))^m$. The definition of $\hat{B}_C(d)$ is written in Definition 6.17:

Definition 6.17. We define $\hat{B}_C(d) \subset J^N(\mathrm{VF}(M))^m \times_M T^*M \times_M T^*M$ by the union of the sets $B_C(d,\sigma)$ with $\sigma \in \mathfrak{S}_m$. Here the definition of $B_C(d,\sigma)$ is below:
For $\sigma \in \mathfrak{S}_m$, let $B_C(d,\sigma)$ be the subset of $J^N(\mathrm{VF}(M))^m \times_M T^*M \times_M T^*M$ of all triples $(j_x^N X, z^{[1]}, z^{[2]})$ such that $q = \pi(z^{[1]}) = \pi(z^{[2]})$:

1). $X_1(x), \cdots, X_m(x)$ are linearly independent;

2). $z^{[1]}, z^{[2]}$ are linearly independent;

3). $\begin{cases} \text{if } m \text{ is odd, then } \Delta_0^{m-1}(\sigma, z^{[1]} \neq 0, \\ \text{if } m \text{ is even, then } \delta_1^{m-1}(\sigma, z^{[1]}) P^{m-2}(\sigma, z^{[2]}) \neq 0; \end{cases}$

4). for $s\ (0 \leqq s \leqq d-1)$, $\begin{cases} \text{if } m \text{ is odd, then } \Theta_s(\sigma, z^{[1]}, z^{[2]}) = 0, \\ \text{if } m \text{ is even, then } \theta_s(\sigma, z^{[1]}, z^{[2]}) = 0. \end{cases}$

§ 6.6. Property of singular controls avoidng the bad set.

We show the important Lemma 6.18 prepared for the proof of the main theorem 2.

Lemma 6.18. *Suppose that $2 \leqq m \leqq n$. Let $d' \geqq 1$ be an integer. We set $d = 2d' - 1$ and $N = d + 1 = 2d'$. Let $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j_x^N X \notin B_{mo}(d') \cup B_C(d)$. Then, if $u : [0,T] \to \Omega$ is a singular X-control for a given initial point $x_0 \in M$ and if the corresponding X-trajectory $x : [0,T] \to M, x(0) = x_0$ is non-trivial and independent, then the control u is of corank one.*

Proof. In order to prove by contradiction, assume that, $u : [0,T] \to \Omega$ is a singular X-control for a given initial point $x_0 \in M$ and $x : [0,T] \to M, x(0) = x_0$ is non-trivial and independent but the control u is not of corank one. Then by Proposition 5.2, there exists two bi-extremals $z^{[1]}, z^{[2]} : [0,T] \to T^*M$ with $z^{[1]} \circ \pi = z^{[2]} \circ \pi = x$ such that $z^{[1]}(t)$ and $z^{[2]}(t)$ are linearly independent for every $t \in [0,1]$.

Case 1 We consider the case m is odd. For $k = 1,2$, we denote by $G^{[k]}$ the Goh matrix $G(z^{[k]}) = (H_{ij}(z^{[k]}))_{1 \leqq i,j \leqq m}$. Since $X = (X_1, \cdots, X_m) \in \mathrm{VF}(M)^m$ is for any $x \in M$, $j_x^N X \notin B_{mo}(d')$, by the proof of Lemma 6.8, there exists $I_{m-1} \subset \{1, \cdots, m\}$ with cardinality $m-1$ such that $\det(H_{ij}(z^{[1]}(t)))_{(i,j) \in I_{m-1}^2} \not\equiv 0$ on $[0,T]$. Then, up to a permutation, we may assume that there exist an open subinterval $K \subset [0,T]$ and $\sigma \in \mathfrak{S}_m$ such that

$$\Delta_0^{[1],m-1}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \text{ for every } t \in K.$$

On the other hand, since x is independent, $I_{\mathrm{indep}}(x)$ has positive measure, where $I_{\mathrm{indep}}(x)$ is the compliment of $I_{\mathrm{dep}}(x)$ in $[0,T]$. Then Let $J := K \cap I_{\mathrm{indep}}(x)$.

Now, since for $k = 1, 2$, $H_i(\sigma, z^{[k]}(t)) = 0$ on $[0, T]$, by differentiating both sides, for $i\,(1 \leqq i \leqq m)$,

$$\sum_{j=1}^{m} u_j(t)\, H_{ij}(\sigma, z^{[k]}(t)) = 0 \text{ a.e. } t \in [0, T].$$

Hence, the matrix

$$\begin{pmatrix} H_{11}(\sigma, z^{[1]}(t)) & \cdots & H_{1m}(\sigma, z^{[1]}(t)) \\ \vdots & & \vdots \\ H_{(m-1)1}(\sigma, z^{[1]}(t)) & \cdots & H_{(m-1)m}(\sigma, z^{[1]}(t)) \\ H_{11}(\sigma, z^{[2]}(t)) & \cdots & H_{1m}(\sigma, z^{[2]}(t)) \end{pmatrix}$$

is not invertible. Therefore $\Theta_0(\sigma, z^{[1]}(t), z^{[2]}(t)) \equiv 0$ on $[0, T]$. By differentiating both sides,

$$\sum_{j=1}^{m} u_j(t)\, \mathcal{L}_{\overrightarrow{H_j}} \Theta_0(\sigma, z^{[1]}(t), z^{[2]}(t)) = 0 \text{ a.e.}\, t \in [0, T].$$

This implies $\Theta_1 \equiv 0$. By proceeding similarly, for $k (0 \leqq k \leqq d-1)$.

$$\begin{cases} \Delta_0^{[1], m-1}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \\ \Theta_k(\sigma, z^{[1]}(t), z^{[2]}(t)) \equiv 0 \end{cases} \text{for every } t \in J.$$

This contradicts the assumption that, for any $x \in M$, $j_x^N X \in B_C(d)$.

Case 2 We consider the case m is even. By proofs of Lemma 6.8, 6.9, up to permutation, there exists an subinterval $K \subset [0, T]$ such that

$$\begin{cases} P^{[1], m-2}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \\ \delta_1^{[1], m-1}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \end{cases} \text{for every } t \in K.$$

Then Let $J := K \cap I_{\mathrm{indep}}(x)$.

We show that $P^{[2], m-2}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0$ for every $t \in J$. For $\alpha \in [0, T]$, consider $z^\alpha = (1-\alpha) z^{[1]} + \alpha z^{[2]}$. Since $P^{m-2}(z^{[\alpha]}(t))$ depends continuously on α, for α small enough, $P^{m-2}(z^{[\alpha]}(t)) = 0$ for every $t \in J$. Moreover, the set of singular X-bi-extremals of singular X-trajectory x is a vector space, $z^{[\alpha]}$ is singular X-bi-extremals of x, and for $\alpha > 0$, $z^{[\alpha]}(t)$ and $z^{[1]}(t)$ are linearly independent for every $t \in J$. Then, it suffices to replace $z^{[2]}$ by z^α, for some $\alpha > 0$ small enough.

Similar to the case m odd, for $k\,(0 \leqq k \leqq d-1)$,

$$\begin{cases} \delta_1^{[1], m-1}(\sigma, z^{[1]}(t), z^{[2]}(t))\, P^{[2], m-2}(\sigma, z^{[1]}(t), z^{[2]}(t)) \neq 0 \\ \theta_k(\sigma, z^{[1]}(t), z^{[2]}(t)) \equiv 0 \end{cases} \text{for every } t \in J.$$

This contradicts the assumption that, for any $x \in M$, $j_x^N X \in B_C(d)$.

□

§ 6.7. Codimension of bad set.

Let d be a positive integer and $N = d+1$. Let $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$ be the m-tuple product space of polynomial vector fields of degree $\leqq N$ over $\mathbb{R}^n$.

In this section, we compute the codimension of the closure of the bad set $B_C(d)$ in $J^N(\mathrm{VF}(M))^m$. In order to prove this Lemma 6.19, we construct the typical fiber $G_C(d) \subset \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$ of $B_C(d)$. $G_C(d)$ and its closure $\overline{G_C(d)}$ are semi-algebraic for d. In particular, dimensions of $G_C(d), \overline{G_C(d)}$ are well-defined. By using the codimension of $\overline{G_C(d)}$ in $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$, we show Lemma 6.19:

Lemma 6.19. $\mathrm{codim}(\overline{B_C(d)}, J^N(\mathrm{VF}(M))^m) \geqq d - 2n.$

Proof. We describe only the outline of the proof of Lemma 6.19 because this bad set $B_C(d)$ is the completely same bad set $B_C(d)$ defined as 3.1.2 in [3]:

Step 1: Construct the typical fiber $G_C(d)$ of $B_C(d)$:

Typical fiber $G_C(d)$ of $B_C(d)$ is the canonical projection of $G_C(d; T_0^*\mathbb{R}^n \times T_0^*\mathbb{R}^n)$ by $\mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n)$. $G_C(d; T_0^*\mathbb{R}^n \times T_0^*\mathbb{R}^n)$ is defined by the set of $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that there exists $\sigma \in \mathfrak{S}_k$ such that $(Q, p^{[1]}, p^{[2]})$ satisfies the conditions 1) to 4):

1). $Q_1(0), \cdots, Q_m(0)$ are linearly independent;

2). $p^{[1]}, p^{[2]}$ are linearly independent;

3). (a) if m is odd, then $\Delta_0^{m-1}(\sigma, z_0^{[1]}) \neq 0$,

(b) if m is even, then $\delta_1^{m-1}(\sigma, z_0^{[1]}) P^{m-2}(\sigma, z_0^{[2]})$;

4). (a) if m is odd, then $\Theta_s(\sigma, z_0^{[1]}, z_0^{[2]})$ for every integer s $(0 \leqq s \leqq d-1)$,

(b) if m is even, then $\theta_s(\sigma, z_0^{[1]}, z_0^{[2]})$ for every integer s $(0 \leqq s \leqq d-1)$,

where $z_0^{[1]}, z_1^{[2]}$ are the elements of $T_0^*\mathbb{R}^n$ given in coordinates by $(0, p_1), (0, p_2)$.

Step 2: Construct the mapping $\phi_\sigma : \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$:

Let $\sigma \in \mathfrak{S}_k$. Then construct the mapping $\phi_{\sigma,V}$ by dividing two cases of m being odd and even:

Case 1 If m be an odd integer, then we define $\phi_\sigma : \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}_{\mathrm{poly}}^{\boldsymbol{N}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\phi_{\sigma,V}(Q, p^{[1]}, p^{[2]}) := \left(\Theta_s(Q)(\sigma, z_0^{[1]}, z_0^{[2]})\right)_{0 \leqq s \leqq d-1},$$

where $z_0^{[1]}, z_0^{[2]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p^{[1]}, p^{[2]})$ and $\Theta_s(Q)$ is the elementary determinants associated to Q.

Case 2 If m be an even integer, then we define $\phi_\sigma : \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^d$ by for $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$,

$$\phi_{\sigma,V}(Q, p^{[1]}, p^{[2]}) := \Big(\theta_s(\sigma, z_0^{[1]}, z_0^{[2]})\Big)_{0 \leqq s \leqq d-1},$$

where $z_0^{[1]}, z_0^{[2]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p^{[1]}, p^{[2]})$ and $\theta_s(Q)$ is the elementary determinants associated to Q.

Step 3: Construct the open subset $V_\sigma \subset \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$:

Let $\sigma \in \mathfrak{S}$. Then, construct V_σ by dividing two cases of m being odd and even:

Case 1 If m is an odd integer, then V_σ is defined by the set of $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that $(Q, p^{[1]}, p^{[2]})$ satisfies the conditions 1) to 2):

1). $p^{[1]}, p^{[2]}$ are linearly independent;

2). $\Delta_0^{m-1}(\sigma, z_0^{[1]}) \neq 0$.

Case 2 If m is an even integer, then V_σ is defined by the set of $(Q, p^{[1]}, p^{[2]}) \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$ such that

1). $p^{[1]}, p^{[2]}$ are linearly independent;

2). $\delta_1^{m-1}(\sigma, z_0^{[1]}) P^{m-2}(\sigma, z_0^{[2]}) \neq 0$,

where $z_0^{[1]}, z_0^{[2]}$ are the elements of $T^*\mathbb{R}^n$ given in coordinates by $(0, p^{[1]}, p^{[2]})$.

Then, V_σ is an open subset of $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n$.

Step 4: $G_C(d; T_0^*\mathbb{R}^n \times T_0^*\mathbb{R}^n)$ is the union of kernels of restriction to V_σ of the mapping ϕ_σ with $\sigma \in \mathfrak{S}_m$.

Step 5: Let Ω_0 be the set of $Q \in \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$ such that $Q_1(0), \cdots, Q_m(0)$ are linearly independent. It is well-known that the local coordinate systems on Ω_0 can be constructed (see Coordinate systems in [2],[3]).Then, if $\sigma \in \mathfrak{S}_m$, then the restriction to the intersection $V_\sigma \cap \hat{V}$ of the mapping ϕ_σ is a submersion for every coordinate neighborhood $\hat{V}$ of $\Omega_0 \times \mathbb{R}^n \times \mathbb{R}^n$. (The proof is Lemma 4.2. in [3].)

Step 6: $\mathrm{codim}(B_C(d), J^N(\mathrm{VF}(M))^m) \geqq d - 2n$:

By Step 4,5, $\mathrm{codim}(G_C(d; T_0^*\mathbb{R}^n \times T_0^*\mathbb{R}^n), \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n) = d$. On the other hand, $G_C(d)$ is the canonical projection of $G_C(d; T_0^*\mathbb{R}^n \times T_0^*\mathbb{R}^n)$ by $\mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)$. Therefore, $\mathrm{codim}(G_C(d), \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n) \geqq d - 2n$. Since $G_C(d)$ is the typical fiber of $B_C(d)$,

$$\mathrm{codim}(B_C(d), J^N(\mathrm{VF}(M)) = \mathrm{codim}(G_C(d), \mathrm{VF}^{\boldsymbol{N}}_{\mathrm{poly}}(\mathbb{R}^n)) \geqq d - 2n.$$

Since the dimensions of $B_C(d)$ and $\overline{B_C(d)}$ are equal,

$$\mathrm{codim}(\overline{B_C(d)}, J^N(\mathrm{VF}(M)) \geqq d - 2n.$$

□

§ 6.8. Proof of main theorem 2

Proof of Theorem 6.12 (Main theorem 2):
Let $d' > 2n$ be an integer. We set $d = 2d' - 1$ and $N = d + 1 = 2d'(> 4n)$.

Then, let G_1 be the set of $X \in \mathrm{VF}(M)^m$ such that for any $x \in M$, $j_x^N X$ is not included in the closure of $B_{mo}(d') \cup B_C(d)$ in $J^N(\mathrm{VF}(M))^m$:

$$G_1 := \left\{ X \in \mathrm{VF}(M))^m \;\middle|\; j_x^N X \notin \overline{B_{mo}(d') \cup B_C(d)} \text{ for any } x \in M. \right\}.$$

By Lemmata 6.11, 6.19,

$$\operatorname{codim}(\overline{B_{mo}(d') \cup B_C(d)}, J^N(\mathrm{VF}(M))^m) \geqq \min\{d' - n, d - 2n\} > n.$$

Then G_1 is an open dense subset of $\mathrm{VF}(M)^m$ by using transversality theorem (see [5]).

Let $X = (X_1, \cdots, X_m) \in G_1$. Then, for any $x \in M$, $j_x^N X \notin B_{mo}(d') \cup B_C(d)$. Therefore, by using Lemma 6.18, if $u : [0, T] \to \Omega$ is a singular X-control for a given initial point $x_0 \in M$ and if the corresponding X-trajectory $x : [0, T] \to M, x(0) = x_0$ is non-trivial and independent, then the control u is of corank one. □

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