RIMS Kôkyûroku Bessatsu
B56 (2016), 11–30

A remark on non-existence results for the semi-linear damped Klein-Gordon equations

By

Masahiro Ikeda* and Takahisa Inui**

Abstract

We consider the Cauchy problem for the semi-linear damped Klein-Gordon equations with a p-th order power nonlinearity in the Euclidean space $\mathbb{R}^d$. It is well-known that the equation is locally well-posed in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ in the energy-subcritical or critical case $1 < p \leq p_1$ for $d \geq 3$ or $1 < p$ for $d = 1, 2$, where $p_1 := 1 + 4/(d-2)$. In the present paper, we give a large data blow-up of energy solution in this case, i.e. $1 < p \leq p_1$ for $d \geq 3$ or $1 < p$ for $d = 1, 2$ (Theorem 2.4). Moreover, we also prove a non-existence of a local weak solution (Definition 2.2) in the energy-supercritical case $p > p_1$ (Theorem 2.7). Our proofs are based on a invariant of a test-function method.

§ 1. Introduction

In the present paper, we study the Cauchy problem for the semi-linear damped Klein-Gordon equations

$$\text{(NLKG)} \qquad \begin{cases} \partial_t^2 u - \Delta u + m^2 u + a\partial_t u = F(u), & (t,x) \in [0,T) \times \mathbb{R}^d, \\ u(0,x) = \lambda f(x),\ \partial_t u(0,x) = \lambda g(x), & x \in \mathbb{R}^d, \end{cases}$$

where $T > 0$, $d \in \mathbb{N}$ is the space dimension, $m, a \in \mathbb{R}$ are constants, $u = u(t,x)$ is a $\mathbb{C}$-valued unknown function of (t,x), $F = F(z)$ denotes a nonlinearity, $\lambda \geq 0$ is a non-negative parameter, and $f = f(x), g = g(x)$ are $\mathbb{C}$-valued prescribed functions of $x \in \mathbb{R}^d$.

Received September 29, 2014. Revised May 11, 2015.
2010 Mathematics Subject Classification(s): 35L15, 35L71
Key Words: *energy-critical, blow-up, large data, non-existence of local solution, energy-supercritical.*
Supported by JSPS Fellows 26·8884
*Department of Mathematics, Graduate School of Science, Kyoto University. Kitashirakawa-Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan
e-mail: `mikeda@math.kyoto-u.ac.jp`
**Department of Mathematics, Graduate School of Science, Kyoto University. Kitashirakawa-Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan
e-mail: `inui@math.kyoto-u.ac.jp`

When $m = a = 0$, the equation is called nonlinear wave equation, when $m > 0, a = 0$, (NLKG) is called nonlinear Klein-Gordon equation and when $m = 0, a > 0$, the equation is called nonlinear damped wave equation. These equations often appear to describe various physical phenomena. Our main aim in this paper is to prove a large data blow-up of energy solution to the equation (NLKG) with $F(z) = \pm|z|^p$ and $1 < p \leq p_1 := 1+\frac{4}{d-2}$ (if $d \geq 3$) or $1 < p$ (if $d = 1, 2$) and to show a non-existence result of local weak solution for the same equation with $p > p_1$ for arbitrary $\lambda > 0$ and a suitable function (f, g) in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$.

Throughout this paper, we assume that the nonlinearity F is continuously differentiable in the sense of functions in real numbers and satisfies the estimates

$$\text{(A)} \quad \begin{cases} F(z) = O(|z|^p), \quad F_z(z),\ F_{\bar{z}}(z) = O(|z|^{p-1}), \\ F_z(z) - F_z(w),\ F_{\bar{z}}(z) - F_{\bar{z}}(w) = O(|z-w|^{\min\{p-1,1\}}(|z|+|w|)^{\max\{0,p-2\}}), \end{cases}$$

for some $p \geq 1$, where F_z and $F_{\bar{z}}$ are the complex derivatives

$$F_z := \frac{1}{2}\left(\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}\right), \quad F_{\bar{z}} := \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

The typical examples which satisfy (A) are the following power type nonlinearities:

$$F(z) = \pm|z|^{p-1}z, \quad \text{or} \quad \pm|z|^p.$$

In the massless and undamped case, i.e. $m = a = 0$, the equation (NLKG) with such p-th order power nonlinearities is invariant under the scale transformation

$$u(t,x) \mapsto u_\gamma(t,x) := \gamma^{\frac{2}{p-1}}u(\gamma t, \gamma x), \quad \text{for } \gamma > 0.$$

Moreover, the direct computation gives

$$\|u_\gamma(0,\cdot)\|_{\dot{H}^1} = \gamma^{\frac{2}{p-1}-\frac{d-2}{2}}\|u(0,\cdot)\|_{\dot{H}^1},$$

where $\dot{H}^1$ denotes the homogeneous Sobolev space. Thus if the order p satisfies

$$\frac{2}{p-1} - \frac{d-2}{2} = 0 \ \Leftrightarrow\ p = p_1 := 1 + \frac{4}{d-2},$$

then the $\dot{H}^1$-norm of the initial data is also invariant. Therefore, the case $p = p_1$ is called H^1-critical case. And the case of $p < p_1$ (resp. $p > p_1$) is called H^1-subcritical case (resp. H^1-supercritical case). Moreover, if u is a real-valued solution to (NLKG), then the following energy identity holds formally:

$$\frac{d}{dt}\{\|\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + m^2\|u(t)\|_{L^2}^2 + G(u(t,x))\} = -a\|\partial_t u(t)\|_{L^2}^2,$$

where $G(u) := \int_0^u F(s)ds$. From the above observations, it is natural to consider the equation (NLKG) in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. It is well known that if the nonlinearity F satisfies (A), then the equation (NLKG) is locally well-posed in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ in the H^1-subcritical or critical case $(1 < p \le p_1)$ (see papers [5, 6, 14, 15] for example, and textbooks [27, 26]). Especially, the final result of the local well-posedness to (NLKG) was obtained in [14]. Recently, global behaviors of solution to (NLKG) with the gauge invariant type nonlinearity $\pm|z|^{p-1}z$ has been extensively studied (see [24, 25, 17, 21, 1]). However the global well-posedness to (NLKG) with general nonlinearities in the subcritical and critical case has been open as well as the local well-posedness in the super critical case. We address these problems and give an answer to them partially in the present paper in the case of $F(z) = \pm|z|^p$. More precisely, as was mentioned before, we prove a large data blow-up result to the equation (NLKG) with $F(z) = \pm|z|^p$ in the subcritical or critical case $1 < p \le p_1$. Moreover we give a non-existence result for local weak solutions in the supercritical case $p > p_1$. Recently, the similar results for the nonlinear Schrödinger equation with $F(z) = |z|^p$ were obtained in [9].

§ 2. Notations and Main results

In order to state our main results, we define energy solution and weak solution to (NLKG) and also introduce several notations.

As was stated in the introduction, we are interested in energy solution to (NLKG), which is defined as follows.

Definition 2.1 (Energy solution, its lifespan). We say that a function $u : [0,T)\times \mathbb{R}^d \to \mathbb{C}$ is energy solution or H^1-solution to (NLKG), if u lies in the class

$$E(T) := C([0,T); H^1(\mathbb{R}^d)) \cap C^1([0,T); L^2(\mathbb{R}^d))$$

and obeys the Duhamel formula in H^1-sense

$$u(t,x) = \lambda(\partial_t + a)D(t)f(x) + \lambda D(t)g(x) + \int_0^t D(t-s)F(u(s,x))ds,$$

where $D(t) := e^{-\frac{a}{2}t}\mathcal{F}^{-1}L(t,\xi)\mathcal{F}$ is the free damped Klein-Gordon evolution group and

$$L(t,\xi) := \begin{cases} \frac{\sin(t\sqrt{m^2-a^2/4+|\xi|^2})}{\sqrt{m^2-a^2/4+|\xi|^2}}, & \text{if } m^2 - a^2/4 + |\xi|^2 > 0, \\ \frac{\sinh(t\sqrt{-m^2+a^2/4-|\xi|^2})}{\sqrt{-m^2+a^2/4-|\xi|^2}}, & \text{if } m^2 - a^2/4 + |\xi|^2 < 0. \end{cases}$$

For the convenience of readers, we give a proof of existence and uniqueness of the local energy solution in Theorem 5.1 in Appendix of this paper. We also define lifespan of the solution as

$$T(\lambda) := \sup\{T \in (0,\infty]; \text{there exists a unique solution } u \text{ to (NLKG) on } [0,T)\}.$$

In this paper, we reduce our problems into whether there exists a weak solution to (NLKG) or not, which is defined as follows.

Definition 2.2. We say that u is a weak solution to (NLKG) on $[0,T)$, if u belongs to $L^p_{loc}([0,T)\times\mathbb{R}^d)$ and the following identity

$$\begin{aligned}(2.1)\qquad &\int_{[0,T)\times\mathbb{R}^d} u(t,x)\{(\partial_t^2\psi)(t,x)-(\Delta\psi)(t,x)+m^2\psi(t,x)-a(\partial_t\psi)(t,x)\}dxdt\\ &=\lambda\int_{\mathbb{R}^d} f(x)(\partial_t\psi)(0,x)dx+\lambda\int_{\mathbb{R}^d}(af(x)+g(x))\psi(0,x)dx\\ &\quad+\int_{[0,T)\times\mathbb{R}^d} F(u(t,x))\psi(t,x)dxdt\end{aligned}$$

holds for any $\psi\in C_0^\infty([0,T)\times\mathbb{R}^d)$. We also define lifespan of the weak solution as

$$T_w(\lambda):=\sup\{T\in(0,\infty];\text{there exists a unique weak solution } u \text{ to (NLKG) on } [0,T)\}.$$

Now we state our main results in the present paper.

Result 1. Large data blow-up in energy-critical or energy-subcritical

First we state a non-existence result for the global weak solution for $p>1$ for a suitable $(f,g)\in(L^1_{loc}(\mathbb{R}^d))^2$ and large $\lambda\gg 1$.

Proposition 2.3 (Non-existence of the global weak solution for $p>1$ and large λ)**.** *Let $m,a\in\mathbb{R}$, $d\in\mathbb{N}$, $p>1$, $F(z)=|z|^p$ and $(f,g)\in(L^1_{loc}(\mathbb{R}^d))^2$. We assume that the function (f,g) satisfies*

$$(2.2)\qquad \Re(af+g)(x)\geq\begin{cases}|x|^{-k}, & (|x|\leq 1),\\ 0, & (|x|>1),\end{cases}$$

where $k<\min\left(d,\frac{p+1}{p-1}\right)$. Then there exist constants $\lambda_0>0$ and $C>0$ depending only on $d,p,k,|m|,|a|$ such that for any $\lambda>\lambda_0$,

$$(2.3)\qquad T_w(\lambda)\leq C\lambda^{-1/\sigma},$$

where $\sigma:=\frac{p+1}{p-1}-k$ (>0).

Next we state one of our main results, which gives a large data blow-up result for the H^1-solution in the H^1-subcritical or critical case, i.e. $1<p\leq p_1$ for $d\geq 3$ or $1<p$ for $d\geq 1,2$.

Theorem 2.4 (Large data blow-up in the H^1-critical or subcritical case)**.** *Let $m,a\in\mathbb{R}$, $d\in\mathbb{N}$, $1<p\leq p_1$ for $d\geq 3$ or $1<p$ for $d=1,2$, $F(z)=|z|^p$ and $(f,g)\in H^1(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$. We assume that the function (f,g) satisfies (2.2) with $k<\frac{d}{2}$.*

Then there exist constants $\lambda_0 > 0$ and $C > 0$ depending only on $d, p, k, |m|, |a|$ such that for any $\lambda > \lambda_0$,

$$T(\lambda) \le C\lambda^{-1/\sigma}, \tag{2.4}$$

where $\sigma := \frac{p+1}{p-1} - k$ (> 0). Moreover, the norm of the solution blows up at $t = T(\lambda)$:

$$\liminf_{t\to T(\lambda)-0} \|(u(t), \partial_t u(t))\|_{H^1\times L^2} = \infty, \quad \text{if } 1 < p < p_1 \text{ for } d \ge 3 \text{ or } 1 < p \text{ for } d = 1, 2,$$

$$\|u\|_{L^{\frac{2(d+1)}{d-2}}_{t,x}([0,T(\lambda))\times\mathbb{R}^d)} = \infty, \quad \text{if } p = p_1.$$

Example 2.5. It should be verified whether there exists the function (f, g) satisfying the assumptions in Theorem 2.4, i.e. $(f, g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ and (f, g) satisfies the estimate (2.2) with $k < \frac{d}{2}$. Indeed, we may choose

$$f(x) := 0 \quad \text{and} \quad g(x) := \begin{cases} |x|^{-k}, & \text{if } |x| \le 1, \\ \text{smooth}, & \text{if } 1 < |x| < 2, \\ 0, & \text{if } |x| \ge 2, \end{cases} \tag{2.5}$$

where $k < \frac{d}{2}$.

Remark 2.1. Lower estimate of lifespan $T(\lambda)$ to the equation (NLKG) can be obtained. However we do not pursue the optimality of the lifespan in this paper (see Remark 5.3).

Result 2. Non-existence of local weak solution in the energy-supercritical

First we prepare a non-existence result for the local weak solution for $p > 1 + \frac{2}{d-1}$ for a suitable $(f, g) \in (L^1_{loc}(\mathbb{R}^d))^2$ and arbitrary $\lambda > 0$.

Proposition 2.6 (Non-existence of local weak solution in $p > 1 + \frac{2}{d-1}$)**.** *Let $m, a \in \mathbb{R}$, $d \ge 2$, $p > 1 + \frac{2}{d-1}$, $F(z) = |z|^p$, $\lambda \ge 0$ and $(f, g) \in (L^1_{loc}(\mathbb{R}^d))^2$. We assume that the function (f, g) satisfies the estimate (2.2) with $\frac{p+1}{p-1} < k < d$. Then if there exist $T > 0$ and a weak-solution u to (NLKG) on $[0, T)$, then $\lambda = 0$.*

Next we give a non-existence result for local weak solution in the energy-supercritical case $p > p_1$ for a suitable (f, g) in the energy space for arbitrary $\lambda > 0$.

Theorem 2.7 (Non-existence of local weak solution in energy-supercritical)**.** *Let $m, a \in \mathbb{R}$, $d \ge 3$, $p > p_1$, $F(z) = |z|^p$, $\lambda \ge 0$ and $(f, g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. We assume that the function (f, g) satisfies (2.2) with $\frac{p+1}{p-1} < k < \frac{d}{2}$. Then, if there exist $T > 0$ and a weak-solution u to (NLKG) on $[0, T)$, then $\lambda = 0$.*

Example 2.8. It should be also verified whether there exists the function (f, g) satisfying the assumptions in Theorem 2.7. Indeed, we may choose the same function (f, g) which appears in Example 2.5.

Remark 2.2. We note that Theorem 2.4 and 2.7 also hold in the massless case $m = 0$ or undamped case $a = 0$, which implies that the mass term $m^2 u$ and the time derivative term $a\partial_t u$ in the equation (NLKG) do not give any effect on the behavior of solutions under the conditions of the theorems, i.e. (NLKG) with $f(z) = |z|^p$ and "$1 < p \leq p_1$ for $d \geq 3$ or $1 < p$ for $d = 1, 2$ and large λ" (Theorem 2.4) or $p > p_1$ (Theorem 2.7).

Remark 2.3. The similar statements as the above results also hold for the negative time direction if the left hand side of (2.2) is replaced by $-\Re(af + g)(x)$.

Next, we explain the strategy of the proof of the theorems. We use a variant of the test-function method used in the papers [29, 30, 22, 13, 23, 28, 12, 11, 10, 8] etc. In the papers, the method was applied to some nonlinear evolution equations (parabolic equations, damped wave equations and Schödinger equations), to prove a small data blow-up result in a subcritical case. Different from their situations, we have to prove a large data blow-up result in the critical case or to treat the supercritical case. Thus we have to modify the argument of their method. The major difference from the previous works comes from how to choose the initial function (f, g). In our result, we take the initial data (f, g) such as $\Re(af + g)$ has a singularity at the origin $x = 0$ different from the previous works (see also [20] for such choice of the initial data). We also note that the test-function method does not seem to work to prove a small data blow-up result for the Klein-Gordon equation because of the presence of the mass term $m^2 u$. With regard to this method, our theorems imply that the method is effective, in order to prove a large data blow-up result or to treat the supercritical case even for the massive Klein-Gordon equation $(m > 0)$.

At the end of this section, we introduce several notations throughout this paper. For $1 \leq p \leq \infty$, we define the Lebesgue space as $L^p = L^p(\mathbb{R}^d)$, with the norm $\|f\|_{L^p} := \left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{1/p}$ if $1 \leq p < \infty$ and $\|f\|_{L^\infty} := \text{ess.sup}_{x \in \mathbb{R}^d} |f(x)|$. For a time interval I and a Banach space X, we use the time-space Lebesgue space $L_t^p(I; X)$, with the norm $\|u\|_{L_t^p(I;X)} := \|\|u(t)\|_X\|_{L_t^p(I)}$. $L_{loc}^p(I \times \mathbb{R}^d)$ denotes the set of measurable functions $u : I \times \mathbb{R}^d \mapsto \mathbb{C}$ such that for every compact interval $J \subset I \times \mathbb{R}^d$, $u|_J \in L^p(J)$. We define $C^r(I; X)$ as the Banach space whose element is an r-times continuously differentiable mapping from I to X with respect to the topology in X. Let $\mathcal{S}(\mathbb{R}^d)$ be the rapidly decaying function space. For $f \in \mathcal{S}(\mathbb{R}^d)$, we define the Fourier transform of f as

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx,$$

and the inverse Fourier transform of f as

$$\mathcal{F}^{-1}[f](x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\, d\xi,$$

and extend them to $\mathcal{S}'(\mathbb{R}^d)$ by duality. We define the inhomogeneous Sobolev spaces as $W^{s,p}(\mathbb{R}^d)$ with the norm $\|f\|_{W^{s,p}} := \|\mathcal{F}^{-1}[\langle\xi\rangle^s \hat{f}]\|_{L^p_x}$, where $\langle\cdot\rangle := (1+|\cdot|^2)^{1/2}$. We also use $H^s(\mathbb{R}^d) := W^{s,2}(\mathbb{R}^d)$. We denote the homogeneous Sobolev space as $\dot{W}^{s,p}(\mathbb{R}^d)$ with the norm $\|f\|_{\dot{W}^{s,p}} := \|\mathcal{F}^{-1}[|\xi|^s \hat{f}]\|_{L^p_x}$. We also define $\dot{H}^s(\mathbb{R}^d) := \dot{W}^{s,2}(\mathbb{R}^d)$.

§ 3. Integral inequalities via a test-function method

In this section, we derive two useful inequalities (Lemmas 3.1, 3.2 below) by using suitable test-functions.

We take the two functions $\eta = \eta(t) \in C_0^\infty([0,\infty))$, $\phi = \phi(x) \in C_0^\infty(\mathbb{R}^d)$ such as $0 \le \eta, \phi \le 1$ and

$$\eta(t) := \begin{cases} 1 & (0 < t < 1/2), \\ \text{smooth} & (1/2 \le t \le 1), \\ 0 & (t > 1), \end{cases} \qquad \phi(x) := \begin{cases} 1 & (0 < |x| < 1/2), \\ \text{smooth} & (1/2 \le |x| \le 1), \\ 0 & (|x| > 1). \end{cases}$$

For a parameter $\tau > 0$, we also define the time-space function

$$\psi_\tau = \psi_\tau(t,x) := \eta_\tau(t)\phi_\tau(x) := \eta(t/\tau)\phi(x/\tau).$$

We define the open ball of radius $r > 0$ at the origin in $\mathbb{R}^d$ as $B(r) := \{x \in \mathbb{R}^d; |x| < r\}$.

Lemma 3.1. *Let $m, a \in \mathbb{R}$, $d \in \mathbb{N}$, $p > 1$, $F(z) = |z|^p$, $l \in \mathbb{N}$ with $l \ge 2q+1$, $\lambda \ge 0$, $F(z) = |z|^p$, $(f,g) \in (L^1_{loc}(\mathbb{R}^d))^2$ and u be a weak solution of (NLKG) on $[0,T)$. Then there exists a constant $C > 0$ depending only on d, p and l, such that the estimate*

$$\lambda \int_{B(\tau)} \Re(af+g)(x)\phi_\tau^l(x)dx \le C\tau^{d+1}(\tau^{-2q} + |m| + |a|\tau^{-q}), \tag{3.1}$$

holds for any $\tau \in (0,T)$, where q is defined by $q = p/(p-1)$.

Proof of Lemma 3.1. We introduce the two positive functions of $\tau \in (0,T)$

$$I(\tau) := \int_{[0,\tau)\times B(\tau)} |u(t,x)|^p \psi_\tau^l(t,x)dxdt,$$
$$J(\tau) := \int_{B(\tau)} (af(x)+g(x))\phi_\tau^l(x)dx.$$

Since u is a weak solution on $[0,T)$ and $\psi_\tau^l \in C_0^\infty([0,T)\times\mathbb{R}^d)$, by substituting the test-function in Definition 2.2 into ψ_τ^l, using the identity $\{\partial_t(\psi_\tau^l)\}(0,x)=0$ and taking the real part of the obtained equation, we have

$$
\begin{aligned}
(3.2)\qquad I(\tau)+\lambda\Re J(\tau) &= \int_{[0,\tau)\times B(\tau)} (\Re u)\partial_t^2(\psi_\tau^l)dxdt + \int_{[0,\tau)\times B(\tau)} (-\Re u)\Delta(\psi_\tau^l)dxdt \\
&\quad + m\int_{[0,\tau)\times B(\tau)} (\Re u)\psi_\tau^l dxdt - a\int_{[0,\tau)\times B(\tau)} (\Re u)\partial_t(\psi_\tau^l)dxdt \\
&=: K_1+K_2+K_3+K_4.
\end{aligned}
$$

We will estimate K_1. Due to $l/q-2\geq 0$ and Hölder's inequality, we have

$$
\begin{aligned}
(3.3)\qquad K_1 &\leq l(l-1)\tau^{-2}\int_{[0,\tau)\times B(\tau)} |u|\eta_\tau^{l-2}\phi_\tau^l|(\partial_t\eta)(t/\tau)|^2 dxdt \\
&\quad + l\tau^{-2}\int_{[0,\tau)\times B(\tau)} |u|\eta_\tau^{l-1}\phi_\tau^l|(\partial_t^2\eta)(t/\tau)|dxdt \\
&\leq C\tau^{-2}\int_{[0,\tau)\times B(\tau)} |u|\psi_\tau^{l/p}dxdt \leq C\tau^{(d+1)/q-2}\{I(\tau)\}^{1/p}.
\end{aligned}
$$

Next we consider K_2. By $l/q-2\geq 0$ and Hölder's inequality, we obtain

$$
\begin{aligned}
(3.4)\qquad K_2 &\leq l(l-1)\tau^{-2}\int_{[0,\tau)\times B(\tau)} |u|\eta_\tau^{l}\phi_\tau^{l-2}|(\Delta\phi)(x/\tau)|dxdt \\
&\quad + l\tau^{-2}\int_{[0,\tau)\times B(\tau)} |u|\eta_\tau^{l}\phi_\tau^{l-1}|(\nabla\phi)(x/\tau)|^2 dxdt \\
&\leq C\tau^{-2}\int_{[0,\tau)\times B(\tau)} |u|\psi_\tau^{l/p}dxdt \leq C\tau^{(d+1)/q-2}\{I(\tau)\}^{1/p}.
\end{aligned}
$$

Next we deal with K_3. From Hölder's inequality, we have

$$
(3.5)\qquad K_3 \leq |m|\int_{[0,\tau)\times B(\tau)} |u|\psi_\tau^{l/p}dxdt \leq C|m|\tau^{(d+1)/q}\{I(\tau)\}^{1/p}.
$$

Finally we estimate K_4. By $l/q-1\geq 0$ and Hölder's inequality, we can get

$$
\begin{aligned}
(3.6)\qquad K_4 &\leq |a|l\tau^{-1}\int_{[0,\tau)\times B(\tau)} |u|\eta_\tau^{l-1}\phi_\tau^l|(\partial_t\eta)(t/\tau)|dxdt \\
&\leq C|a|\tau^{-1}\int_{[0,\tau)\times B(\tau)} |u|\psi_\tau^{l/p}dxdt \leq C|a|\tau^{(d+1)/q-1}\{I(\tau)\}^{1/p}.
\end{aligned}
$$

By combining the estimates (3.2)–(3.6), we obtain

$$
(3.7)\qquad \lambda\Re J(\tau) \leq (C\tau^{(d+1)/q-2}+C|m|\tau^{(d+1)/q}+C|a|\tau^{(d+1)/q-1})\{I(\tau)\}^{1/p}-I(\tau).
$$

We note that since $p, q > 1$ and $1/p + 1/q = 1$, we have $ab \leq a^p/p + b^q/q$ for $a, b > 0$. By combining this estimate and (3.7), we have

$$\lambda \Re J(\tau) \leq C\tau^{d+1-2q} + C|m|\tau^{d+1} + C|a|\tau^{d+1-q}, \tag{3.8}$$

where C is a positive constant dependent only on d, p and l, which completes the proof of the lemma. □

Lemma 3.2. *We assume the same assumptions as in Lemma 3.1. Furthermore we assume that the function (f, g) satisfies (2.2) with $k < d$. Then the estimate*

$$\lambda \leq C\tau^{k+1}(\tau^{-2q} + |m| + |a|\tau^{-q}) \left(\int_{|x| \leq 1/\tau} |x|^{-k} \phi^l(x) dx \right)^{-1} \tag{3.9}$$

holds for any $\tau \in (0, T)$, where $C > 0$ is the same constant as in Lemma 3.1.

Proof of Lemma 3.2. By changing variables and (2.2), we have

$$\Re J(\tau) = \tau^d \int_{\mathbb{R}^d} \Re(af + g)(\tau x)\phi^l(x) dx \geq \tau^{d-k} \int_{|x| \leq 1/\tau} |x|^{-k} \phi^l(x) dx$$

for any $\tau \in (0, T)$. By combining Lemma 3.1 and the above estimate, we obtain (3.1), which completes the proof of the lemma. □

§ 4. Proof of the main results

§ 4.1. Large data blow-up for the energy-critical or subcritical case

First we give a proof of Proposition 2.3.

Proof of Proposition 2.3. By Lemma 3.2, we have

$$\lambda \leq C_1\tau^{k+1}(\tau^{-2q} + |m| + |a|\tau^{-q})\{L(\tau)\}^{-1}, \tag{4.1}$$

for any $\tau \in (0, T_w(\lambda))$, where

$$L(\tau) := \int_{|x| \leq 1/\tau} |x|^{-k} \phi^l(x) dx. \tag{4.2}$$

Claim. There exists $\lambda_0 > 0$ depending only on $|m|, |a|, d, p, l, k$ such that if $\lambda > \lambda_0$, then the estimate holds:

$$T_w(\lambda) \leq 2. \tag{4.3}$$

Indeed, on the contrary, we assume that $T_w(\lambda) > 2$. Then by (4.1) with $\tau = 2$,

$$\lambda \leq C_1 2^{k+1}(2^{-2q} + |m| + |a|2^{-q})\{L(2)\}^{-1}. \tag{4.4}$$

By changing variables and $k < d$, we have

$$L(2) = \int_{|x|\le 1/2} |x|^{-k} dx = C \int_0^{1/2} r^{-k+d-1} dr =: C_2(< \infty). \tag{4.5}$$

By combining (4.4)-(4.5), we obtain

$$\lambda \le C_1 C_2^{-1} 2^{k+1} (2^{-2q} + |m| + |a| 2^{-q}) =: \lambda_0,$$

which leads to a contradiction to $\lambda > \lambda_0$. Thus the claim is proved.
Since $L(\tau)$ is monotone decreasing on $[0,\infty)$ and $k < d$, we obtain

$$L(\tau) > L(2) = \int_{|x|<1/2} |x|^{-k} dx = C_2, \tag{4.6}$$

for any $\tau \in (0,2)$. In the case of $\lambda > \lambda_0$, let $\tau \in (0, T_w(\lambda))$. Noting that $0 < \tau < T_w(\lambda) \le 2$, by (4.1) and (4.6), we can get

$$\lambda < C_1 C_2^{-1} \tau^{k+1} (\tau^{-2q} + |m| + |a|\tau^{-q}) \le C_3 \tau^{-\kappa},$$

where C_3 is a positive constant depending only on $d, p, l, k, |m|, |a|$ and $\sigma = 2q - k - 1$. Since $\sigma > 0$ due to $k < \frac{p+1}{p-1}$, we have

$$\tau \le C_4 \lambda^{-1/\kappa}.$$

Since τ is arbitrary in $(0, T_w(\lambda))$, the above inequality implies $T_w(\lambda) \le C_4 \lambda^{-1/\kappa}$, which completes the proof of the proposition. □

Now we give a proof of Theorem 2.4. We need the following proposition.

Proposition 4.1. *Let $a, m \in \mathbb{R}$, $d \in \mathbb{N}$, $T > 0$. Assume that the nonlinearity F satisfies (A) with $1 \le p \le p_1$ for $d \ge 3$ or $1 \le p$ for $d = 1, 2$. If u is a energy solution to (NLKG) with $(u(0,\cdot), \partial_t u(0,\cdot)) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ on $[0,T)$ which belongs to $X(T)$ given in Theorem 5.1, then u becomes weak solution on $[0,T)$ in the sense of Definition 2.2.*

The proof of this proposition is due to a standard density argument. Thus we omit the detail (see Appendix in [11] for example).
Now we prove Theorem 2.4.

Proof of Theorem 2.4. We note that by $k < d/2$, we can find the function $(f, g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ satisfying (2.2) (see Example 2.5). We also note that p satisfies $1 < p \le p_1$ for $d \ge 3$ or $1 < p$ for $d = 1, 2$. Thus by the existence of the uniqueness of the local energy solution, we can see that $T(\lambda) > 0$. Let $\tau \in (0, T(\lambda))$ and u be the energy-solution on $[0, \tau)$. By Proposition 4.1, we can see that u is a weak solution on $[0, \tau)$. We also note that by the assumptions $1 < p \le p_1$ for $d \ge 3$ or $1 < p$ for $d = 1, 2$ and $k < d/2$, we have $k < d/2 \le \min(d, \frac{p+1}{p-1})$. Thus we can apply Proposition 2.3 to obtain $\tau \le C\lambda^{-\kappa}$, $\lambda > \lambda_0$, where λ_0 is given in Proposition 2.3. Since τ is arbitrary in $(0, T(\lambda))$, we obtain

$$T(\lambda) \le C\lambda^{-\kappa},$$

for $\lambda > \lambda_0$. The divergence of the norm $\|(u(t), \partial_t u(t))\|_{H^1 \times L^2}$ if $1 < p < p_1$ for $d \ge 3$ or if $1 < p$ for $d = 1, 2$ or $\|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}((0,T(\lambda))\times\mathbb{R}^d)}$ if $p = p_1$ and $d \ge 3$ for the solution follows from the standard blow-up criterion (see Theorem 5.1 for example), which completes the proof of the theorem. □

§ 4.2. Non-existence of the local weak solution in the supercritical case

We prove Proposition 2.6.

Proof of Proposition 2.6. By Lemma 3.2, we have

$$\lambda \le C_5 \tau^{k+1-2q} \{L(\tau)\}^{-1}, \tag{4.7}$$

for any $\tau \in (0, \min(1, T))$, where $L(\tau)$ is defined by (4.2) and C_5 is a positive constant dependent only on $d, p, l, k, |m|, |a|$. Since $L(\tau)$ is monotone decreasing on $[0, \infty)$ and $k < d$, we obtain

$$L(\tau) > L(1) > \int_{|x|<1/2} |x|^{-k} dx =: C_2 (< \infty), \tag{4.8}$$

for any $\tau \in (0, 1)$. Thus by (4.7)–(4.8), we can get

$$0 \le \lambda \le C_5 C_2^{-1} \tau^{k+1-2q}, \tag{4.9}$$

for any $\tau \in (0, \min(1, T))$. By the assumption $\frac{p+1}{p-1} < k$, we have $k + 1 - 2q > 0$. Therefore, taking the limit $\tau \to +0$ in (4.9), we can conclude $\lambda = 0$, which completes the proof of the proposition. □

Finally we give a proof of Theorem 2.7.

Proof of Theorem 2.7. By $p > p_1$, we have $\frac{p+1}{p-1} < \frac{d}{2}$. Thus we can see that there exists a function $(f, g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ satisfying (2.2) with $\frac{p+1}{p-1} < k < \frac{d}{2}$. Thus we can apply Proposition 2.6 and obtain $\lambda = 0$. □

§ 5. Appendix

In this appendix, we recall the local well-posedness (L.W.P) result in the energy space for (NLKG) in the energy-subcritical or critical case $1 \le p \le p_1$ for $d \ge 3$ or

$1 \le p$ for $d = 1, 2$ (see [5, 6, 14, 15, 27, 26, 17, 21] and their references). L.W.P for more general hyperbolic equations was obtained in [14]. However, for the convenience of the readers, we give a proof of this result. The proof can be done via Strichartz's estimates (Lemma 5.2) and Sobolev's inequality (Lemma 5.3).

We define the closed ball in the energy space at the origin of the radius r as

$$B_r(H^1 \times L^2) := \{(f, g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d); \|(f, g)\|_{H^1 \times L^2} \le r\}.$$

For $T > 0$, we also introduce the resolution space

$$X(T) := \begin{cases} E(T) & \text{if } 1 \le p \le \frac{d}{d-2} \text{ for } d \ge 3 \text{ or } 1 < p \text{ for } d = 1, 2, \\ E(T) \cap Y(T) & \text{if } \frac{d}{d-2} < p \le p_1 \text{ for } d \ge 3, \end{cases}$$

where $Y(T)$ is an auxiliary space given by

$$Y(T) := \begin{cases} L_t^\gamma(0, T; W^{\frac{1}{2}, \gamma}) & \text{if } \frac{d}{d-2} < p \le 1 + \frac{4d}{(d-2)(d+1)}, \\ L_t^\gamma(0, T; W^{\frac{1}{2}, \gamma}) \cap L_t^\mu(0, T; L_x^\varrho), & \text{if } 1 + \frac{4d}{(d-2)(d+1)} < p \le p_1, \end{cases}$$

where γ, μ and ϱ are defined by $\gamma := \frac{2(d+1)}{d-1}$, $\frac{1}{\mu} := \frac{d-2}{2} - \frac{2d}{(p-1)(d+1)}$ and $\varrho := \frac{(p-1)(d+1)}{2}$ respectively.

Theorem 5.1 (L.W.P. in the energy space in H^1-subcritical or critical case)**.** *Let $m, a, \in \mathbb{R}$ and $d \in \mathbb{N}$. Assume that the nonlinearity F satisfies (A) with $1 \le p \le p_1$ for $d \ge 3$ or $1 \le p$ for $d = 1, 2$. Then the Cauchy problem (NLKG) is locally well-posed in $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ for arbitrary $(u(0, \cdot), \partial_t u(0, \cdot)) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. More precisely, the following statememts hold:*

- *(Existence) For any $r > 0$ and arbitrary data $(u(0, \cdot), \partial_t u(0, \cdot)) \in B_r(H^1 \times L^2)$, there exists $T = T(r, u(0, \cdot)) > 0$ such that there exists a energy solution $u \in X(T)$ to (NLKG) on $[0, T)$.*
- *(Uniqueness) Let $u \in X(T)$ be the above solution, $0 < T_1 \le T$ and $v \in X(T_1)$ be another energy solution of (NLKG) on $[0, T_1)$. If $v(0, \cdot) = u(0, \cdot)$, then $v = u|_{[0,T_1)}$.*
- *(Continuity of the flow map) The flow map $B_r(H^1 \times L^2) \mapsto X(T)$, $u(0, \cdot) \mapsto u$ is Lipshitz continuous.*
- *(Blow-up criterion) Either (i) $T(\lambda) = \infty$ or (ii) $T(\lambda) < \infty$ and*

$$\liminf_{t \to T(\lambda) - 0} \|(u(t), \partial_t u(t))\|_{H^1 \times L^2} = \infty, \quad \text{if } \ 1 < p < p_1 \text{ for } d \ge 3 \text{ or } 1 < p \text{ for } d = 1, 2,$$

$$\|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0, T(\lambda)) \times \mathbb{R}^d)} = \infty, \quad \text{if } \ p = p_1,$$

is valid.

The similar statements also hold in the negative time direction.

Remark 5.1. In the subcritical case, i.e. $1 \leq p < p_1$ for $d \geq 3$ or $1 \leq p$ for $d = 1, 2$, the existence time of the local solution depends only on the size of the initial data. On the other hand, in the critical case $p = p_1$, it may depend not only on the size of the data but also on its profile.

Remark 5.2. In the energy critical, massless and undamped case $(p, m, a) = (p_1, 0, 0)$, the similar result as Theorem 5.1 holds, even if the inhomogeneous spaces $H^1(\mathbb{R}^d)$ and $W^{\frac{1}{2},\gamma}$ are replaced by the homogeneous ones $\dot{H}^1(\mathbb{R}^d)$ and $\dot{W}^{\frac{1}{2},\gamma}$ respectively. Moreover, in this case, global well-posedness in $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ holds for small initial data $(u(0,\cdot), \partial_t u(0,\cdot))$ in $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$.

Even in the case when there exists a time derivative in the equation (NLKG), i.e. $a \neq 0$, it suffices to consider the Cauchy problem for the following Klein-Gordon equations without time derivative:

$$\begin{cases} \partial_t^2 v - \Delta v + v = F(v), & (t,x) \in [0,T) \times \mathbb{R}^d, \\ v(0,x) = f(x),\ \partial_t v(0,x) = g(x), & x \in \mathbb{R}^d, \end{cases} \tag{5.1}$$

where $v = v(t,x)$ is a $\mathbb{C}$-valued unknown function of (t,x). Indeed, let u be a solution to (NLKG). If we set $v(t,x) := e^{a/t}u(t,x)$, then we can easily see that v satisfies

$$\begin{cases} \partial_t^2 v - \Delta v + v = (\frac{a^2}{4} + m^2 + 1)v + e^{\frac{a}{2}t}F(e^{-\frac{a}{2}t}v), & (t,x) \in [0,T) \times \mathbb{R}^d, \\ v(0,x) = \lambda f(x),\ \ \partial_t v(0,x) = \lambda(\frac{a}{2}f + g)(x), & x \in \mathbb{R}^d. \end{cases} \tag{5.2}$$

If T is bounded, then so are $e^{\frac{a}{2}t}$, $e^{-\frac{a}{2}t}$ in (5.2). Thus we can see that the right hand side of (5.2) also satisfies (A) with the same p. Hereafter we consider the equation (5.1) instead of the original equation (NLKG) or (5.2). And we convert the equation (5.1) into the integral equation as follows:

$$v(t,x) = \partial_t K(t)f(x) + K(t)g(x) + \int_0^t K(t-s)F(v(s,x))ds, \tag{5.3}$$

where $K(t)$ is the free Klein-Gordon evolution group defined by

$$K(t) := \frac{\sin(t\langle\nabla\rangle)}{\langle\nabla\rangle} := \mathcal{F}^{-1}\frac{\sin(t\langle\xi\rangle)}{\langle\xi\rangle}\mathcal{F}, \quad \text{and } \partial_t K(t) = \cos(t\langle\nabla\rangle) := \mathcal{F}^{-1}\cos(t\langle\xi\rangle)\mathcal{F}.$$

Next we state Strichartz estimates, which is used to treat the case $\frac{d}{d-2} < p \leq p_1$ and $d \geq 3$.

Lemma 5.2 (Strichartz estimates (see [7, 27, 26, 18, 21])). *Let $d \in \mathbb{N}$, $s \in [0,1]$, $2 < q, \tilde{q} \leq \infty$ and $2 \leq r, \tilde{r} < \infty$ be exponents satisfying the scaling* *and admissibility conditions:*

$$\frac{1}{q}+\frac{d}{r}=\frac{d}{2}-s=\frac{1}{\tilde{q}'}+\frac{d}{\tilde{r}'}-2 \quad \textit{and} \quad \frac{1}{q}+\frac{d-1}{2r},\ \frac{1}{\tilde{q}}+\frac{d-1}{2\tilde{r}}\leq\frac{d-1}{4},$$

and $T>0$. Then the estimates

$$\begin{aligned}\|\partial_t K(t)f\|_{L_t^\infty(0,T;H^s)}+\|\partial_t K(t)f\|_{L_t^q(0,T;L_x^r)}&\leq C\|f\|_{H^s},\\ \|K(t)g\|_{L_t^\infty(0,T;H^s)}+\|K(t)g\|_{L_t^q(0,T;L_x^r)}&\leq C\|g\|_{H^{s-1}},\\ \left\|\int_0^t K(t-s)F(s)ds\right\|_{L_t^\infty(0,T;H^\gamma)}+\left\|\int_0^t K(t-s)F(s)ds\right\|_{L_t^q(0,T;L_x^r)}&\leq C\|F\|_{L_t^{\tilde{q}'}(0,T;L_x^{\tilde{r}'})}\end{aligned}$$

are valid for any $f\in H^s(\mathbb{R}^d)$, $g\in H^{s-1}(\mathbb{R}^d)$ and $F\in L_t^{\tilde{q}'}(0,T;L_x^{\tilde{r}'})$, where C is a constant independent of T, and $\tilde{q}'$ and $\tilde{r}'$ are defined by $\tilde{q}':=\frac{\tilde{q}}{\tilde{q}-1}$ and $\tilde{r}':=\frac{\tilde{r}}{\tilde{r}-1}$.

We also use the Gagliardo-Nirenberg inequality.

Lemma 5.3 (Gagliardo-Nirenberg's inequality)**.**
Let $\nu,\eta\in[1,\infty]$ and $\alpha,\beta\in\mathbb{R}$ with $0\leq\alpha<\beta$. Then the following inequality is valid:

$$\|(-\Delta)^{\alpha/2}f\|_{L^\rho}\leq C\|(-\Delta)^{\beta/2}f\|_{L^\eta}^{\theta}\|f\|_{L^\nu}^{1-\theta},$$

where C is a constant depending only on d,α,β,ν,η and θ. Here $\rho\geq 1$ is such that

$$\frac{1}{\rho}=\frac{\alpha}{d}+\theta\left(\frac{1}{\eta}-\frac{\beta}{d}\right)+\frac{1-\theta}{\nu}$$

and the parameter θ is any from the interval $\frac{\alpha}{\beta}\leq\theta\leq 1$, with the following exception: if the value $\beta-\alpha-\frac{d}{\eta}$ is a nonnegative integer, then the parameter θ is any from the interval $\frac{\alpha}{\beta}\leq\theta<1$.

For the proof of this lemma, see [4] for example. We also need the chain rule for fractional derivatives.

Lemma 5.4. *Let $F:\mathbb{C}\mapsto\mathbb{C}$ be continuously differentiable in the sense of functions in real numbers and $\alpha\in(0,1]$ and $1<\rho,\nu,\eta<\infty$ are such that $\frac{1}{\rho}=\frac{1}{\nu}+\frac{1}{\eta}$. Then*

$$\||\nabla|^\alpha F(u)\|_{L^\rho}\leq C\|F'(u)\|_{L^\nu}\||\nabla|^\alpha u\|_{L^\eta},$$

provided that the right hand side is finite, where $C>0$ is a constant independent of u.

For the proof of this lemma, see [3] for example.

Now we give a proof of Theorem 5.1.

Proof of Theorem 5.1. (Existence) We define the nonlinear mapping J as

$$J[v](t) := \partial_t K(t)f + K(t)g + \int_0^t K(t-s)F(v)(s)ds.$$

Let $r > 0$, $(f,g) \in B_r(H^1 \times L^2)$ and let $M > 0$ that will be determined later. We define the complete metric space

$$X(T,M) := \{v \in X(T); \|v\|_{X(T)} \le M\},$$

with the metric

$$d(u,v) := \begin{cases} \|u-v\|_{L_t^\infty(0,T;H^1)} + \|\partial_t u - \partial_t v\|_{L_t^\infty(0,T;L_x^2)}, & \text{if } 1 \le p \le \frac{d}{d-2} \text{ for } d \ge 3 \\ & \text{or } 1 \le p \text{ for } d = 1,2, \\ \|u-v\|_{L_t^\infty(0,T;H^{\frac{1}{2}})} + \|u-v\|_{L_{t,x}^\gamma([0,T)\times\mathbb{R}^d)}, & \text{if } \frac{d}{d-2} < p \le \frac{d+5}{d+1} \text{ for } d \ge 3, \\ \|u-v\|_{L_{t,x}^\gamma([0,T)\times\mathbb{R}^d)}, & \text{if } \frac{d+5}{d+1} \le p \le p_1 \text{ for } d \ge 3. \end{cases}$$

We will prove that J is contractive from $X(T,M)$ into itself if T is sufficiently small. Thus by the contraction mapping principle, we can find a energy solution v to (5.3)
Case 1. $1 \le p \le \frac{d}{d-2}$ for $d \ge 3$ or $1 \le p$ for $d = 1,2$. In this case, the Sobolev embedding $H^1(\mathbb{R}^d) \subset L^{2p}(\mathbb{R}^d)$ holds. Thus we obtain

$$\begin{aligned} &\|J[v]\|_{L_t^\infty(0,T;H^1)} + \|\partial_t J[v]\|_{L_t^\infty(0,T;L_x^2)} \le C_0(\|f\|_{H^1} + \|g\|_{L^2}) + C\|F(v)\|_{L_t^1(0,T;L_x^2)} \\ &\le C_0 r + C\|\|v(t)\|_{L_x^{2p}}^p\|_{L_t^1(0,T)} \le C_0 r + C_1 T\|v\|_{L_t^\infty(0,T;H^1)}^p \le C_0 r + C_1 T M^p \le M, \end{aligned}$$

and

$$\begin{aligned} d(J[u],J[v]) &\le C\|F(u)-F(v)\|_{L_t^1(0,T;L_x^2)} \le C\|(|u|^{p-1}+|v|^{p-1})(u-v)\|_{L_t^1(0,T;L_x^2)} \\ &\le C\|(\|u(s)\|_{L_x^{2p}}^{p-1} + \|v(s)\|_{L_x^{2p}}^{p-1})\|u(s)-v(s)\|_{L_x^{2p}}\|_{L_t^1(0,T)} \\ &\le C_1 T M^{p-1} d(u,v) \le \frac{1}{2} d(u,v), \end{aligned}$$

if we choose M, T such as $M \ge 2C_0 r$ and $0 < T \le \frac{1}{2C_1 M^{p-1}}$.
Case 2. $1 + \frac{2}{d-2} < p < 1 + \frac{4}{d+1}$ and $d \ge 6$. We choose ρ, θ such as

$$\frac{1}{\rho} := \frac{1}{2} + \frac{(d-2)(p-1)}{2d}, \quad \theta := \frac{d(d+1)(p-1)+2}{2(d+1)(p+1)}.$$

By $\frac{d}{d-2} < p < 1 + \frac{4}{d+1} < \frac{d(d+3)}{(d+1)(d-2)}$, we can see $\theta \in (\frac{1}{2}, 1)$. By Lemma 5.3, 5.4 and the Sobolev embedding $H^1(\mathbb{R}^d) \subset L^{\frac{2d}{d-2}}(\mathbb{R}^d)$, we have

$$\begin{aligned} &\|\langle\nabla\rangle^{\frac{1}{2}}(F(u))\|_{L_x^{\frac{2(d+1)}{d+3}}} \le C\|\langle\nabla\rangle(F(u))\|_{L_x^\rho}^\theta \|F(u)\|_{L_x^{\frac{2}{p}}}^{1-\theta} \\ &\le C\|u\|_{L_x^{\frac{2d}{d-2}}}^{\theta(p-1)} \|\langle\nabla\rangle u\|_{L_x^2}^\theta \|u\|_{L_x^2}^{p(1-\theta)} \le C\|u\|_{H^1}^p. \end{aligned}$$

From this inequality, we obtain
(5.4)

$$\|\langle\nabla\rangle^{\frac{1}{2}}F(u)\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}} \leq C\|\|u(t)\|_{H^1}^p\|\|_{L_t^{\frac{2(d+1)}{d+3}}} \leq C_2T^{\frac{d+3}{2(d+1)}}\|u\|_{L_t^\infty(0,T;H^1)}^p \leq C_2T^{\frac{d+3}{2(d+1)}}M^p.$$

By Lemma 5.2 and (5.4), we have

$$\begin{aligned}(5.5)\quad &\|J[v]\|_{L_t^\infty(0,T;H^1)} + \|\partial_t J[v]\|_{L_t^\infty(0,T;L_x^2)} + \|\langle\nabla\rangle^{1/2}J[v]\|_{L_{t,x}^\gamma}\\ &\leq C_3(\|f\|_{H^1}+\|g\|_{L^2}) + C\|\langle\nabla\rangle^{1/2}F(v)\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}} \leq C_3r + C_2T^{\frac{d+3}{2(d+1)}}M^p \leq M,\end{aligned}$$

if we choose M, T such as $M \geq 2C_3r$ and $0 < T \leq (\frac{1}{2C_2M^{p-1}})^{\frac{2(d+1)}{d+3}}$.
In the same manner as the proof of (5.4), we can get

$$\begin{aligned}&\||u|^{p-1}(u-v)\|_{L_x^{\frac{2(d+1)}{d+3}}} \leq \||u|^{p-1}(u-v)\|_{L_x^\varrho}^\theta\||u|^{p-1}(u-v)\|_{L_x^{\frac{2}{p}}}^{1-\theta}\\ &\leq \||u|^{p-1}\|_{L_x^{\frac{2d}{(d-2)(p-1)}}}^\theta\|u-v\|_{L_x^2}^\theta\||u|^{p-1}\|_{L_x^{\frac{2}{p-1}}}^{1-\theta}\|u-v\|_{L_x^2}^{1-\theta}\\ &\leq C\|u\|_{H^1}^{p-1}\|u-v\|_{L_x^2}.\end{aligned}$$

By Lemma 5.2 and the above inequality, we obtain

$$\begin{aligned}&d(J[u],J[v]) \leq C\|F(u)-F(v)\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}([0,T)\times\mathbb{R}^d)}\\ &\leq C\|(|u|^{p-1}+|v|^{p-1})|u-v|\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}([0,T)\times\mathbb{R}^d)}\\ &\leq C_4\|(\|u(t)\|_{H^1}^{p-1}+\|v(t)\|_{H^1}^{p-1})\|u(t)-v(t)\|_{L_x^2}\|_{L_t^{\frac{2(d+1)}{d+3}}(0,T)}\\ &\leq C_4T^{\frac{d+3}{2(d+1)}}(\|u\|_{L_t^\infty(0,T;H^1)}^{p-1}+\|v\|_{L_t^\infty(0,T;H^1)}^{p-1})\|u-v\|_{L_t^\infty(0,T;L_x^2)}\\ &\leq C_4T^{\frac{d+3}{2(d+1)}}M^{p-1}d(u,v) \leq \frac{1}{2}d(u,v),\end{aligned}$$

if we choose T such as $0 < T \leq (\frac{1}{2C_4M^{p-1}})^{\frac{2(d+1)}{d+3}}$.
Case 3. $1+\frac{4}{d+1} \leq p \leq 1+\frac{4d}{(d+1)(d-2)}$ and $d \geq 3$. In this case, we have $2 \leq \varrho \leq \frac{2d}{d-2}$. We set $\theta := \frac{2d}{(p-1)(d+1)} - \frac{d-2}{2} \in [0,1]$. By the interpolation and the Sobolev embedding $H^1(\mathbb{R}^d) \subset L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)$, we have

$$(5.6)\qquad \|v\|_{L_{t,x}^\varrho([0,T)\times\mathbb{R}^d)} \leq \|\|v(t)\|_{L_x^2}^\theta\|v(t)\|_{L_x^{\frac{2d}{d-2}}}^{1-\theta}\|_{L_t^\varrho[0,T)} \leq CT^{\frac{1}{\varrho}}M.$$

By Lemma 5.2, 5.4 and (5.7), we have

$$\begin{aligned}
&\|J[v]\|_{L_t^\infty(0,T;H^1)} + \|\partial_t J[v]\|_{L_t^\infty(0,T;L_x^2)} + \|\langle\nabla\rangle^{1/2} J[v]\|_{L_{t,x}^\gamma([0,T)\times\mathbb{R}^d)} \\
&\leq C_5(\|f\|_{H^1} + \|g\|_{L^2}) + C\|\langle\nabla\rangle^{1/2} F(v)\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}([0,T)\times\mathbb{R}^d)} \\
&\leq C_5 r + C\|v\|_{L_{t,x}^\varrho([0,T)\times\mathbb{R}^d)}^{p-1} \|\langle\nabla\rangle^{\frac{1}{2}} v\|_{L_{t,x}^\gamma([0,T)\times\mathbb{R}^d)} \\
&\leq C_5 r + C_6 T^{\frac{2}{d+1}} M^p \leq M, \qquad (5.7)
\end{aligned}$$

and

$$\begin{aligned}
&d(J[u],J[v]) \leq C\|F(u)-F(v)\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}([0,T)\times\mathbb{R}^d)} \leq C\|(|u|^{p-1}+|v|^{p-1})(u-v)\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}} \\
&\leq C(\|u\|_{L_{t,x}^\varrho([0,T)\times\mathbb{R}^d)}^{p-1} + \|v\|_{L_{t,x}^\varrho([0,T)\times\mathbb{R}^d)}^{p-1})\|u-v\|_{L_{t,x}^\gamma([0,T)\times\mathbb{R}^d)} \\
&\leq C_6 T^{\frac{2}{d+1}} M^{p-1} d(u,v) \leq \frac{1}{2} d(u,v),
\end{aligned}$$

if we choose M, T such as $M \geq 2C_5 r$ and $0 < T \leq (\frac{1}{2C_6 M^{p-1}})^{\frac{d+1}{2}}$.

Case 4. $1 + \frac{4}{(d+1)(d-2)} < p < p_1$ and $d \geq 3$. In this case, we have $\mu > 0$ and $\beta := \frac{4-(p-1)(d-2)}{2(p-1)} > 0$. By the Hölder inequality, we have

$$\|v\|_{L_{t,x}^\varrho([0,T)\times\mathbb{R}^d)} \leq T^\beta \|v\|_{L_t^\mu(0,T;L_x^\varrho)} \leq T^\beta M. \qquad (5.8)$$

By Lemma 5.2, 5.4 and (5.9), we obtain

$$\begin{aligned}
&\|J[v]\|_{L_t^\infty(0,T;H^1)} + \|\partial_t J[v]\|_{L_t^\infty(0,T;L_x^2)} + \|\langle\nabla\rangle^{\frac{1}{2}} J[v]\|_{L_{t,x}^\gamma([0,T)\times\mathbb{R}^d)} + \|J[v]\|_{L_t^\mu(0,T;L_x^\varrho)} \\
&\leq C_7(\|f\|_{H^1} + \|g\|_{L^2}) + C\|\langle\nabla\rangle^{\frac{1}{2}} F(v)\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}([0,T)\times\mathbb{R}^d)} \\
&\leq C_7 r + C\|v\|_{L_{t,x}^\varrho([0,T)\times\mathbb{R}^d)}^{p-1} \|\langle\nabla\rangle^{\frac{1}{2}} v\|_{L_{t,x}^\gamma([0,T)\times\mathbb{R}^d)} \leq C_7 r + C_8 T^\alpha M^p \leq M,
\end{aligned}$$

where α is defined by $\alpha := \beta(p-1) > 0$, and

$$\begin{aligned}
d(J[u],J[v]) &\leq C\|F(u)-F(v)\|_{L_{t,x}^{\frac{2(d+1)}{d+3}}([0,T)\times\mathbb{R}^d)} \\
&\leq C\|(|u|^{p-1}+|v|^{p-1})\|_{L_{t,x}^{\frac{d+1}{2}}([0,T)\times\mathbb{R}^d)} \|u-v\|_{L_{t,x}^\gamma([0,T)\times\mathbb{R}^d)} \\
&\leq C_8 T^\alpha M^{p-1} d(u,v) \leq \frac{1}{2} d(u,v),
\end{aligned}$$

if we choose M, T such as $M \geq 2C_7 r$ and $0 < T \leq (\frac{1}{2C_8 M^{p-1}})^{\frac{1}{\alpha}}$.

Case 5. $p = p_1$ and $d \geq 3$. In this case, we have $\alpha = 0$. Thus we have to modify the argument of the previous case. For $L, M > 0$, which will be determined later, we define the complete metric space

$$X(T,L,M) := \left\{ v \in L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T)\times\mathbb{R}^d); \|\langle\nabla\rangle^{\frac{1}{2}} v\|_{L_{t,x}^{\gamma}([0,T)\times\mathbb{R}^d)} \le L, \right.$$
$$\left. \|v\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T)\mathbb{R}^d)} \le M \right\},$$

with the metric

$$d(u,v) := \|u-v\|_{L_{t,x}^{\gamma}([0,T)\times\mathbb{R}^d)}.$$

We prove that J is contractive from $X(T,L,M)$ into itself if T is sufficiently small. Since $(f,g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, by Lemma 5.2, we can find a small $T = T(L,M) > 0$ such that

$$\|\langle\nabla\rangle^{\frac{1}{2}} \partial_t K(t) f\|_{L_{t,x}^{\gamma}([0,T)\times\mathbb{R}^d)} + \|\partial_t K(t) f\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T)\times\mathbb{R}^d)}$$
$$+ \|\langle\nabla\rangle^{\frac{1}{2}} K(t) g\|_{L_{t,x}^{\gamma}([0,T)\times\mathbb{R}^d)} + \|K(t) g\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T)\times\mathbb{R}^d)} \le \frac{1}{2}\min(L,M). \tag{5.9}$$

Let $v \in X(T,L,M)$. In the same manner as the proof of the estimate (5.8), we have

$$\|\langle\nabla\rangle^{\frac{1}{2}} J[v]\|_{L_{t,x}^{\gamma}([0,T)\times\mathbb{R}^d)} \le \|\langle\nabla\rangle^{\frac{1}{2}} \partial_t K(t) f\|_{L_{t,x}^{\gamma}([0,T)\times\mathbb{R}^d)} + \|\langle\nabla\rangle^{\frac{1}{2}} K(t) g\|_{L_{t,x}^{\gamma}([0,T)\times\mathbb{R}^d)}$$
$$+ C_9 \|v\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T)\times\mathbb{R}^d)}^{\frac{4}{d-2}} \|\langle\nabla\rangle^{\frac{1}{2}} v\|_{L_{t,x}^{\gamma}([0,T)\times\mathbb{R}^d)}$$
$$\le \frac{1}{2}L + C_9 L M^{\frac{4}{d-2}} \le L, \tag{5.10}$$

if we choose M such as $0 < M \le (\frac{1}{2C_9})^{\frac{d-2}{4}}$. We also have

$$\|J[v]\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T)\times\mathbb{R}^d)} \le \|\partial_t K(t) f\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T)\times\mathbb{R}^d)} + \|K(t) g\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T)\times\mathbb{R}^d)}$$
$$+ C_{10} \|v\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T)\times\mathbb{R}^d)}^{\frac{4}{d-2}} \|\langle\nabla\rangle^{\frac{1}{2}} v\|_{L_{t,x}^{\gamma}([0,T)\times\mathbb{R}^d)}$$
$$\le \frac{1}{2}M + C_{10} L M^{\frac{4}{d-2}} \le M,$$

if we choose L such as $0 < L \le \frac{1}{2C_{10}} M^{\frac{d-6}{d-2}}$.
Moreover, in the same manner as the proof of (5.8), we obtain

$$d(J[u],J[v]) = \left\| \int_0^t K(t-s)\{F(u)-F(v)\}ds \right\|_{L_{t,x}^{\gamma}([0,T)\times\mathbb{R}^d)}$$
$$\le C(\|u\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T)\times\mathbb{R}^d)}^{\frac{4}{d-2}} + \|v\|_{L_{t,x}^{\frac{2(d+1)}{d-2}}([0,T)\times\mathbb{R}^d)}^{\frac{4}{d-2}}) \|u-v\|_{L_{t,x}^{\gamma}([0,T)\times\mathbb{R}^d)}$$
$$\le C_9 M^{\frac{4}{d-2}} d(u,v) \le \frac{1}{2} d(u,v).$$

(Uniqueness, Continuity of the flow-map) The uniqueness in $X(T)$ and the Lipchitz continuity of the flow-map can be proved in the same manner as the existence part. Thus we omit the details.
(Blow-up criterion) The blow-up criterion can be also proved in a standard manner (see [2] for the subcritical case and see the proof of Lemma 2.11 in [16] for the critical case, for example). $\square$

Remark 5.3. The proof of the existence part of Theorem 5.1 implies the lower estimate of lifespan $T(\lambda)$ is estimated as $T(\lambda) \geq C\lambda^{-\omega}$ for any $\lambda > 0$, where C is independent of λ. Here ω is defined by

$$\omega := \begin{cases} p-1 & \text{if } 1 \leq p \leq 1 + \frac{2}{d-2}, \\ \frac{2(d+1)(p-1)}{d-3} & \text{if } 1 + \frac{2}{d-2} < p < 1 + \frac{4}{d+1}, \\ \frac{2(p-1)}{d+1} & \text{if } 1 + \frac{4}{d+1} \leq p \leq 1 + \frac{4d}{(d+1)(d-2)}, \\ \frac{p+1}{p-1} - \frac{d}{2} & \text{if } 1 + \frac{4}{(d+1)(d-2)} < p < p_1. \end{cases}$$

In the last case $1 + \frac{4}{(d+1)(d-2)} < p < p_1$, ω is almost same to σ in Theorem 2.4 if we take k sufficiently close to $\frac{d}{2}$. In the other cases, ω is quite different from σ.

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