RIMS Kôkyûroku Bessatsu
B56 (2016), 31–41

Remarks on the ill-posedness results for the drift diffusion system

By

Tsukasa IWABUCHI* and Takayoshi OGAWA**

Abstract

The ill-posedness issue is considered for the drift-diffusion system of bipolar type. It is shown that the continuous dependence on initial data does not hold generally in the scaling invariant Besov spaces. The scaling invariant Besov spaces are $\dot{B}^{-2+\frac{n}{p}}_{p,\sigma}(\mathbb{R}^n)$ with $1 \le p, \sigma \le \infty$, and the reason on the optimality of the case $p = 2n$ is explained to obtain the well-posedness and the ill-posedness for the drift diffusion system of bipolar type with the attention to the divergence form structure of the nonlinear term. The scaling invariant spaces for the heat equation with the nonlinear term u^2 are same as those for the drift diffusion system and the difference is also indicated on the ill-posedness results between the heat equation and the drift diffusion system.

§ 1. Introduction

The purpose of this paper is surveying the result in [11] on the ill-posedness for the drift diffusion equation of bipolar type in the Besov spaces to compare the ill-posedness results for other equations with quadratic nonlinear terms such as the drift diffusion equation of monopolar type, and the heat equation with the nonlinear term u^2, where the scaling invariant spaces are common. We shall explain the essential difference of the ill-posedness results among these equations due to the divergence form structure of nonlinear terms.

Received September 30, 2014. Revised April 21, 2015.
2010 Mathematics Subject Classification(s): 35K08, 35K55
Key Words: *Ill-posedness*, *Besov spaces*, *Drift diffusion equation.*
*Osaka City University, Osaka 558-8585, Japan.
e-mail: iwabuchi@sci.osaka-cu.ac.jp
**Tohoku University, Sendai 980-8578, Japan.
e-mail: ogawa@math.tohoku.ac.jp

Let us consider the initial value problems of a drift-diffusion equation of bipolar type:

$$
(1.1) \qquad \begin{cases} \partial_t u - \Delta u + \kappa \nabla \cdot (u \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ \partial_t v - \Delta v - \kappa \nabla \cdot (v \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ -\Delta \psi = v - u, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), v(0,x) = v_0(x), & x \in \mathbb{R}^n, \end{cases}
$$

where u and v are the particle density of negative and positive electric charge, κ is the coupling constant and we assume $\kappa = \pm 1$. u_0 and v_0 are given initial data. The system (1.1) was originally considered for an initial boundary value problem with Dirichlet or Neumann boundary condition as the one of the simplest models for simulation of a semi-conductor device and we refer to [1, 6, 8, 12, 20, 25] for the related results. As the model of the semiconductor device simulation, the monopolar model is also considered;

$$
(1.2) \qquad \begin{cases} \partial_t u - \Delta u + \kappa \nabla \cdot (u \nabla \psi) = 0, & t > 0, x \in \mathbb{R}^n, \\ -\Delta \psi = u, & t > 0, x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases}
$$

This model is also regarded as the limitting case of the chemotaxis problem known as the Keller-Segel system [13]. This system was introduced by Keller-Segel [13] in 1970, describes the motion of the chemotaxis molds, and we refer to [3, 5, 9, 15, 16, 21, 22, 23, 29] and references theirin on the results of the system. The well-posedness issue was considered in both models and results are along the similar line. For instance, on the initial value problems in whole space, the existence of local solutions and global solutions was shown by Kurokiba-Ogawa [18, 19] in Lebesgue spaces and weighted Lebesgue spaces. Ogawa-Shimizu [23] showed global well-posedness in the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$. Zhao-Liu-Ciu [31] showed global well-posedness in the Besov spaces $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ $(n/2 < p < 2n)$. Iwabuchi [9] obtained the global well-posedness for the larger Besov spaces $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ $n/2 \le p < \infty$. See also [15], [16].

The purpose of this paper is to prove the ill-posedness for the problem (1.1) by showing that the continuous dependence on initial data does not hold generally in the Besov spaces $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ $(2n \le p \le \infty)$, where there is the invariant scaling to the system (1.1), namely for $\lambda > 0$,

$$
(1.3) \qquad \begin{cases} u_\lambda(t,x) = \lambda^2 u(\lambda^2 t, \lambda x), \\ v_\lambda(t,x) = \lambda^2 v(\lambda^2 t, \lambda x), \\ \psi_\lambda(t,x) = \psi(\lambda^2 t, \lambda x), \end{cases}
$$

the equations in system (1.1) is maintained invariant. Here, we note that the difference of the scaling transformations between u_λ, v_λ and ψ_λ comes from the third equation

$-\Delta\psi = v - u$ in (1.1). The invariant function space is defined by the Bochner space $L^\theta(\mathbb{R}_+; X)$ with a Banach scale X, where the invariant scaling (1.3) is invariant. If X is the Lebesgue space $L^p(\mathbb{R}^n)$, then the exponents (θ, p) necessarily satisfies

$$2 = \frac{2}{\theta} + \frac{n}{p}$$

so that $L^\theta(\mathbb{R}_+; L^p(\mathbb{R}^n))$ is the invariant space. In particular $\theta = \infty$, the consistency property holds and we reach the invariant function spaces for (1.1) as $(u, v, \psi) \in L^{n/2}(\mathbb{R}^n) \times L^{n/2}(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ with the boundedness of the solution in time variable. Such a critical space can be generalized by term of the homogeneous Besov spaces with negative regularity indeces $\dot{B}^{-2+\frac{n}{p}}_{p,\sigma}(\mathbb{R}^n)$.

Almost all basic nature of the models (1.1) and (1.2) are common and well-posedness properties coinsides as far as we consider rather regular initial data in the known liturature. Indeed, the well-posedness is obtained in the Besov spaces $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ with $p < 2n$ for both equations (1.1) and (1.2). However, in the case $p \geq 2n$, we shall show that there appears a difference for low regularity initial data between the bipolor and monopolar systems (1.1) and (1.2) for the well-posedness issue in the function spaces where the scaling is invariant. To specify the critical space for the well-posedness, it is not enough to describe in term of the homogeneous Sobolev space $\dot{H}^s_p$ (with the Riesz potential of $s \in \mathbb{R}$ and $1 \leq p \leq \infty$) and we introduce the scaling invariant Besov spaces $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ with $1 \leq p \leq \infty$, below. The initial value problem (1.2) of the monopolar type is well-posed in the Besov spaces $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ with $\frac{n}{2} < p < \infty$ and the ill-posedness in $\dot{B}^{-2}_{\infty,\infty}(\mathbb{R}^n)$ was shown in Iwabuchi [9]. This stands that the case $p = \infty$ is threshold for the monopolar type (1.2) in the sense of the well-posedness and the ill-posedness.

On the other hand for the equation (1.1) of bipolar type, Zhang-Liu-Ciu [31] showed that the problem is well-posed in $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ with $\frac{n}{2} < p < 2n$. Then, we show that the critical space for the well-posedness and the ill-posedness to the equation (1.1) is identified as $p = 2n$ through the study of the ill-posedness in the Besov spaces $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ $(2n \leq p \leq \infty)$. We note the following inclusion relations; for $\frac{n}{2} \leq p_1 \leq 2n \leq p_2$ it holds that

$$L^{\frac{n}{2}}(\mathbb{R}^n) \subset \dot{B}^{-2+\frac{n}{p_1}}_{p_1,\infty}(\mathbb{R}^n) \subset \dot{B}^{-\frac{3}{2}}_{2n,\infty}(\mathbb{R}^n) \subset \dot{B}^{-2+\frac{n}{p_2}}_{p_2,\infty}(\mathbb{R}^n) \subset \dot{B}^{-2}_{\infty,\infty}(\mathbb{R}^n)$$

We define the homogeneous Sobolev and the Besov spaces and state our theorems. We denote the function spaces of rapidly decreasing functions by $\mathcal{S}(\mathbb{R}^n)$, tempered distributions by $\mathcal{S}'(\mathbb{R}^n)$, and polynomials by $\mathcal{P}(\mathbb{R}^n)$.

Definition (*the homogeneous Sobolev spaces*). *For any* $s \in \mathbb{R}$, $1 \leq p \leq \infty$, *the homoge-*

neous Sobolev space $\dot{H}^s_p(\mathbb{R}^n)$ is defined by

$$\dot{H}^s_p = \dot{H}^s_p(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) \,\Big|\, \|f\|_{\dot{H}^s_p} := \Big\| \mathcal{F}^{-1}\big[|\xi|^s \hat{f}(\xi)\big] \Big\|_{L^p(\mathbb{R}^n)} < \infty \Big\},$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Definition (the homogeneous Besov spaces). Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be satisfying the following:

$$\operatorname{supp}\widehat{\phi} \subset \{\, \xi \in \mathbb{R}^n \,|\, 2^{-1} \le |\xi| \le 2\}, \quad \sum_{j\in\mathbb{Z}} \widehat{\phi}\Big(\frac{\xi}{2^j}\Big) = 1 \textit{ for } \xi \in \mathbb{R}^n \setminus \{0\},$$

where $\widehat{\phi}$ is the Fourier transform of ϕ, and let $\{\phi_j\}_{j\in\mathbb{Z}}$ be defined by

$$\phi_j(x) := 2^{nj}\phi(2^j x) \quad \textit{for } j \in \mathbb{Z}, x \in \mathbb{R}^n.$$

Then, for any $s \in \mathbb{R}$, $1 \le p, \sigma \le \infty$, the homogeneous Besov space $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ is defined by

$$\dot{B}^s_{p,\sigma}(\mathbb{R}^n) := \Big\{ f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n) \,\Big|\, \|f\|_{\dot{B}^s_{p,\sigma}} := \Big\| \big\{ 2^{sj} \|\phi_j * f\|_{L^p(\mathbb{R}^n)} \big\}_{j\in\mathbb{Z}} \Big\|_{\ell^\sigma(\mathbb{Z})} < \infty \Big\}.$$

Remark. It is possible to regard the above homogeneous spaces as certain subspaces of $\mathcal{S}'(\mathbb{R}^n)$ for some s, p, q. Indeed, if s and p satisfy $s < n/p$, then the homogeneous Sobolev space $\dot{H}^s_p(\mathbb{R}^n)$ is equivalent to

$$\big\{\, u \in \mathcal{S}'(\mathbb{R}^n) \,\big|\, \|u\|_{\dot{H}^s_p} < \infty,\, u = \sum_{j\in\mathbb{Z}} \phi_j * u \text{ in } \mathcal{S}'(\mathbb{R}^n) \,\big\}.$$

If $s < n/p$ with $1 \le q \le \infty$, or $s = n/p$ with $q = 1$, the Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is also equivalent to

$$\big\{\, u \in \mathcal{S}'(\mathbb{R}^n) \,\big|\, \|u\|_{\dot{B}^s_{p,q}} < \infty,\, u = \sum_{j\in\mathbb{Z}} \phi_j * u \text{ in } \mathcal{S}'(\mathbb{R}^n) \,\big\}.$$

These equivalence are due to the argument by Kozono-Yamazaki [17].

Theorem 1.1. [11] *Let $n \ge 2$, $\kappa = \pm 1$ and let p, σ satisfy*

$$2n < p \le \infty \textit{ and } 1 \le \sigma \le \infty, \quad \textit{or} \quad p = 2n \textit{ and } 2 < \sigma \le \infty. \tag{1.4}$$

Then, there exist a sequence of time $\{T_N\}_N$ with $T_N \to 0\,(N \to \infty)$ and a sequence of smooth and rapidly decreasing initial data $\{u_{0,N}\}_N, \{v_{0,N}\}_N$ $(N = 1, 2, \cdots)$ such that the corresponding sequence of smooth solutions $\{u_N\}_N, \{v_N\}_N$ to (1.1) with $u_N(0) = u_{0,N}$ and $v_N(0) = v_{0,N}$ satisfies

$$\lim_{N\to\infty} \|u_{0,N}\|_{\dot{B}^{-2+\frac{n}{p}}_{p,\sigma}} = 0, \quad \lim_{N\to\infty} \|v_{0,N}\|_{\dot{B}^{-2+\frac{n}{p}}_{p,\sigma}} = 0,$$

$$\lim_{N\to\infty} \|u_N(T_N)\|_{\dot{B}^{-2+\frac{n}{p}}_{p,\sigma}} = \infty, \quad \lim_{N\to\infty} \|v_N(T_N)\|_{\dot{B}^{-2+\frac{n}{p}}_{p,\sigma}} = \infty.$$

Remark. Solutions in the above theorem are smooth. The problem (1.1) is locally well-posed in the Besov space $\dot{B}^{-1}_{n,1}(\mathbb{R}^n)$ by the result [31], and the solution (u_N, v_N) is in $C([0,\widetilde{T}_N), \dot{B}^{-1}_{n,1}(\mathbb{R}^n))$, where $\widetilde{T}_N$ denotes the maximal existence time of the solution (u_N, v_N), and $T_N < \widetilde{T}_N$ for all N.

Remark. We note that when $n = 2$, $p = 4$ and $\sigma = 2$, the Besov space $\dot{B}^{-\frac{3}{2}}_{4,2}(\mathbb{R}^2)$ is the critical space for the well-posedness and the ill-posedness. It is indeed possible to show the global well-posedness for small initial data in the same space.

We refer to the paper [11] on the precise proof, and explain an idea of the proof in this paper. we introduce a formal expansion of the solution to (1.1) by some small parameter $\varepsilon > 0$ as

$$u = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \varepsilon^3 U_3 + \cdots , \tag{1.5}$$

$$v = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \varepsilon^3 V_3 + \cdots . \tag{1.6}$$

Then, we regard the parameter ε depending only on the inverse of the frequency parameter N for the initial data. Namely we expand u_0 and v_0 by

$$u_0 = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3 + \cdots ,$$
$$v_0 = \rho_0 + \varepsilon\rho_1 + \varepsilon^2\rho_2 + \varepsilon^3\rho_3 + \cdots$$

by $\varepsilon \cong N^{-1}$ and consider the equations for $U_k = U_k[u_0, v_0]$, $V_k = V_k[u_0, v_0]$ in (1.5) and (1.6) of the solution to (1.1) with the initial data $U_k(0) = \varphi_k$, $V_k(0) = \rho_k$. If we take the initial data which low frequency part is vanishing and it has only some frequency component φ_1 and ρ_1, the initial data u_0 and v_0 are given by $u_0 = 0 + \varepsilon\varphi_1 + \varepsilon^2 \times 0 + \cdots$ and $v_0 = 0 + \varepsilon\rho_1 + \varepsilon^2 \times 0 + \cdots$, and we find $U_0 \equiv V_0 \equiv 0$. Then, if u and v satisfy the equation (1.1), we have by the term of order ε^k with $k = 1, 2, 3, \cdots$, such as

$$\varepsilon : \quad \begin{cases} (\partial_t - \Delta)U_1 = 0, \\ (\partial_t - \Delta)V_1 = 0, \\ \quad U_1(0) = \varphi_1, V_1(0) = \rho_1, \end{cases}$$

$$\varepsilon^2 : \quad \begin{cases} (\partial_t - \Delta)U_2 = -\kappa\nabla\cdot\big(U_1\nabla(-\Delta)^{-1}(V_1 - U_1)\big), \\ (\partial_t - \Delta)V_2 = \kappa\nabla\cdot\big(V_1\nabla(-\Delta)^{-1}(V_1 - U_1)\big), \\ \quad U_2(0) = 0, V_2(0) = 0, \end{cases}$$

$$\vdots \qquad\qquad \vdots$$

respectively. In general, we have on the term of order $k \geq 2$

$$\varepsilon^k : \begin{cases} (\partial_t - \Delta)U_k = -\kappa \displaystyle\sum_{k_1+k_2=k,\ k_1,k_2\geq 1} \big(U_{k_1}\nabla(-\Delta)^{-1}(V_{k_2} - U_{k_2})\big), \\ (\partial_t - \Delta)V_k = \kappa \displaystyle\sum_{k_1+k_2=k,\ k_1,k_2\geq 1} \big(V_{k_1}\nabla(-\Delta)^{-1}(V_{k_2} - U_{k_2})\big), \\ \quad U_k(0) = 0, V_k(0) = 0, \end{cases} \tag{1.7}$$

formally. Therefore, it is natural to introduce $U_k = U_k[\varphi_1, \rho_1]$ and $V_k = V_k[\varphi_1, \rho_1]$ for $k = 1, 2, 3, \cdots$ inductively

(1.8)

$$\begin{cases} U_1[\varphi_1, \rho_1](t) := e^{t\Delta}\varphi_1, \\ U_k[\varphi_1, \rho_1](t) := -\kappa \displaystyle\sum_{k_1+k_2=k,\ k_1,k_2\geq 1} \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(U_{k_1}\cdot\nabla(-\Delta)^{-1}(V_{k_2} - U_{k_2}))d\tau \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } k = 2, 3, \cdots, \end{cases}$$

(1.9)

$$\begin{cases} V_1[\varphi_1, \rho_1](t) := e^{t\Delta}\rho_1, \\ V_k[\varphi_1, \rho_1](t) := \kappa \displaystyle\sum_{k_1+k_2=k,\ k_1,k_2\geq 1} \int_0^t e^{(t-\tau)\Delta}\nabla\cdot(V_{k_1}\cdot\nabla(-\Delta)^{-1}(V_{k_2} - U_{k_2}))d\tau \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for any } k = 2, 3, \cdots. \end{cases}$$

In view of the expansion (1.5) and (1.6), we consider

$$\begin{aligned} u(t) &= U_0[0,0](t) + \sum_{k=1} \varepsilon^k U_k[\varphi_1, \rho_1] = \sum_{k=1} \varepsilon^k U_k[\varphi_1, \rho_1], \\ v(t) &= V_0[0,0](t) + \sum_{k=1} \varepsilon^k V_k[\varphi_1, \rho_1] = \sum_{k=1} \varepsilon^k V_k[\varphi_1, \rho_1], \end{aligned}$$

as a solution to (1.1) for the initial data $u_0 = \varepsilon\varphi_1$, $v_0 = \varepsilon\rho_1$.

Under such a formal expansion, we investigate that each term in the expansion of the solution can stay modest way or diverge as $t \to 0$ by some initial test functions. One simplest test function is the monochromatic data that Fourier transform is supported locally at particular frequency. We let such a test function as $u_{0,N}, v_{0,N}$ with $\operatorname{supp}\widehat{u_{0,N}}, \operatorname{supp}\widehat{v_{0,N}} \cong \{\xi; |\xi| \cong 2^N\}$ for $N = 1, 2, \cdots$. By those test functions, one may find that all the higher terms from the expansion $U_k[u_{0,N}, v_{0,N}]$, $V_k[u_{0,N}, v_{0,N}]$ $k \geq k_0$ for some k_0 remains bounded under $N \to \infty$, while some lower order term such as $U_2[u_{0,N}, v_{0,N}]$, $V_2[u_{0,N}, v_{0,N}]$ blow up in the larger space $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ for $p > p_0$ as $N \to \infty$ with $t \to 0$, where p_0 is the threshold index for the well-posedness and the ill-posedness. Then fixing the expansion parameter ε as a constant, we see the equation is ill-posed in the space $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$ with $p > p_0$ or $p \geq p_0$. For the system (1.2) of monopolar type, the threshold index for the well-posedness is $p = \infty$. On the other hand for (1.1) of bipolar type, there appears a saturation because of the difference on the divergence form of the nonlinear part, and one can not attain the well-posedness up to the case $p = \infty$.

The main reason why the well-posedness class differs between the system (1.1) and (1.2) is that the nonilnear structure of each system has an essential difference with its derivative structure. For the equation (1.2) of monopolar type, the nonlinear term $u\nabla(-\Delta)^{-1}u$ satisfies the following identity:

$$\begin{aligned}\partial_{x_j} u \partial_{x_j}(-\Delta)^{-1}u = &\frac{1}{2}\partial_{x_j}(-\Delta)\big\{\big((-\Delta)^{-1}u\big)\big(\partial_{x_j}(-\Delta)^{-1}u\big)\big\} \\ &+ \partial_{x_j}\nabla\cdot\big\{\big((-\Delta)^{-1}u\big)\big(\nabla\partial_{x_j}(-\Delta)^{-1}u\big)\big\} \\ &+ \frac{1}{2}\partial_{x_j}^2\big\{\big((-\Delta)^{-1}u\big)u\big\},\end{aligned} \tag{1.10}$$

which was observed in [9]. The most important point is that we can regard the nonlinear term $\nabla\cdot(u\nabla(-\Delta)^{-1}u)$ such as $|\nabla|^2\big\{\big(|\nabla|^{-2}u\big)u\big\}$ roughly by (1.10). This expression shows that we may regard the nonlinear term for the monopolar type can be treated as the divergence form of the second order. This enables us to treat the equation in the weaker Besov space up to $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}$ with $2 \leq p < \infty$ and $\dot{B}^{-2}_{\infty,2}$. On the other hand for the bipolar type (1.1), the nonlinear term $\nabla\cdot\big(u\nabla(-\Delta)^{-1}v\big)$ has lack of the symmetry in the nonlinear structure which prevents to have such a good expression as is seen in (1.10).

To be more precise, let χ be a characteristic function which support is $[-1,1]$ and $e_1 := (1,0,\cdots,0)$, and we take initial data as

$$u_0 = N^{2-\frac{n}{p}}\mathcal{F}^{-1}\big[\chi(\cdot - Ne_1)\big], \quad v_0 = N^{2-\frac{n}{p}}\mathcal{F}^{-1}\big[\chi(\cdot + Ne_1)\big]. \tag{1.11}$$

We note that the Fourier transforms of u_0 and v_0 are supported locally at particular frequency N and $-N$, respectively, and $\|u_0\|_{\dot{H}_p^{-2+\frac{n}{p}}}$, $\|v_0\|_{\dot{H}_p^{-2+\frac{n}{p}}}$ are independent of N, where $\dot{H}^s_p(\mathbb{R}^n)$ is the homogeneous Sobolev space. By the Duhamel formula, we write the solution

$$\begin{aligned}u(t) &= e^{t\Delta}u_0 - \kappa\int_0^t e^{(t-\tau)\Delta}\nabla\cdot\Big(u\nabla(-\Delta)^{-1}(v-u)\Big)d\tau, \\ v(t) &= e^{t\Delta}v_0 + \kappa\int_0^t e^{(t-\tau)\Delta}\nabla\cdot\Big(v\nabla(-\Delta)^{-1}(v-u)\Big)d\tau.\end{aligned}$$

Since both of u and v are treated similarly, we consider the nonlinear term of u only. Then, we approximate the nonlinear part in right hand side by the use of the approximation with the linear parts $u \cong e^{\tau\Delta}u_0$, $v \cong e^{\tau\Delta}v_0$ to see that

(1.12)
$$
\begin{aligned}
&\left\| \int_0^t e^{(t-\tau)\Delta}\nabla\cdot\Big(e^{\tau\Delta}u_0\nabla(-\Delta)^{-1}e^{t\Delta}(v_0-u_0)\Big)d\tau\right\|_{\dot{H}_p^{-2+\frac{n}{p}}}\\
&= \left\|\mathcal{F}^{-1}\left[|\xi|^{-2+\frac{n}{p}}\xi\cdot\int_{\mathbb{R}^n}\widehat{u_0}(\xi-\eta)\Big(\widehat{v_0}(\eta)-\widehat{u_0}(\eta)\Big)\frac{\eta}{|\eta|^2}\frac{e^{-t|\xi|^2}\big(e^{2t(\xi-\eta)\cdot\eta}-1\big)}{2(\xi-\eta)\cdot\eta}d\eta\right]\right\|_{L^p}\\
&\cong \left\|\mathcal{F}^{-1}\left[|\xi|^{-2+\frac{n}{p}}\xi\cdot\int_{\mathbb{R}^n}\widehat{u_0}(\xi-\eta)\widehat{v_0}(\eta)\frac{N}{N^2}\frac{1}{2N^2}d\eta\right]\right\|_{L^p}\\
&\cong \left\|\mathcal{F}^{-1}\big[|\xi|^{-2+\frac{n}{p}}\xi\chi(\xi)\big]\right\|_{L^p}N^{2(2-\frac{n}{p})}N^{-1}N^{-2}\\
&\cong N^{1-\frac{2n}{p}},
\end{aligned}
$$

and the last term diverges as $N\to\infty$ if $p>2n$. In the third line of (1.12), we suppose that the convolution of $\widehat{u_0}$ and $\widehat{v_0}$ should have less regulariy than that of $\widehat{u_0}$ and itself due to the difference of the divergence form structures of two terms to ignore the convolution of $\widehat{u_0}$ and itself, and we utilized the frequency restriction $|\xi|\le 2$, $|\xi-\eta|\simeq N$ and $|\eta|\simeq N$ obtained for $\xi-\eta\in\operatorname{supp}\widehat{u_0}$, $\eta\in\operatorname{supp}\widehat{v_0}$ with the initial data given by (1.11). On the other hand for the part with the convolution of $\widehat{u_0}$ and $\widehat{u_0}$, we have from the structure $\nabla\cdot(u\nabla(-\Delta)^{-1}u)\cong|\nabla|^2\{(|\nabla|^{-2}u)u\}$ by (1.10) and the support of $\widehat{u_0}*\widehat{u_0}$ being in the neighborhood of the frequency $2N$, namely, $\xi-\eta\simeq N$, $\eta\simeq N$ and $\xi\simeq 2N$,

$$
\begin{aligned}
&\left\|\mathcal{F}^{-1}\left[|\xi|^{-2+\frac{n}{p}}\xi\cdot\int_{\mathbb{R}^n}\widehat{u_0}(\xi-\eta)\widehat{u_0}(\eta)\frac{\eta}{|\eta|^2}\frac{e^{-t|\xi|^2}\big(e^{2t(\xi-\eta)\cdot\eta}-1\big)}{2(\xi-\eta)\cdot\eta}d\eta\right]\right\|_{L^p}\\
&\cong \left\|\mathcal{F}^{-1}\left[|\xi|^{-2+\frac{n}{p}}|\xi|^2\int_{\mathbb{R}^n}\widehat{u_0}(\xi-\eta)\widehat{u_0}(\eta)\frac{1}{|\eta|^2}\frac{e^{-t|\xi|^2}\big(e^{2t(\xi-\eta)\cdot\eta}-1\big)}{2(\xi-\eta)\cdot\eta}d\eta\right]\right\|_{L^p}\\
&\cong \left\|\mathcal{F}^{-1}\big[|\xi|^{\frac{n}{p}}\chi(\xi-2N)\big]\right\|_{L^p}N^{2(2-\frac{n}{p})}N^{-2}N^{-2}\\
&\cong (2N)^{\frac{n}{p}}N^{-\frac{2n}{p}},\\
&\cong N^{-\frac{n}{p}}
\end{aligned}
\tag{1.13}
$$

and the last term is bounded for all $N\in\mathbb{N}$. Therefore, we can expect the divergence of the nonlinear term u as $N\to\infty$ in the case $p>2n$ by (1.12) and (1.13) although the norm of the initial data is bounded. Then one can expect the existence of solutions such that the solution is large and the initial datum is small.

§ 2. Comparison with the nonlinear term u^2

We give a remark that the optimal space of well-posedness for the nonlinear heat equation $\partial_t u-\Delta u=u^2$ is different from that for the drift diffusion equation, where scaling invariant spaces are common and $\dot{B}_{p,q}^{-2+\frac{n}{p}}(\mathbb{R}^n)$ $(1\le p,q\le\infty)$. In conclusion, we shall explain that the critical space for the nonlinear term u^2 is worse than that for the drift diffusion equation to obtain the well-posedness since the term u^2 is not a divergence form.

Let us consider 2 space dimensions for simplicity. For the following heat equation;

$$\begin{cases} \partial_t u - \Delta u = u^2 & t > 0, x \in \mathbb{R}^2, \\ u(0,x) = u_0(x) & x \in \mathbb{R}^2, \end{cases}$$

it is possible to show the ill-posedness in the space $H^s(\mathbb{R}^2)$ with $s \le -1$ by the result [10]. The space $H^{-1}(\mathbb{R}^2)$ corresponds to the case $p = n = 2$ for the scaling critical space $\dot{B}_{p,q}^{-2+\frac{2}{p}}(\mathbb{R}^2)$, and let us explain the essential reason for the optimality of the case $p = n$ through the analogous observation to (1.12). Similarly to (1.11), let u_0 be defined by

$$u_0 := N^{2-\frac{2}{p}} \mathcal{F}[\chi(\cdot - Ne_1) + \chi(\cdot + Ne_1)], \tag{2.1}$$

and we see that

$$\sup_{N \ge 1} \|u_0\|_{H_p^{-2+\frac{2}{p}}} < \infty.$$

Here we note that the initial data (2.1) is taken such that u_0 have the frequency component N and $-N$, and there is a high-high frequency interaction to low frequency for the estimate (2.2) below. Let us consider the solution u satisfying the integral equation;

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\tau)\Delta} u(\tau)^2 d\tau.$$

Approximating the nonlinear term by $u(\tau)^2 \simeq (e^{\tau\Delta} u_0)^2$, we have from the ristriction to the low frequencey part by $\chi(\xi)$

$$\begin{aligned} &\Big\| \int_0^t e^{(t-\tau)\Delta} (e^{\tau\Delta} u_0)^2 d\tau \Big\|_{H^{-2+\frac{2}{p}}} \\ &= \Big\| \mathcal{F}^{-1} \Big[(1 + |\xi|)^{-2+\frac{2}{p}} \int_{\mathbb{R}^n} \widehat{u_0}(\xi - \eta) \widehat{u_0}(\eta) \frac{e^{-t|\xi|^2} (e^{2t(\xi-\eta)\cdot\eta})}{2(\xi-\eta)\cdot\eta} d\eta \Big] \Big\|_{L^p} \\ &\ge \Big\| \mathcal{F}^{-1} \Big[\chi(\xi) \cdot 1^{-2+\frac{2}{p}} \cdot \int_{\mathbb{R}^n} \widehat{u_0}(\xi - \eta) \widehat{u_0}(\eta) \frac{e^{-t|\xi|^2} (e^{2t(\xi-\eta)\cdot\eta})}{2(\xi-\eta)\cdot\eta} d\eta \Big] \Big\|_{L^p} \\ &\ge c (N^{2-\frac{2}{p}})^2 \cdot \frac{1}{N^2} \\ &= c N^{2-\frac{4}{p}} \\ &\to \infty \quad \text{as } N \to \infty \quad \text{if } p > 2. \end{aligned} \tag{2.2}$$

In the third line of the above estimate, we used the restriction $|\xi| \le 1$, $|\xi - \eta| \simeq N$ and $|\eta| \simeq N$ obtained for the initial data u_0 given by (2.1). By this observation, one can expect that the problem is ill-posed for the case $p > 2(= n)$ since the nonlinear term can be arbitrary large for initial data which are bounded.

Drift diffusion equations of monopolar type and bipolar type and heat equation with the nonlinear term u^2 have the divergence form of the two order, the one order and the zero order, respectively. Although the scaling invariant spaces are common for all equations, the observations (1.12), (1.13) and (2.2) imply that smaller order of divergence structure causes the ill-posedness in the smaller spaces. Indeed, in Besov spaces $\dot{B}^{-2+\frac{n}{p}}_{p,\infty}(\mathbb{R}^n)$, ill-posedness are obtained with the indices $p = \infty$ for the drift diffusion equation of monopolar type, $p \geq 2n$ for the drift diffusion equation of bipolar type and $p \geq n$ for the heat equation with the nonlinear term u^2. The following embeddings are known

$$\dot{B}^{-1}_{n,q}(\mathbb{R}^n) \subset \dot{B}^{-\frac{3}{2}}_{2n,q}(\mathbb{R}^n) \subset \dot{B}^{-2}_{\infty,q}(\mathbb{R}^n),$$

where $1 \leq q \leq \infty$. Therefore, the spaces of well-posedness for the nonlinearity u^2, which does not have the divergence form structure, should be smaller than those for the drift diffusion equations.

Acknowledgments. The work of T. Iwabuchi is partially supported by JSPS Grant in Aid for Young Scientists (B) #25800069. The work of T. Ogawa is partially supported by JSPS Grant in Aid Scientific Research (S) #25220702.

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