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A note on multilinear fractional integral operators

By

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Abstract

This article is organized in the following way. In Section 1 we state a brief history of multilinear operators, in particular the bilinear Hilbert transform and bilinear fractional integral operator. In Section 2 we summarize our recent results in [6].

§ 1. Introduction

Suppose that Γ is a curve in the complex plane $\mathbb{C}$ given by $z(x) = x + iA(x)$. We consider the following classical problem: given a continuous, bounded function f on Γ, does there exist a function $F(z)$, analytic in $\mathbb{C} \setminus \Gamma$, such that

$$\lim_{\varepsilon \to +0} F(z(x) + i\varepsilon) - F(z(x) - i\varepsilon) = f(z(x)).$$

Our first approach is to consider the Cauchy integral of f on Γ: $Cf(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w-z}\, dw$. For $z \in \Gamma$, we write

$$C_\Gamma f(z) = \lim_{\varepsilon \to +0} \frac{1}{2\pi i} \int_{w \in \Gamma : |w-z| > \varepsilon} \frac{f(w)}{w - z}\, dw,$$

$$C^+ f(z) = \lim_{\varepsilon \to +0} Cf(z + i\varepsilon) \quad \text{and} \quad C^- f(z) = \lim_{\varepsilon \to +0} Cf(z - i\varepsilon),$$

whenever the limits exist. If both f and Γ are sufficiently smooth, we have

$$C_\Gamma f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w) - f(z)}{w - z}\, dw + \frac{f(z)}{2}.$$

By similar calculation we obtain the following Plemelj's formulae:

$$C^+ f(z) = C_\Gamma f(z) + 1/2\, f(z) \quad \text{and} \quad C^- f(z) = C_\Gamma f(z) - 1/2\, f(z).$$

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By a simple change of variable, the problem is reduced to consider the following singular integral operator:

$$C_A f(x) = \text{p.v.} \int_{\mathbb{R}^1} \frac{f(y)}{x-y+i(A(x)-A(y))}\, dy, \quad \text{where} \quad A' \in L^\infty(\mathbb{R}^1).$$

This is called the Cauchy integral operator. If we can prove the L^p boundedness of C_A, by a standard argument of singular integral operators, we can also prove the boundedness of the maximal Cauchy integral operator

$$C_A^* f(x) = \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y+i(A(x)-A(y))}\, dy \right|,$$

and we can show that $C_A f$ converges almost everywhere when $f \in L^p$. C_A can be written, at least formally, as the following.

$$C_A f(x) = \sum_{k=0}^{\infty} (-i)^k T_A^k f(x), \quad \text{where} \quad T_A^k f(x) = \text{p.v.} \int_{\mathbb{R}^1} \frac{(A(x)-A(y))^k}{(x-y)^{k+1}} f(y)\, dy.$$

T_A^0 is nothing but the Hilbert transform $Hf(x) = \text{p.v.} \int f(y)/(x-y)\, dy$, therefore T_A^1 is very important and this is called the Calderón's commutator, and we denote

$$T_A f(x) = \text{p.v.} \int_{\mathbb{R}^1} \frac{A(x)-A(y)}{(x-y)^2} f(y)\, dy.$$

The name "commutator" comes from the formula: $T_A = [|D|, M_A]$, where $D = -i\frac{d}{dx}$ and $M_A f(x) = A(x)f(x)$. Note that $\frac{d}{dx}[H, M_A]f(x) + A'(x)Hf(x) = T_A f(x)$.

Calderón proposed the next problem.

Calderón's problem. Prove the L^p boundedness of T_A when $A' \in L^\infty(\mathbb{R}^1)$ and $1 < p < \infty$.

To solve this problem he wrote

$$T_A f(x) = \int_0^1 \left(\text{p.v.} \int_{\mathbb{R}^1} \frac{a(x-ty)f(x-y)}{y}\, dy \right) dt = \int_0^1 H_t(a,f)(x)\, dt,$$

where $a = A'$ and H_t is called parameterized bilinear Hilbert transforms. Note that $H_0(a,f)(x) = a(x)Hf(x)$ and $H_1(a,f)(x) = H(af)(x)$, therefore

$$\|H_t(a,f)\|_{L^p} \le C_p \|a\|_{L^\infty} \|f\|_{L^p} \quad \text{when} \quad t = 0, 1.$$

By this observation, Calderón conjectured the following:

$$\|H_t(a,f)\|_{L^p} \le C_{p,t} \|a\|_{L^\infty} \|f\|_{L^p} \quad \text{and} \quad \int_0^1 C_{p,t}\, dt < \infty.$$

Calderón also considered more general cases where $t \in \mathbb{R}^1$. Since all the operators H_t behave similarly for any real number t, for symmetry reasons the traditional approach was to consider the following particular formula which corresponds to $t = -1$.

Definition 1.1 (the bilinear Hilbert transform)**.**

$$BH(f,g)(x) = \text{p.v.} \int_{\mathbb{R}^1} \frac{f(x+y)g(x-y)}{y}\, dy.$$

Calderón conjectured the following:

Calderón's conjecture. $\|BH(f,g)\|_{L^r} \le C_{p,q}\|f\|_{L^p}\|g\|_{L^q}$, where $1/r = 1/p + 1/q$ and $1 < p, q \le \infty$, but $(p,q) \ne (\infty,\infty)$.

Calderón's problem was solved by Coifman and Meyer [2], [3], and the L^p boundedness of the Cauchy integral operator was proved by Coifman, McIntosh and Meyer [1].

Theorem 1.2 (Coifman and Meyer [2], [3])**.**

$$\|T_A\|_{L^p} \le C_p\|A'\|_{L^\infty}\|f\|_{L^p} \quad \textit{where} \quad 1 < p < \infty.$$

To prove this theorem they wrote T_A as the following bilinear Fourier multiplier operator. Let $a = A'$.

$$T_A f(x) = -\text{p.v.} \int_{\mathbb{R}^1} \Big(\int_0^1 a(x+ty)\, dt \Big) f(x+y)\, \frac{dy}{y}.$$

Since $a(x+ty) = \int_{\mathbb{R}^1} \widehat{a}(\eta) e^{2\pi i(x+ty)\eta}\, d\eta$, $f(x+y) = \int_{\mathbb{R}^1} \widehat{f}(\xi) e^{2\pi i(x+y)\xi}\, d\xi$ and

$$\text{p.v.} \int_{\mathbb{R}^1} \frac{e^{2\pi i y(\xi+t\eta)}}{y}\, dy = -\text{sign}(\xi + t\eta),$$

$$T_A f(x) = \iint_{\mathbb{R}^2} \widehat{f}(\xi)\widehat{a}(\eta) m(\xi,\eta) e^{2\pi i x(\xi+\eta)}\, d\xi d\eta, \tag{1.1}$$

where $m(\xi,\eta) = \int_0^1 \text{sign}(\xi + t\eta) dt$. Note that this is the restriction of two-dimensional Fourier multiplier operator to the diagonal. By this observation we define another bilinear Hilbert transform.

Definition 1.3.

$$\widetilde{BH}_j(f,g)(x) = \iint_{\mathbb{R}^2} \frac{\Omega_j(y_1,y_2)}{|y_1|^2 + |y_2|^2} f(x-y_1)g(x-y_2)\, dy_1 dy_2, \quad j = 1, 2,$$

where $\Omega_j(y_1,y_2) = y_j/\sqrt{y_1^2 + y_2^2}$.

Calderón' s problem was solved, but Calderón's conjecture remained unsolved. The symbol of T_A (see (1.1)) is continuous except for the origin, but that of BH is discontinuous on the line $\xi + \eta = 0$. Therefore the bilinear Hilbert transform is strongly singular integral operator, and $\widetilde{BH}_j$ is less singular than BH.

Around 1990 there was no result for the bilinear Hilbert transform, so Grafakos [4] introduced bilinear fractional integral operators.

Definition 1.4.

$$BH_\alpha(f,g)(x) = \int_{\mathbb{R}^1} \frac{f(x+y)g(x-y)}{|y|^{1-\alpha}}\,dy, \quad 0<\alpha<1, \tag{1.2}$$

$$\widetilde{BH}_\alpha(f,g)(x) = \iint_{\mathbb{R}^2} \frac{f(x-y_1)g(x-y_2)}{(|y_1|+|y_2|)^{2-\alpha}}\,dy_1dy_2, \quad 0<\alpha<2. \tag{1.3}$$

These operators are less singular than BH. However if we have nice estimates for these operators, we may be able to obtain new knowledge for BH. This is a motive of introducing bilinear fractional integral operators, but nowadays these operator are interesting themselves and many studied have been done for bilinear fractional integral operators. In the next section we shall show our results for $\widetilde{BH}_\alpha$.

Calderón's conjecture was solved in the affirmative when $r > 2/3$ by Lacey and Thiele [7], [8].

§ 2. Endpoint estimates for multilinear fractional integral operators

We recall some elementary properties for the ordinary fractional integral operators. In the following we consider on $\mathbb{R}^n$.

Definition 2.1.

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad 0<\alpha<n.$$

Proposition 2.2.

$$\begin{aligned}
&I_\alpha : L^p \to L^q \quad \textit{when} \quad 1/q = 1/p - \alpha/n > 0 \quad \textit{and} \quad p>1,\\
&I_\alpha : L^1 \to L^{q,\infty} \quad \textit{when} \quad 1/q = 1-\alpha/n,\\
&I_\alpha : L^{n/\alpha} \not\to L^\infty.
\end{aligned}$$

Multilinear fractional integral operators are defined as follows.

Definition 2.3.

$$I_{m,\alpha}(f_1,f_2,\dots,f_m)(x) = \int_{\mathbb{R}^{mn}} \frac{f_1(y_1)f_2(y_2)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}\,dy, \quad 0<\alpha<mn.$$

Kenig and Stein [5] proved the next theorem.

Theorem 2.4 (Kenig and Stein [5])**.** *Let* $1/q = \sum_{i=1}^m 1/p_i - \alpha/n > 0$. *If each* $p_i > 1$, *then* $I_{m,\alpha}$ *is bounded from* $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$. *If* $p_i = 1$ *for some* i *and* $p_j > 1$ *for* $j \neq i$, *then* $I_{m,\alpha}$ *is bounded from* $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ *to* $L^{q,\infty}(\mathbb{R}^n)$.

In [6] we prove that $I_{m,\alpha}$ is bounded from $\prod_{i=1}^{m} L^{p_i}$ to L^q even if some p_i are equal to one under additional conditions on α.

We state our results. In the following we always assume that $1 \le p_1 \le p_2 \le \cdots \le p_m \le \infty$.

Theorem 2.5. *Let $p_1 = \cdots = p_k = 1, 1 < p_{k+1}, \dots, p_{m-l} < \infty$ and $p_{m-l+1} = \cdots = p_m = \infty$ for some $0 \le k < m - l \le m$, and $1/q = k + \sum_{i=k+1}^{m-l} 1/p_i - \alpha/n > 0$. Assume that*

$$kn \le \alpha < (m - l)n. \tag{2.1}$$

Then

$$\|I_{m,\alpha}(f_1, \dots, f_m)\|_{L^q} \le C \prod_{i=1}^{k} \|f_i\|_{L^1} \prod_{i=k+1}^{m-l} \|f_i\|_{L^{p_i}} \prod_{i=m-l+1}^{m} \|f_i\|_{L^\infty}.$$

If $k = 0$ then we assume that $1 < p_i$ for all i and $\alpha > 0$. If $l = 0$ then we assume that $p_i < \infty$ for all i.

Remark. $I_{m,\alpha}$ is not bounded from $L^1 \times \cdots \times L^1 \times L^\infty \times \cdots \times L^\infty$ to L^q.

When $q = \infty$ we obtain the following result.

Theorem 2.6. *Let $p_1 = \cdots = p_k = 1$, $1 < p_{k+1}, \dots, p_{m-l} < \infty$ and $p_{m-l+1} = \cdots = p_m = \infty$ for some $0 \le k < m - l \le m$, and $k + \sum_{i=k+1}^{m-l} 1/p_i = \alpha/n$. Assume that*

$$(k + 1)n \le \alpha < (m - l)n. \tag{2.2}$$

Then

$$\|I_{m,\alpha}(f_1, \dots, f_m)\|_{L^\infty} \le C \prod_{i=1}^{k} \|f_i\|_{L^1} \prod_{i=k+1}^{m-l} \|f_i\|_{L^{p_i}} \prod_{i=m-l+1}^{m} \|f_i\|_{L^\infty}.$$

If $k = 0$ then we assume that $1 < p_i$ for all i. If $l = 0$ then we assume that $p_i < \infty$ for all i.

Remark. $I_{m,\alpha}$ is not bounded from $L^1 \times \cdots \times L^1 \times L^{p_i} \times L^\infty \times \cdots \times L^\infty$ to L^∞.

We shall give a proof of Theorem 2.5 for $m = 2$. We prove only easy cases: $p_1 = 1$ and $n < \alpha < 2n$, or $p_2 = \infty$ and $0 < \alpha < n$.

Proof of Theorem 2.5 ($m = 2$). Let $1/q = 1 + 1/p_2 - \alpha/n$ and $n < \alpha < 2n$. Since

$$|I_{2,\alpha}(f_1, f_2)(x)| \le \|f_1\|_{L^1} \int_{\mathbb{R}^n} \frac{|f_2(y_2)|}{|x - y_2|^{n-(\alpha-n)}} \, dy_2 = \|f_1\|_{L^1} I_{\alpha-n}(|f_2|)(x),$$

and $1/q = 1/p_2 - (\alpha - n)/n$, we have

$$\|I_{2,\alpha}(f_1, f_2)\|_{L^q} \le C \|f_1\|_{L^1} \|f_2\|_{L^{p_1}}$$

by Proposition 2.2.

Next we consider the case $p_2 = \infty, 1/q = 1/p_1 - \alpha/n$ and $0 < \alpha < n$. Since

$$|I_{2,\alpha}(f_1, f_2)(x)| \le \|f_2\|_{L^\infty} \int_{\mathbb{R}^n} |f_1(y_1)| \left(\int_{\mathbb{R}^n} \frac{1}{(|x-y_1| + |x-y_2|)^{n+n-\alpha}} \, dy_2 \right) dy_1$$
$$\le C\|f_2\|_{L^\infty} \int_{\mathbb{R}^n} \frac{|f_1(y_1)|}{|x-y_1|^{n-\alpha}} \, dy_1 \le C\|f_2\|_{L^\infty} I_\alpha(|f_1|)(x).$$

We obtain the desired result by Proposition 2.2. □

The main interest of Theorem 2.5 is the case $\alpha = kn$. We prove it by induction on m. The following inequality is essential.

$$\int_{\mathbb{R}^{2n}} \frac{|f_1(y_1) f_2(y_2)|}{(|y_1| + |y_2|)^n} dy_1 dy_2 \le C\|f_1\|_{L^p} \|f_2\|_{L^{p'}}, \quad 1 < p < \infty.$$

We show that the condition $(k+1)n \le \alpha$ in Theorem 2.6 is optimal by giving a counterexample when $n = 1$ and $m = 3$. If $1 + 1/p_2 = \alpha$ then $I_{3,\alpha}$ is not bounded from $L^1 \times L^{p_2} \times L^\infty$ to L^∞. Note that $\alpha < 2$.

Let $f_1(x) = \chi_{(0,1)}(x), f_2(x) = x^{-1/p_2} (\log x)^{-1} \chi_{\{x \ge 10\}}$ and $f_3(x) \equiv 1$. Then

$$\int_{\mathbb{R}^3} \frac{f_1(y_1) f_2(y_2) f_3(y_3)}{(|y_1| + |y_2| + |y_3|)^{3-\alpha}} dy_1 dy_2 dy_3 \ge C \int_{10}^\infty \frac{f_2(y_2)}{y_2^{2-\alpha}} dy_2 = \infty.$$

To prove Theorem 2.6 we use the following inequality.

$$\int_{\mathbb{R}^{3n}} \frac{|f_1(y_1) f_2(y_2) f_3(y_3)|}{(|y_1| + |y_2| + |y_3|)^{2n}} dy_1 dy_2 dy_3 \le C\|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|f_3\|_{L^{p_3}},$$

where $1/p_1 + 1/p_2 + 1/p_3 = 1$.

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