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Asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation

By

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Abstract

The integrable discrete nonlinear Schrödinger equation was introduced by Ablowitz-Ladik. We study the initial value problem for the defocusing version of the equation. The solution shows different asymptotic behaviors in three regions $|n|/t < 2, |n|/t \approx 2$ and $|n|/t > 2$. The present article is meant to be a short introduction to nonlinear steepest descent.

§ 1. Introduction

The defocusing integrable nonlinear Schrödinger equation (NLS)

$$iu_t + u_{xx} - 2|u|^2 u = 0 \tag{1.1}$$

can be solved by the inverse scattering transform. This technique can be used to study the asymptotic behavior of the solutions. Deift-Its-Zhou ([5, 8]) proved that the behavior is

$$u(x,t) \sim \alpha t^{-1/2} \exp(4iz_0^2 t - i\nu \log 8t) \quad \text{as } t \to \infty \text{ (decaying oscillation)}, \tag{1.2}$$

where $\alpha = \alpha(z_0) \in \mathbb{C}$ and $\nu = \nu(z_0) \in \mathbb{R}$ are constants determined by $z_0 = -x/4t$ and the reflection coefficient corresponding to the initial value $u(x,0)$. The point z_0 is the saddle point of the phase function that appears in a kind of nonlinear Fourier representation of u. Notice that (1.2) was derived in [19] by a formal calculation.

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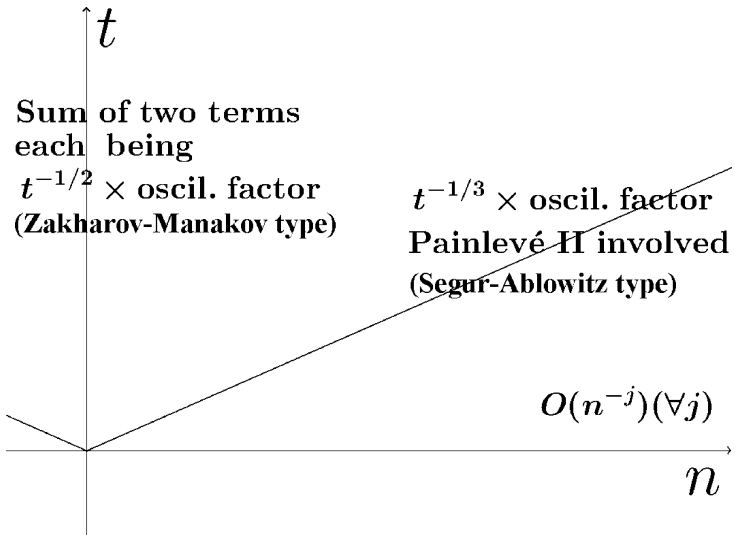

Figure 1. Three regions $|n|/t < 2, \approx 2, > 2$

The following discrete version of (1.1), called the (defocusing) integrable discrete nonlinear Schrödinger equation (IDNLS), was introduced by Ablowitz-Ladik ([1, 2]):

$$i\frac{d}{dt}R_n + (R_{n+1} - 2R_n + R_{n-1}) - |R_n|^2(R_{n+1} + R_{n-1}) = 0 \quad (t \geq 0, n \in \mathbb{Z}). \tag{1.3}$$

The nonlinear term is so chosen that the equation admits a Lax pair representation and can be solved by the inverse scattering transform. The present author studied the behavior of $R_n = R_n(t)$ satisfying (1.3) as $t \to \infty$ or $n \to \infty$ in three regions (Figure 1). Roughly speaking (see §4.4 for a more detailed statement):

- in $|n|/t < 2$, $R_n(t)$ is asymptotic to a sum of two terms, each being $t^{-1/2}\times$(oscillatory factor) as $t \to \infty$. This is a variation of the Zakharov-Manakov type formula in (1.2).

- in $|n|/t \approx 2$, $R_n(t)$ is asymptotic to $t^{-1/3}\times$(oscillatory factor) as $t \to \infty$.

  The oscillatory factor involves a quantity written in terms of a solution of the Painlevé II equation $u'' - su(s) - 2u^3(s) = 0$. This kind of behavior was first found by Segur-Ablowitz ([14]) concerning the KdV and the MKdV equations near $x = 0$.

- in $|n|/t > 2$, $R_n(t) = O(n^{-j})$ as $n \to \infty$ for any j.

The reader might ask why there are three regions. The answer shall be given in §4.4 in terms of the nonlinear Fourier representation of $R_n(t)$ involving a phase function; the configuration of its stationary points depends on the ratio n/t and different configurations lead to different asymptotic behaviors. Our proof is based on the nonlinear

steepest descent method of Deift-Zhou ([7]). Notice that a formal calculation was given in [13] for a slightly different equation (the focusing case).

The outline of this article is as follows. In § 2, we briefly explain Riemann-Hilbert problems and § 3 is devoted to the Beals-Coifman formula. In § 4, we explain nonlinear steepest descent. After reviewing known results about (1.1) and the MKdV equation, we give our own result about (1.3).

§ 2. Riemann-Hilbert problem and contour deformation

Let Γ be an oriented (reasonably good) contour in the complex plane. The left-hand side is called the $+$ side. Let $v(z)$ be a given 2×2 matrix on Γ. For an unknown 2×2 matrix $m(z)$ whose components are holomorphic in $\mathbb{C} \setminus \Gamma$, its boundary values on Γ from the $\pm$ sides are denoted by $m_{\pm}(z)$. We consider

$$m_+(z) = m_-(z)v(z) \text{ on } \Gamma, \tag{2.1}$$

$$m(z) \to I \text{ (the identity matrix) as } z \to \infty. \tag{2.2}$$

This kind of boundary value problem is called a Riemann-Hilbert problem (RHP).

Remark. An RHP can replace the role of a Gelfand-Levitan-Marchenko equation in the study of an integrable equation. In that context, v has parameters (x, t) or (n, t), and some components oscillate as $t \to \infty$. The contour Γ is the real axis and the circle $|z| = 1$ in the continuous and discrete cases respectively.

One can deform the contour in a Riemann-Hilbert problem. We give two examples. We consider $m_+ = m_- v$ on Γ as is indicated in Figure 2 and assume that $v = v(z)$ is holomorphic in a sufficiently large domain. We introduce a new unknown n as in the figure. Then it is holomorphic near Γ (no jump there), and $m_+ = m_- v$ on Γ is equivalent to $n_+ = n_- v$ on $\tilde{\Gamma}$.

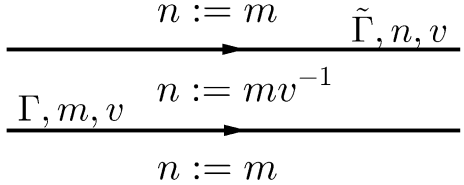

Figure 2. deformation

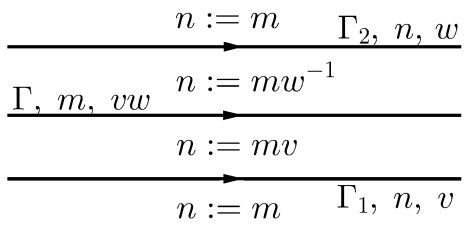

Figure 3. factorization

Next, we consider $m_+ = m_- vw$ on Γ in Figure 3. If we introduce a new unknown n as in the figure, then the original RHP is equivalent to $n_+ = n_- v$ on Γ_1 and $n_+ = n_- w$

on Γ_2. Assume additionally that w contains a parameter t and that it approaches I as $t \to \infty$. Then n has a very small jump on Γ_2 and is almost holomorphic there. Up to some error, w and Γ_2 can be neglected (§3). The asymptotic behavior of m is determined by v alone. Summing up, if one can get a nice factorization of a given jump matrix, one can remove insignificant factors by contour deformation and the original RHP is reduced (up to some error) to a simpler one.

Conjugation technique is useful in obtaining a good factorization. When $m_+ = m_- v$, set $n = m\Delta$, where Δ is an invertible matrix with $\Delta \to I (z \to \infty)$. It implies $n_+ = n_-(\Delta_-^{-1} v \Delta_+)$. We have a new jump matrix $\tilde{v} = \Delta_-^{-1} v \Delta_+$. In some cases, there exists a suitable matrix Δ with $\Delta_+ \neq \Delta_-$ such that $\tilde{v}$ has a good factorization.

§ 3. Beals-Coifman formula

If the jump matrix admits a certain kind of factorization, there is an integral representation for the solution of (2.1) and (2.2). It is continuous with respect to small perturbations of the coefficients.

Let C be the Cauchy integral along Γ: for an arbitrary function f on Γ, set

$$Cf(z) = \int_\Gamma \frac{f(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i}.$$

We denote its boundary values on Γ by $C_\pm f$. Then $C_+ f - C_- f = f$ holds. If v in (2.1) admits a factorization of the form $v = (I - w_-)^{-1}(I + w_+)$, we set $w = w_+ + w_-$ and define the operator C_w by $C_w f = C_+(f w_-) + C_-(f w_+)$. If μ solves $\mu = I + C_w \mu$, i.e. $\mu = (1 - C_w)^{-1} I$ (we assume that the resolvent exists), we set

$$m(z) = I + (C(\mu w))(z) = I + \int_\Gamma \frac{\mu(\zeta) w(\zeta)}{\zeta - z} \frac{d\zeta}{2\pi i}. \tag{3.1}$$

Then $m(z)$ satisfies (2.2). Moreover, (2.1) also holds because

$$\begin{aligned} m_+ &= I + C_+(\mu w) = I + C_+(\mu w_+) + C_+(\mu w_-) \\ &= I + \mu w_+ + C_-(\mu w_+) + C_+(\mu w_-) = I + C_w \mu + \mu w_+ = \mu(I + w_+), \end{aligned}$$

and similarly $m_- = \mu(I - w_-)$. The formula (3.1) is due to Beals-Coifman ([4]). It is useful in asymptotic analysis because m is continuous with respect to $w_\pm$. When $w_\pm$ is almost 0 on some part Γ' of Γ, then we can neglect Γ' if we admit a small error in evaluating $m(z)$. Another usage of continuity is the reduction to an exactly solvable case. Assume that the RHP corresponding to $w_\pm^0$ can be solved in closed form* and let

*A typical example is the use of the parabolic cylinder equation $d^2 g/d\zeta^2 + (1/2 - \zeta^2/4 + a)g = 0$ and connection formulas associated with it. See [7].

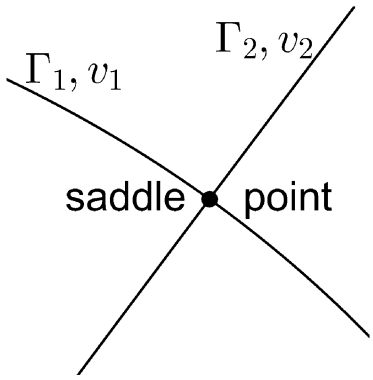

Figure 4. saddle point and a new contour

$m^0(z)$ be the solution. If $w_\pm$ is close to $w_\pm^0$, then the solution $m(z)$ corresponding to $w_\pm$ is close to $m^0(z)$.

This observation, together with contour deformation, leads to the Riemann-Hilbert version of the classical method of steepest descent, which is called *the Deift-Zhou method* or *nonlinear steepest descent.*

§ 4. Nonlinear steepest descent and application

§ 4.1. Overview

We consider a Riemann-Hilbert problem $m_+ = m_- v$ on a contour Γ. Assume that v involves $\exp(\pm it\psi)$ and that the function ψ is real at its saddle point S. Let $\Gamma_1 \cup \Gamma_2$ be a new contour as in Figure 4. Following the classical method of steepest descent, we draw Γ_j so that $\Gamma_1 \setminus \{S\} \subset \{\operatorname{Im}\psi > 0\}$ and $\Gamma_2 \setminus \{S\} \subset \{\operatorname{Im}\psi < 0\}$. It implies that $\exp(it\psi(z)) \to 0$ as $t \to \infty$ on $\Gamma_1 \setminus \{S\}$. There is a uniform estimate if one removes a small neighborhood of S. An analogous statement holds about $\exp(-it\psi(z))$ on Γ_2. Assume that the original RHP is, by the argument in § 2, equivalent to $n_+ = n_- v_j$ on Γ_j $(j = 1, 2)$, where $n = n(z)$ is a new unknown matrix. Moreover, suppose that $v_1 - I$ and $v_2 - I$ are of orders $O(\exp(it\psi))$ and $O(\exp(-it\psi))$ respectively. Then v_1 and v_2 are almost I except in a small neighborhood of S, if t is large. By continuity in §3, we conclude that $n(z)$ and $m(z)$ are almost determined by $v(z)$, where z is near S. Then we can approximate $v(z)$ by a simple matrix. If a solution of an integrable equation is calculated from $m(z)$, the saddle point argument as above is useful in deriving the asymptotic expansion of the solution. This technique is called *nonlinear steepest descent.*

If there is only one saddle point, the steepest descent argument is simple. That is the case with NLS. There are two saddle points in the MKdV case. Their contributions

are separated out in [7, § 3][†]and shown to be conjugate because of symmetry. In the IDNLS case, there are four saddle points on the circle $|z| = 1$. They are two pairs of antipodal points and the contribution from a point is identical with that of its antipodal point.

§ 4.2. Nonlinear Schrödinger equation

We summarize the result of Deift-Its-Zhou [5].

Let $r(z), z \in \mathbb{R}$, be the reflection coefficient determined by the initial value $u(x, 0)$. Then the solution $u(x, t)$ to the initial value problem of (1.1) is obtained from the solution $m(z)$ of an RHP on the real axis with the jump matrix

$$v_1 = \begin{bmatrix} 1 - |r(z)|^2 & -e^{-2it\psi_1}\overline{r(z)} \\ e^{2it\psi_1} r(z) & 1 \end{bmatrix}, \quad \psi_1 = \psi_1(z) = 2z^2 + \frac{xz}{t}.$$

The function $\psi_1(z)$ has only one saddle point $z_0 = -x/4t \in \mathbb{R}$. It plays the role of ψ in §4.1. The solution u is reconstructed from $m(z)$ by using the formula

$$u(x, t) = 2i \lim_{z\to\infty} z\, m(z; x, t)_{12},$$

where $m(z; x, t)_{12}$ is the $(1, 2)$-component of the matrix $m(z; x, t)$. Deift-Its-Zhou ([5]) replaced the original contour, the real axis, by a cross like the one in Figure 4. The geometry is not too complicated and they obtained (1.2).

§ 4.3. MKdV equation

We sketch some results of Deift-Zhou [7] about the MKdV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0. \tag{4.1}$$

Let $r(z), z \in \mathbb{R}$, be the reflection coefficient determined by the initial value $u(x, 0)$. Then the solution $u(x, t)$ to the initial value problem of (4.1) is obtained from the solution $m(z)$ of the RHP on the real axis with the jump matrix

$$v_2 = \begin{bmatrix} 1 - |r(z)|^2 & -e^{-2it\psi_2}\overline{r(z)} \\ e^{2it\psi_2} r(z) & 1 \end{bmatrix}, \quad \psi_2 = \psi_2(z) = 4z^3 + \frac{xz}{t}.$$

The reconstruction formula is

$$u(x, t) = 2 \lim_{z\to\infty} z\, m(z; x, t)_{21} = 2 \lim_{z\to\infty} z\, m(z; x, t)_{12}.$$

[†]An alternative idea is the use of cut-off functions. It would work only if the existence of some resolvents are assured. See § 5.

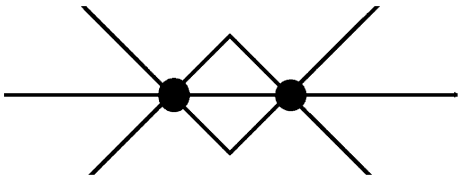

Figure 5. $x < 0$

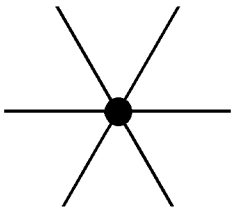

Figure 6. $x = 0$

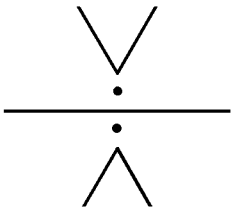

Figure 7. $x > 0$

The function ψ_2 has saddle points $\pm z_0 = \pm\sqrt{-x/12t}$ if $x \neq 0$. They are not always on the contour $\mathbb{R}$ and it makes geometry more complicated than in the NLS case. We introduce new contours as in Figures 5-7. The stationary points are indicated by large dots.

If $x < 0$, there are two distinct saddle points on $\mathbb{R}$. By symmetry, their contributions are conjugate and the leading part of the asymptotic expansion of u reduces to a single term. There exist real numbers $\alpha = \alpha(z_0)$, $\nu = \nu(z_0)$ and $\beta = \beta(z_0)$ such that

$$u \sim \alpha t^{-1/2} \cos(16tz_0^3 - \nu \log t + \beta) \text{ as } t \to \infty.$$

If $x = 0$, then $z_0 = 0$ is a double root of $\psi_2' = 0$ and is no longer a saddle point. One has $\psi_2 = 4z^3$ in this case. If $x = O(t^{1/3})$, one has

$$u \sim (3t)^{-1/3} p(x/(3t)^{1/3}) \text{ as } t \to \infty.$$

Here p is a solution of the Painlevé II equation $p''(s) - sp(s) - 2p^3(s) = 0$ and $p(x/(3t)^{1/3})$ is constant along a curve $x^3/t = \text{const}$.

If $x > 0$, there are two saddle points off the real axis. In $x > \text{const}.t$, u decays more rapidly than any negative power of x as $x \to \infty$.

§ 4.4. Discrete nonlinear Schrödinger equation

We assume $n \geq 0$ without loss of generality for the time being: the equation (1.3) is invariant under the reflection $n \mapsto -n$. In order to solve (1.3), one has only to consider an RHP on $|z| = 1$ (clockwise), as opposed to the real axis, with an unknown 2×2 matrix $m = m(z; n, t)$. Let $r(z), |z| = 1$, be the reflection coefficient determined by the initial value $R_n(0)$. Then the jump matrix is

$$v_3 = \begin{bmatrix} 1 - |r(z)|^2 & -e^{-2it\psi_3(z)}\overline{r(z)} \\ e^{2it\psi_3(z)} r(z) & 1 \end{bmatrix}, \quad \psi_3(z) = \psi_3(z, n, t) = \frac{1}{2}(z - z^{-1})^2 + \frac{in}{t} \log z.$$

The reconstruction formula is ([1, 2, 3])

$$R_n(t) = -\lim_{z \to 0} \frac{1}{z} m(z)_{21} = -\left. \frac{d}{dz} m(z)_{21} \right|_{z=0}$$

The stationary points of ψ_3 are $z = S_j$ $(j = 1, 2, 3, 4)$, where

$$S_1 = e^{-\pi i/4}A,\ S_2 = e^{-\pi i/4}\bar{A},\ S_3 = -S_1,\ S_4 = -S_2,$$
$$A = 2^{-1}\big(\sqrt{2 + n/t} - i\sqrt{2 - n/t}\,\big).$$

Their configuration is as follows:

- $n/t < 2$: four saddle points (two pairs of antipodal points) on $|z| = 1$ (simple zeros of ψ_3').
- $n/t = 2$: two stationary points (a pair of antipodal double zeros of ψ_3').
- $n/t > 2$: four saddle points *off* $|z| = 1$.

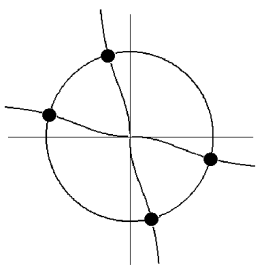

Figure 8. $n/t < 2$

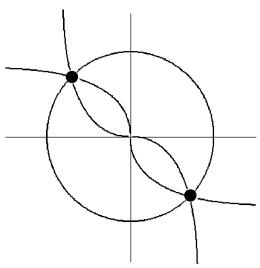

Figure 9. $n/t = 2$

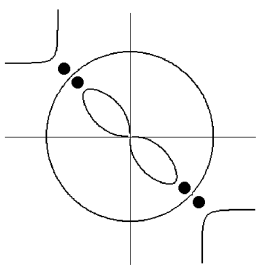

Figure 10. $n/t > 2$

In order to introduce new contours, we have to draw the curve $\operatorname{Im}\psi_3(z) = 0$. See Figures 8-10. We calculate the asymptotic behavior of $R_n(t)$ by using the new contours shown in Figures 11-13. How many terms does the leading part consist of? The contributions of antipodal points coincide. So we should count the number of *pairs* of antipodal points, rather than the number of *points*.

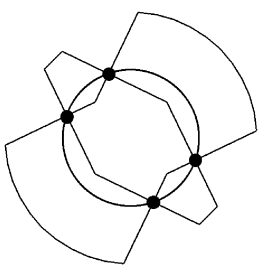

Figure 11. $n/t < 2$

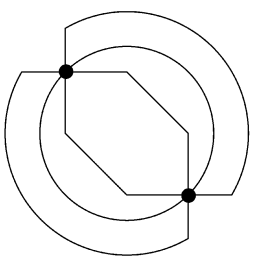

Figure 12. $n/t = 2$

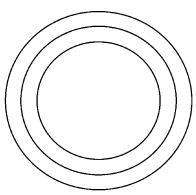

Figure 13. $n/t > 2$

Our result, including the case $n < 0$, is as follows (see [16, 17] for details).

Assume that $\sum_{n\in\mathbb{Z}} |n|^k |R_n(0)|$ is finite for any k and that $\sup_{n\in\mathbb{Z}} |R_n(0)| < 1$ holds. Then we have:

- In the region $|n|/t < 2$, there exist $C_j = C_j(n/t) \in \mathbb{C}$, $p_j = p_j(n/t) \in \mathbb{R}$ and $q_j = q_j(n/t) \in \mathbb{R}$ such that

$$R_n(t) = \sum_{j=1}^{2} C_j t^{-1/2} \exp\Big(-i(p_j t + q_j \log t)\Big) + O(t^{-1}\log t) \quad \text{as } t \to \infty.$$

  The leading part is the contributions of two pairs of saddle points.

- In the region $n/t \approx 2$, we consider a curve $2 - n/t = \text{const.}t^{-2/3}(6 - n/t)^{1/3}$. Then, up to a time shift $t \mapsto t - t_0$, we have $n/t \to 2$ and

$$R_n(t) = \text{const.}t^{-1/3}e^{i(-4t+\pi n)/2} + O(t^{-2/3}) \tag{4.2}$$

  as $t \to \infty$ on this curve. This constant is written in terms of the Painlevé II function. The leading part is the contribution of one pair of stationary points.

- In the region $n/t \approx -2$, the behavior is a variation of (4.2): we have only to replace n by $-n$.

- In the region $|n|/t > 2$, we have $|R_n(t)| = O(n^{-j})$ for any j as $n \to \infty$.

§ 5. Open problem

The next step is to study the focusing case:

$$i\frac{d}{dt}R_n + (R_{n+1} - 2R_n + R_{n-1}) + |R_n|^2(R_{n+1} + R_{n-1}) = 0 \ (t \geq 0, n \in \mathbb{Z}). \tag{5.1}$$

The difficulty and fun lie in the fact that it admits solitons. They can be dealt with by incorporating poles in RHPs, as is carried out, for example, in [6, 9, 10, 11, 12].

The absolute value of the reflection coefficient may be large in the focusing case and the existence of some resolvents is more difficult to prove than in the defocusing case. The present author thinks that this difficulty can be overcome by using results in [6, 15, 18]. The shortest path is probably through [6, Lemma 5.9]. In it, the authors studied an integral operator on $\mathbb{R}_z$ associated with the matrix

$$\begin{bmatrix} 1 + |r(z_0)|^2 & r(z_0)e^{i\theta} \\ \overline{r(z_0)}e^{-i\theta} & 1 \end{bmatrix}, \ z_0 = \frac{x}{t}, \ \theta = \theta(z) = xz - \frac{tz^2}{2},$$

and showed the existence of its resolvent. (The present author is mainly interested in the case $x = 0$.) Their proof based on a Fredholm argument works even if $|r(z_0)| \geq 1$ and is a significant improvement over a simple one based on the smallness of $|r(z_0)|$ ([7, (3.93)], [16, §11.1]). On the other hand, the calculations in [15, 18] can be applied

if $r(z_0)$ is replaced by $r(z), z \in \mathbb{R}$. They do not need the smallness of $|r(z)|_{L^\infty(\mathbb{R})}$ but they do need decay assumptions on $r(z)$ as $|z| \to \infty$. The reduction of our problem to such a case might be possible by partition of unity. Notice that their methods are more like the method of stationary phase in the $\mathcal{C}^\infty$-category than the method of steepest descent: they rely on integration by parts rather than on contour deformation.

References

[1] Ablowitz, M. J. and Ladik, J. F., Nonlinear differential-difference equations, *J. Math. Phys.*, **16** (1975), 598-603.

[2] Ablowitz, M. J. and Ladik, J. F., Nonlinear differential-difference equations and Fourier analysis, *J. Math. Phys.*, **17** (1976), 1011-1018.

[3] Ablowitz, M. J., Prinari, B. and Trubatch, A. D., *Discrete and continuous nonlinear Schrödinger systems*, Cambridge University Press, 2004.

[4] Beals, R. and Coifman, R., Scattering and inverse scattering for first order systems, *Comm. Pure Appl. Math.*, **37** (1984), 39-90.

[5] Deift, P. A., Its, A. R. and Zhou, X., Long-time asymptotics for integrable nonlinear wave equations, *Important developments in soliton theory, 1980-1990* edited by A. S. Fokas and V. E. Zakharov, Springer-Verlag (1993), 181-204.

[6] Deift, P. and Park, J., Long-time asymptotics for solutions of the NLS equation with a delta potential and even initial data, *Int. Math. Res. Not. IMRN*, **24** (2011), 5505-5624.

[7] Deift, P. A. and Zhou, X., A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, *Ann. of Math.*(2), **137**(2) (1993), 295-368.

[8] Deift, P. A. and Zhou, X., *Long-time behavior of the non-focusing nonlinear Schrödinger equation – a case study*, Lectures in Mathematical Sciences No. 5, The University of Tokyo, 1994.

[9] Fokas, A. S. and Its, A. R., The linearization of the initial-boundary value problem of the nonlinear Schrödinger equation, *SIAM J. Math. Anal.*, **27**(3)(1996), 738-764.

[10] Grunert, K. and Teschl, G., *Long-time asymptotics for the Korteweg-de Vries equation via nonlinear steepest descent*, *Math. Phys. Anal. Geom. M*, **12** (2009), 287-324.

[11] Kamvissis S., Focusing nonlinear Schrödinger equation with infinitely many solitons, *J. Math. Phys*, **36** (1995), 4175-4180.

[12] Krüger, H. and Teschl, G., Long-time asymptotics of the Toda lattice in the soliton region, *Math. Z.*, **262**(3) (2009), 585-602.

[13] Novokshënov, V. Yu., Asymptotic behavior as $t \to \infty$ of the solution of the Cauchy problem for a nonlinear differential-difference Schrödinger equation, *Differentsialnye Uravneniya*, **21** (1985), no. 11, 1915-1926. (in Russian); *Differential Equations*, **21** (1985), no. 11, 1288-1298.

[14] Segur, H. and Ablowitz, M. J. , Asymptotic solutions of nonlinear evolution equations and a Painlevé Transcendent, *Physica D*, **3** (1981), 165-184.

[15] Varzugin, G. G., Asymptotics of oscillatory Riemann-Hilbert problems, *J. Math. Phys*, **37** (1996), 5869-5892.

[16] Yamane, H., Long-time asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation, *J. Math. Soc. Japan*, **66** (2014), 765 - 803.

[17] Yamane, H., Long-time asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation II, arXiv:1407.5751 [math-ph].

[18] Do, Y., A nonlinear stationary phase method for oscillatory Riemann-Hilbert Problems, *Int. Math. Res. Not. IMRN*, **12**, (2011), 2650-2765.

[19] Zakharov, V. E. and Manakov, S. V., Asymptotic behavior of nonlinear wave systems integrated by the inverse method, *Zh. Eksp. Teor. Fiz.*, **71** (1976), 203-215 (in Russian); *Sov. Phys.-JETP*, **44** (1976), 106-112.