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The hypergeometric function and WKB solutions

By

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Abstract

The hypergeometric function is a constant multiple of the Borel sum of the WKB solution of the hypergeometric differential equation with a large parameter which is recessive at the origin. The constant can be computed explicitly.

Introduction

The aim of this article is to relate the hypergeometric function containing a large parameter to the WKB solutions of the hypergeometric differential equation with the large parameter. It is well known that the hypergeometric differential equation

$$x(1-x)\frac{d^2w}{dx^2}+(c-(a+b+1)x)\frac{dw}{dx}-abw=0, \tag{0.1}$$

has a solution defined by the hypergeometric series

$$F(a,b,c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}x^n \tag{0.2}$$

which is convergent in the unit disk with the center at the origin in the complex plane and defines a holomorphic function called the hypergeometric function (cf. [4], [5]). Here the Pochhammer symbol $(a)_n$ stands for $a(a+1)(a+2)\cdots(a+n-1)$. If a solution w of (0.1) is holomorphic at the origin, then it is a constant multiple of the

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hypergeometric function. The hypergeometric function can be analytically continued to a multi-valued holomorphic function defined in $\mathbb{C} - \{0, 1\}$, which is also called the hypergeometric function. We put a large parameter η in (0.1) by setting

$$a = \frac{1}{2} + \alpha\eta, \quad b = \frac{1}{2} + \beta\eta, \quad c = 1 + \gamma\eta. \tag{0.3}$$

Here α, β and γ are complex constants. Next we eliminate the first-order term of (0.1) by introducing a new unknown function ψ:

$$w = x^{-\frac{1}{2}-\frac{\gamma\eta}{2}}(1-x)^{-\frac{1}{2}-\frac{(\alpha+\beta-\gamma)\eta}{2}}\psi. \tag{0.4}$$

The equation for ψ has the form

$$\Big(-\frac{d^2}{dx^2} + \eta^2 Q\Big)\psi = 0, \tag{0.5}$$

where the explicit form of Q is given in §2. The WKB solutions of this equation are, by definition, formal solutions of the form

$$\psi = \exp\left(\int S dx\right).$$

Here $S = \sum_{j=-1}^{\infty} \eta^{-j} S_j$ is a formal solution of the Riccati equation

$$\frac{dS}{dx} + S^2 = \eta^2 Q \tag{0.6}$$

associated with (0.5). If we take a suitable normalization of the integral in a WKB solution ψ, it is Borel summable under some assumptions and the Borel sum of ψ is an analytic solution of (0.5). Thus we arrive at the following natural question: What is the relation between the solution of (0.5) coming from the hypergeometric function and the Borel sums of WKB solutions?

Partial answer to this question has been obtained by the third author [10], where the relation between a basis of the solution space consisting of hypergeometric functions and another basis given by the Borel sums of WKB solutions is obtained up to a multiplicative constant by comparing the monodromy matrices.

In this article, we employ another method and give an answer to our question for $F(a, b, c; x)$ including the determination of the constant that is left undermined in [10]. We use a special way of normalization for WKB solutions, namely, normalization at the origin, one of the regular singular points. Using this normalization and the discussion by Koike and Schäfke (cf. [7]) concerning the Borel summability of WKB solutions, we find that one of the WKB solutions is corresponding to a holomorphic solution of

the hypergeometric differential equation. Thus the Borel sum of the WKB solution is a constant multiple of the hypergeometric function. We determine the constant by evaluating the WKB solution at the origin. This idea is inspired by [8]. As an application, we obtain a formula which gives an asymptotic expansion of the hypergeometric function with respect to the large parameter.

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§ 1. WKB solutions normalized at a regular singular point

We consider the differential equation

$$\Big(-\frac{d^2}{dx^2}+\eta^2 Q\Big)\psi=0, \tag{1.1}$$

where $Q=Q(x,\eta)=\sum_{j=0}^{N}\eta^{-j}Q_j$ is a polynomial of η^{-1} whose coefficients are rational functions of x. The logarithmic derivative $S:=\psi'/\psi$ of the unknown function satisfies the following Riccati-type equation:

$$\frac{dS}{dx}+S^2=\eta^2 Q. \tag{1.2}$$

There are two formal solutions $S^{\pm}=\sum_{j=-1}^{\infty}\eta^{-j}S_j^{\pm}$ of the above equation corresponding to the choice of the branch of the leading term $S_{-1}^{\pm}=\pm\sqrt{Q_0}$. Here the branch of $\sqrt{Q_0}$ is chosen suitably. The WKB solutions of (1.1) have the form (cf. [6])

$$\exp\left(\int S^{\pm}dx\right).$$

If we set

$$S_{\mathrm{odd}}=\frac{1}{2}(S^+-S^-),\quad S_{\mathrm{even}}=\frac{1}{2}(S^++S^-), \tag{1.3}$$

we have

$$S^+=S_{\mathrm{odd}}+S_{\mathrm{even}},\quad S^-=-S_{\mathrm{odd}}+S_{\mathrm{even}} \tag{1.4}$$

and

$$S_{\mathrm{even}}=-\frac{1}{2}\frac{d}{dx}\log S_{\mathrm{odd}}. \tag{1.5}$$

In other words, we can take the following special normalization of the integral of S_{even}:

$$\int S_{\mathrm{even}}dx=-\frac{1}{2}\log S_{\mathrm{odd}}.$$

Hence

$$\frac{1}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int S_{\mathrm{odd}}dx\right) \tag{1.6}$$

are WKB solutions of (1.1).

In the sequel, we assume that x^2Q is holomorphic in a neighborhood of the origin. Then (1.1) has a regular singularity at $x=0$. We set

$$\operatorname*{Res}_{x=0}\sqrt{Q(x,\eta)} =: \rho = \rho_0 + \eta^{-1}\rho_1 + \eta^{-2}\rho_2 + \cdots$$

and assume that $\rho_0 \neq 0$. It follows from the discussion given in Proposition 3.6 of [6] that

$$\operatorname*{Res}_{x=0} S_{\mathrm{odd}} = c\eta \tag{1.7}$$

with

$$c = \rho\sqrt{1+\frac{1}{4\rho^2\eta^2}}. \tag{1.8}$$

Note that ρ and c are convergent power series of η^{-1} by the assumptions. Now we set

$$T(x,\eta) = S_{\mathrm{odd}}(x,\eta) - \frac{c\eta}{x}. \tag{1.9}$$

Then $T(x,\eta)$ is a formal power series in η^{-1} whose coefficients are holomorphic in a neighborhood of the origin and

$$\int_0^x T(x,\eta)dx$$

is well-defined. We may take $c\eta\log x$ as the normalization of the integral $\displaystyle\int\frac{c\eta}{x}dx$.

Definition 1.1. Formal solutions of (1.1) of the form

$$\psi_{\pm}^{(0)} = \frac{x^{c\eta}}{\sqrt{S_{\mathrm{odd}}}}\exp\left(\pm\int_0^x T(x,\eta)dx\right) \tag{1.10}$$

are called the WKB solutions normalized at the origin.

Under the above notation, we have

Theorem 1.2. *Assume* $\operatorname{Re}\rho_0 > 0$ *and set* $\tilde{\psi}_+^{(0)} = x^{-\frac{1}{2}-c\eta}\psi_+^{(0)}$. *Then* $\tilde{\psi}_+^{(0)}$ *is Borel summable in a neighborhood* U *of the origin and its Borel sum* $\tilde{\Phi}_+^{(0)}$ *is holomorphic in* $U\times\{\eta;\ \operatorname{Re}\eta \gg 0\}$. *Moreover,* $\tilde{\Phi}_+^{(0)}(0,\eta) = (c\eta)^{-\frac{1}{2}}$ *holds.*

Proof. Since $\operatorname{Re}\rho_0 > 0$, the real part of $\rho_0 \log x$ tends to $-\infty$ when $x \to 0$ along any integral curve of $\operatorname{Im}\sqrt{Q_0}dx = 0$ flowing into the origin. It follows from the arguments given in [7] that $\psi_+^{(0)}$ is Borel summable in $U - \{0\}$ for a neighborhood U of the origin and that the Borel sum $\Psi_+^{(0)}$ is holomorphic in $(U - \{0\}) \times \{\eta;\ \operatorname{Re}\eta \gg 0\}$. Hence $\tilde{\psi}_+^{(0)}$ is also Borel summable in $U - \{0\}$ and $\tilde{\Phi}_+^{(0)}$ is holomorphic in $(U - \{0\}) \times \{\eta;\ \operatorname{Re}\eta \gg 0\}$.

By the definition, $\tilde{\psi}_+^{(0)}$ has the form

$$\exp\left(\eta \int_0^x T_{-1}(x)dx\right) \sum_{j=0}^{\infty} \eta^{-\frac{1}{2}-j}\phi_j(x), \tag{1.11}$$

where T_{-1} denotes the leading term of $T(x,\eta)$ with respect to η and $\phi_j(x)$ are holomorphic functions defined in a neighborhood of the origin. We can take termwise evaluation:

$$\begin{aligned}
\tilde{\psi}_+^{(0)}(0,\eta) &= \sum_{j=0}^{\infty} \eta^{-\frac{1}{2}-j}\phi_j(0) \\
&= \frac{1}{\sqrt{c\eta + O(x)}} \exp\left(\int_0^x T(x,\eta)dx\right)\Bigg|_{x=0} \\
&= (c\eta)^{-\frac{1}{2}}.
\end{aligned}$$

By the definition, the Borel transform of $\tilde{\psi}_+^{(0)}$ is

$$\tilde{\psi}_{+,\mathrm{B}}^{(0)}(x,y) = \sum_{j=0}^{\infty} \frac{\phi_j(x)}{\Gamma(j+\frac{1}{2})}(y + t(x))^{j-\frac{1}{2}}, \tag{1.12}$$

where we set $t(x) = \displaystyle\int_0^x T_{-1}(x)dx$. The radius of convergence of the above series can be taken uniformly when x tends to the origin and constants concerning the estimates for discussing Borel summability can be taken uniformly in $\{x;\ 0 < |x| \le \delta\}$ for sufficiently small $\delta > 0$. Thus $\tilde{\psi}_+^{(0)}(x,\eta)$ is Borel summable in a neighborhood of $x = 0$. This implies $\{0\} \times \{\eta;\ \operatorname{Re}\eta \gg 0\}$ is a removable singularity of the Borel sum $\tilde{\Phi}_+^{(0)}(x,\eta)$. Since $\eta^{\frac{1}{2}}\tilde{\psi}_+^{(0)}(0,\eta)$ is a convergent power series with a finite radius of convergence, $y^{\frac{1}{2}} * \tilde{\psi}_{+,\mathrm{B}}^{(0)}(0,y)$ is an entire function of y of exponential type. Hence we can evaluate $\tilde{\Psi}_+^{(0)}(x,\eta)$ at $x = 0$ directly for sufficiently large η:

$$\begin{aligned}
\tilde{\Psi}_+^{(0)}(0,\eta) &= \frac{1}{2\pi i}\int_{|x|=\delta} \frac{\Phi_{+,\mathrm{B}}^{(0)}(x,\eta)}{x}dx \\
&= \frac{1}{2\pi i}\int_{|x|=\delta} \frac{1}{x}\int_0^{\infty} \tilde{\psi}_{+,\mathrm{B}}^{(0)}(x,y)\exp(-\eta y)dydx \\
&= \int_0^{\infty} \tilde{\psi}_{+,\mathrm{B}}^{(0)}(0,y)\exp(-\eta y)dy \\
&= \tilde{\psi}_+^{(0)}(0,\eta).
\end{aligned}$$

Thus we have $\tilde{\Psi}_+^{(0)}(0,\eta) = \tilde{\psi}_+^{(0)}(0,\eta) = (c\eta)^{-\frac{1}{2}}$. $\square$

§ 2. Exact WKB analysis of the hypergeometric differential equation

In this section, we consider (1.1) for the case where $Q = Q_0 + \eta^{-1}Q_1$ with

$$Q_0 = \frac{(\alpha-\beta)^2x^2 + 2(2\alpha\beta - \alpha\gamma - \beta\gamma)x + \gamma^2}{4x^2(x-1)^2}, \tag{2.1}$$

$$Q_1 = -\frac{x^2 - x + 1}{4x^2(x-1)^2}. \tag{2.2}$$

As is mentioned in Introduction, this comes from the hypergeometric differential equation (0.1) by introducing a large parameter η as (0.3) and changing the dependent variable w to a new one ψ as (0.4).

We assume that the triplet of parameters (α,β,γ) does not belong to the exceptional sets E_j $(j = 0,1,2)$ defined by

$$\begin{aligned} E_0 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid \alpha\,\beta\,\gamma\,(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)(\alpha+\beta-\gamma) = 0\}, \\ E_1 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid \mathrm{Re}\,\alpha\,\mathrm{Re}\,\beta\,\mathrm{Re}(\gamma-\alpha)\mathrm{Re}(\gamma-\beta) = 0\}, \\ E_2 &= \{(\alpha,\beta,\gamma) \in \mathbb{C}^3 \mid \mathrm{Re}(\alpha-\beta)\mathrm{Re}(\alpha+\beta-\gamma)\mathrm{Re}\,\gamma = 0\}. \end{aligned}$$

Under this assumption we see that there are two distinct zeros a_0, a_1 of Q_0dx^2 which do not coincide with the regular singular points 0, 1 ∞ and which are called the turning points of (1.1). Moreover, the Stokes curves

$$\mathrm{Im}\int_a^x \sqrt{Q_0}dx = 0 \quad (a = a_0 \text{ or } a_1) \tag{2.3}$$

are non-degenerate (cf. [1], [6]).

In the sequel, we assume further that $\mathrm{Re}\,\gamma > 0$. There is at least one Stokes curve flowing into the origin, one of the regular singular points (cf. [1]). We take the branch of $\sqrt{Q_0}$ so that $\sqrt{Q_0} \sim \dfrac{\gamma}{2x}$ holds near the origin. Then the residue of S_{odd} at the origin is

$$\operatorname*{Res}_{x=0} S_{\mathrm{odd}} = \frac{\gamma\eta}{2} \tag{2.4}$$

and the WKB solution $\psi_+^{(0)}$ defined as in the preceding section is recessive (cf. [6], [11]) on any Stokes curve that flows into the origin. A typical configuration of the Stokes curves of our equation are given in Figure 1 for the case $0 < \mathrm{Re}\ \alpha < \mathrm{Re}\ \gamma < \mathrm{Re}\ \beta$.

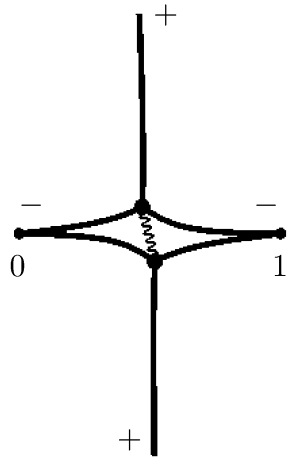

Figure 1.

Here the larger dots designate the turning points, the wavy line connecting the turning points indicates the branch cut for $\sqrt{Q_0}$ and the plus or the minus signs on the Stokes curves show which WKB solution is dominant.

Applying Theorem 1.2, we see that $x^{-\frac{1}{2}-\frac{\gamma\eta}{2}}\psi_+^{(0)}$ is Borel summable in a neighborhood of the origin and the Borel sum $x^{-\frac{1}{2}-\frac{\gamma\eta}{2}}\Psi_+^{(0)}(x,\eta)$ is holomorphic in $U\times\{\operatorname{Re}\eta \gg 0\}$ for a neighborhood U of the origin. Going back to the unknown of the hypergeometric differential equation, we see that

$$w_+(x,\eta) = x^{-\frac{1}{2}-\frac{\gamma\eta}{2}}(1-x)^{-\frac{1}{2}-\frac{(\alpha+\beta-\gamma)\eta}{2}}\Psi_+^{(0)}(x,\eta) \tag{2.5}$$

is a holomorphic solution of (0.1). Thus we conclude that there is a constant C_0 so that

$$F\Big(\frac{1}{2}+\alpha\eta, \frac{1}{2}+\beta\eta, 1+\gamma\eta; x\Big) = C_0 w_+(x,\eta) \tag{2.6}$$

holds. By the second statement of Theorem 1.2, we can evaluate the right-hand side at $x=0$ and find C_0. Hence we have

Theorem 2.1. *Under the assumptions and notation given above, the following relation holds in a neighborhood of the origin*:

$$F\Big(\frac{1}{2}+\alpha\eta, \frac{1}{2}+\beta\eta, 1+\gamma\eta; x\Big) = \sqrt{\frac{\gamma\eta}{2}}\,x^{-\frac{1}{2}-\frac{\gamma\eta}{2}}(1-x)^{-\frac{1}{2}-\frac{(\alpha+\beta-\gamma)\eta}{2}}\Psi_+^{(0)}(x,\eta).$$

It follows immediately from Watson's lemma (cf. [3]) that

Corollary 2.2. *Under the same assumption as Theorem* 2.1, *we have the following asymptotic expansion formula in a neighborhood of the origin*:

$$F\Big(\frac{1}{2}+\alpha\eta, \frac{1}{2}+\beta\eta, 1+\gamma\eta; x\Big) \sim \sqrt{\frac{\gamma\eta}{2}}\,x^{-\frac{1}{2}-\frac{\gamma\eta}{2}}(1-x)^{-\frac{1}{2}-\frac{(\alpha+\beta-\gamma)\eta}{2}}\psi_+^{(0)}(x,\eta),$$

as $\eta\to\infty$.

Remark. To obtain asymptotic expansion formulas for x which is far from the origin, we need to solve the connection problems of the WKB solution normalized at the origin. The connection problem for WKB solutions normalized at a turning point can be solved by using the standard theory (cf. [6]) and the relation between those two normalizations of WKB solutions can be described in terms of the Voros coefficient of the origin (cf. [2], [9], [11]; see also [10]). To carry this plan out, we must modify slightly the definition of the Voros coefficients for the hypergeometric differential equation given in [2], [9]. These will be done in our forthcoming paper.

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