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Unique solvability of coupling equations in holomorphic functions

By

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Abstract

The theory of coupling equations was introduced by the third author [4], as a theory of a class of transformations between some nonlinear partial differential equations in complex domains. There, he constructed, to the initial value problem of a coupling equation, a formal power series solution of a special form in infinitely many variables, satisfying suitable estimates. It would be desirable, from several aspects, to study the coupling equations and their solvability as functional equations for "holomorphic functions".

In this report, we consider coupling equations for partial differential equations of normal form in the t variable. After preparing and recalling some notions of holomorphy on infinite dimensional spaces, we announce our recent result on the unique solvability of the initial value problem of a coupling equation, using the contraction mapping principle.

§ 1. Introduction

The coupling theory is a theory of a class of transformations between some nonlinear partial differential equations in complex domains, due to the third author [4]. In this paper, he introduced the notion of coupling equations for partial differential equations of normal form in the t variable

$$\frac{\partial u}{\partial t} = F(t, x, u, \frac{\partial u}{\partial x}),$$

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with holomorphic functions F in a neighborhood of $0 \in \mathbb{C}^4_{(t,x,u_0,u_1)}$. Moreover, in his subsequent papers [5] and [6], he introduced the notion of coupling equations for partial differential equations of Briot-Bouquet type

$$t\frac{\partial u}{\partial t} = F(t,x,u,\frac{\partial u}{\partial x}),$$

where $F(t,x,u_0,u_1)$ is a holomorphic function in a neighborhood of $0 \in \mathbb{C}^4$ satisfying $F(0,x,0,0) = 0$ and $(\partial F/\partial u_1)(0,x,0,0) = 0$.

Let us recall the notion of coupling equations for partial differential equations of normal form. Consider two equations

$$\frac{\partial u}{\partial t} = F(t,x,u,\frac{\partial u}{\partial x}), \qquad \frac{\partial w}{\partial t} = G(t,x,w,\frac{\partial w}{\partial x}), \tag{1.1}$$

with $F(t,x,u_0,u_1)$, $G(t,x,w_0,w_1) \in \mathcal{O}_{\mathbb{C}^4,0}$, and correspondences between unknown functions $u(t,x)$ and $w(t,x)$:

$$\Phi : u \mapsto w, \quad w(t,x) = \phi(t,x,u(t,x),\frac{\partial u}{\partial x}(t,x),\frac{\partial^2 u}{\partial x^2}(t,x),\dots),$$
$$\Psi : w \mapsto u, \quad u(t,x) = \psi(t,x,w(t,x),\frac{\partial w}{\partial x}(t,x),\frac{\partial^2 w}{\partial x^2}(t,x),\dots),$$

given in terms of $\phi(t,x,u_0,u_1,\dots)$ and $\psi(t,x,w_0,w_1,\dots)$. For such Φ and Ψ to become transformations between solution spaces of two equations in (1.1), ϕ and ψ should formally satisfy

$$\frac{\partial \phi}{\partial t} + \sum_{m\geq 0} D^m[F](t,x,u_0,\dots,u_{m+1})\frac{\partial \phi}{\partial u_m} = G(t,x,\phi,D[\phi]), \tag{Φ}$$

$$\frac{\partial \psi}{\partial t} + \sum_{m\geq 0} D^m[G](t,x,w_0,\dots,w_{m+1})\frac{\partial \psi}{\partial w_m} = F(t,x,\psi,D[\psi]). \tag{Ψ}$$

Here D is a formal vector field of infinitely many variables given by

$$D := \frac{\partial}{\partial x} + \sum_{m\geq 0} u_{m+1}\frac{\partial}{\partial u_m}, \quad \left(\text{or} \quad D := \frac{\partial}{\partial x} + \sum_{m\geq 0} w_{m+1}\frac{\partial}{\partial w_m}\right).$$

The equations (Φ) and (Ψ) are called the coupling equations, which we want to solve under the additional initial value conditions: $\phi|_{t=0} = u_0$ and $\psi|_{t=0} = w_0$.

In [4], $\phi(t,x,u_0,u_1,\dots)$ was treated as a formal power series of form

$$\phi = u_0 + \sum_{k\geq 1} \phi_k(x,u_0,\dots,u_k)t^k \in \sum_{k\geq 0} \mathcal{O}_{\mathbb{C}}(\{|x| \leq R\})[[u_0,\dots,u_k]]t^k,$$

and the author discussed in the case $G \equiv 0$ (i.e., coupling $\partial u/\partial t = F \leftrightarrow \partial w/\partial t = 0$) about

- the unique existence of a formal power series solution ϕ to (Φ).
- the estimate of ϕ so that $w = \phi(t, x, u, \partial u/\partial x, \dots)$ makes sense as a transformation between some classes of solutions.
- similar statements for the solution ψ to (Ψ).
- "reversibility" of ϕ and ψ, (i.e., Φ and Ψ are inverses each other).

For the purpose of further applicability, it seems better to consider coupling equations as functional equations using a functional analytic method. For example, we want their "holomorphic solutions" rather than their power series solutions.

In this report, we announce our recent result on the unique solvability of the initial value problem of a coupling equation (Φ). We prepare notions of holomorphy and chain rules on infinite dimensional spaces in the section 2, and give our main result Theorem 3.3 in the section 3.

In what follows, we use $z = (z_i)_{i\in\mathbb{N}} = (z_0, z_1, \dots)$ as independent variables instead of $(u_0, u_1, \dots)$ and $(w_0, w_1, \dots)$, and use ∂_t, ∂_x, ∂_{z_i}, instead of $\partial/\partial t$, $\partial/\partial x$, $\partial/\partial z_i$, and so on. Therefore, F and G in (1.1) shall be considered as holomorphic functions in a neighborhood of $0 \in \mathbb{C}^4$ with variables (t, x, z_0, z_1), and similarly, ϕ, ψ and D shall be written as $\phi(t, x, z)$, $\psi(t, x, z)$ and $D = \partial_x + \sum_{i\in\mathbb{N}} z_{i+1}\partial_{z_i}$.

§ 2. Holomorphy, admissibility, and chain rules in $\mathbb{C}^{\mathbb{N}}$

We regard $\phi(t, x, z) = \phi(t, x, z_0, z_1, \dots)$ as a function defined on a subset on $\mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}_z^{\mathbb{N}}$. For the study of such functions, we recall some locally convex spaces X continuously embedded as $\mathbb{C}_z^{(\mathbb{N})} \hookrightarrow X \hookrightarrow \mathbb{C}_z^{\mathbb{N}}$. Here, $\mathbb{C}^{\mathbb{N}}$ and $\mathbb{C}^{(\mathbb{N})}$ denote

$$\mathbb{C}^{\mathbb{N}} := \{z = (z_i)_{i\in\mathbb{N}} = (z_0, z_1, \dots) \mid z_i \in \mathbb{C}\},$$
$$\mathbb{C}^{(\mathbb{N})} := \{z = (z_i)_{i\in\mathbb{N}} \mid z_i = 0, \text{ for all but finitely many } i\},$$

endowed with the product topology and the inductive limit topology, respectively. (Both are locally convex.)

Definition 2.1. A subspace X of $\mathbb{C}^{\mathbb{N}}$ including $\mathbb{C}^{(\mathbb{N})}$ endowed with a locally convex topology is called *a locally convex space between* $\mathbb{C}^{(\mathbb{N})}$ *and* $\mathbb{C}^{\mathbb{N}}$, if both inclusions $\mathbb{C}^{(\mathbb{N})} \hookrightarrow X$ and $X \hookrightarrow \mathbb{C}^{\mathbb{N}}$ are continuous.

Note that for any subspace X of $\mathbb{C}^{\mathbb{N}}$ including $\mathbb{C}^{(\mathbb{N})}$ endowed with a locally convex topology, $\mathbb{C}^{(\mathbb{N})} \hookrightarrow X$ is always continuous, while $X \hookrightarrow \mathbb{C}^{\mathbb{N}}$ is not necessarily continuous.

As examples, we introduce ℓ^1 and ℓ^∞ spaces with weights. A sequence $c = (c_i)_{i\in\mathbb{N}}$ of positive numbers is called a weight sequence.

Example 2.2 (weighted ℓ^1 and ℓ^∞ spaces)**.** The ℓ^1 space and the ℓ^∞ space with weight c is defined by

$$\ell^1(c) := \{z = (z_i)_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \|z\|_{\ell^1(c)} := \sum_{i\in\mathbb{N}} c_i|z_i| < +\infty\},$$
$$\ell^\infty(c) := \{z = (z_i)_{i\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}} \mid \|z\|_{\ell^\infty(c)} := \sup_{i\in\mathbb{N}} c_i|z_i| < +\infty\}.$$

They are Banach spaces, and locally convex spaces between $\mathbb{C}^{(\mathbb{N})}$ and $\mathbb{C}^{\mathbb{N}}$.

We consider $\ell^1(\sigma(r))$ with the weight sequences $\sigma(r) := (r^i/i!)_{i\in\mathbb{N}}$ for $r > 0$, and take inductive limits with respect to r.

Example 2.3. We define the spaces $X[\eta]$ for $0 \le \eta < +\infty$ by

$$X[\eta] := \varinjlim_{r>\eta} \ell^1(\sigma(r)), \quad (0 \le \eta < +\infty),$$

which are also locally convex spaces between $\mathbb{C}^{(\mathbb{N})}$ and $\mathbb{C}^{\mathbb{N}}$. Under the correspondence $(z_i)_{i\in\mathbb{N}} \mapsto \sum_{i\in\mathbb{N}} z_i x^i/i!$, they are isomorphic to the spaces of convergent power series as follows.

$$X[\eta] \xrightarrow{\sim} \mathcal{O}_{\mathbb{C}}(\{|x| \le \eta\}).$$

We may replace $\ell^1(\sigma(r))$ in the definition by $\ell^\infty(\sigma(r))$, and we can also take a countable inductive system consisting of Banach spaces and compact maps, which is cofinal to the one given above. (Consider, for example, $\varinjlim_k \ell^1(\sigma(\eta+1/k))$.) Therefore, $X[\eta]$ are DFS spaces. Refer to Komatsu [2] for DFS spaces and their properties.

Definition 2.4 (locally admissible functions)**.** Let X be a locally convex space between $\mathbb{C}^{(\mathbb{N})}$ and $\mathbb{C}^{\mathbb{N}}$, W an open subset of X, and f a $\mathbb{C}$-valued function on W.

(1) f is said to be *admissible on* W, if f is a uniform limit on W of some sequence of holomorphic functions of finitely many variables, i.e., there exist holomorphic functions $f_k(z_0, z_1, \dots, z_{n_k-1})$ on a suitable domains in $\mathbb{C}^{n_k}$ with some $n_k \in \mathbb{N}$ for $k = 0, 1, 2, \dots$, such that

$$f(z) = \lim_{k\to\infty} f_k(z_0, z_1, \dots, z_{n_k-1}), \quad \text{uniformly on } W.$$

(2) f is said to be *locally admissible on* W, if for any point $\dot{z} \in W$, there exists a neighborhood $U \subset W$ of $\dot{z}$ such that f is admissible on U.

Note that a locally admissible function $f(z)$ is *separately holomorphic*, that is, for any $i \in \mathbb{N}$, the univariate function $z_i \mapsto f(\dots, z_i, \dots)$ is holomorphic when the other variables $(z_j)_{j\ne i}$ are fixed.

The notion of (local) admissibility is very close to the usual holomorphy in finitely many variables, and we can expect that many formulas in calculus and in complex analysis will be extended. For example, a composition of an admissible function and holomorphic functions is holomorphic provided the substitution is well-defined.

Lemma 2.5 (compositions). *Let X and W be as above. Consider an admissible function $f : W \to \mathbb{C}$, and holomorphic functions $u_i(t)$ $(i \in \mathbb{N})$ on an open set $\Omega \subset \mathbb{C}$ such that $u(t) := (u_i(t))_i \in W$ for any $t \in \Omega$. Then,*

$$g(t) := f(u(t)) = f(u_0(t), u_1(t), u_2(t), \dots)$$

is holomorphic on Ω.

On the other hand, admissibility is not stable under partial differentiations, and it seems not straightforward to define the notion of (local) admissibility for $\mathbb{C}^{\mathbb{N}}$-valued maps so that it is stable under compositions.

We recall the notion of holomorphy on locally convex spaces. See, for example, Dineen [1].

Definition 2.6 (Gâteaux holomorphy and holomorphy). Let X and Y be locally convex spaces, and $W \subset X$ an open set.

(1) A map $f : W \to Y$ is said to be *Gâteaux holomorphic*, or *G-holomorphic* for short, if for any $x_0 \in W$, $x_1 \in X$, $g \in Y'$, a function $t \mapsto g(f(x_0 + x_1 t)) \in \mathbb{C}$ is holomorphic in a neighborhood of $t = 0$.

(2) A map $f : W \to Y$ is said to be *holomorphic*, if it is G-holomorphic and continuous. We denote by $\mathcal{O}_X(W)$ the space of $\mathbb{C}$-valued holomorphic functions on W.

Holomorphy is stable under compositions, while Gâteaux holomorphy is not.

In general, local admissibility implies holomorphy. On the other hand, holomorphy does not always imply local admissibility. In fact, there exist holomorphic functions on $\ell^1(c)$ and on $\ell^\infty(c)$, which are not locally admissible.

In $X[\eta]$, however, both notions are equivalent.

Theorem 2.7. *For a function defined on an open subset of $X[\eta]$, holomorphy implies local admissibility.*

By virtue of this theorem, when we work on $X[\eta]$, we can enjoy the merits of local admissibility and those of holomorphy simultaneously.

Definition 2.8. We define a formal vector field D of infinitely many variables $(x, z) = (x, z_0, z_1, \dots)$ by

$$D := \partial_x + \sum_{i \in \mathbb{N}} z_{i+1} \partial_{z_i}. \tag{2.1}$$

For a holomorphic function $f(x,z)$ on $U \subset \mathbb{C}_x \times X$ with $\mathbb{C}_z^{(\mathbb{N})} \subset X \subset \mathbb{C}_z^{\mathbb{N}}$, we consider a formal sum

$$D[f](x,z) = \partial_x f(x,z) + \textstyle\sum_{i\in\mathbb{N}} z_{i+1}\partial_{z_i} f(x,z).$$

If it converges, we regard $D[f]$ as a function on U.

Consider a holomorphic function $f(x,z)$ of finitely many variables, i.e., a holomorphic function $f(x,z_0,\dots,z_k)$ defined on an open set $U \subset \mathbb{C}_x \times \mathbb{C}^{k+1}_{(z_0,\dots,z_k)}$ for some k. Then, $D[f] = (\partial_x + \sum_{i=0}^k z_{i+1}\partial_{z_i})f$ defines a holomorphic function of $(x,z_0,\dots,z_{k+1})$ on $U \times \mathbb{C}_{z_{k+1}} \subset \mathbb{C}_x \times \mathbb{C}^{k+2}_{(z_0,\dots,z_{k+1})}$. Consider also a holomorphic function $u(x)$ on an open set $\Omega \subset \mathbb{C}_x$, satisfying

$$(x, u(x), \partial_x u(x), \dots, \partial_x^k u(x)) \in U \quad \text{for } x \in \Omega.$$

Then, the composition $g(x) := f(x,u(x),\partial_x u(x),\dots,\partial_x^k u(x))$ is holomorphic on Ω, and satisfies

$$\partial_x g(x) = D[f](x,u(x),\partial_x u(x),\dots,\partial_x^{k+1}u(x)), \quad \text{for } x \in \Omega,$$

and moreover,

$$\partial_x^m g(x) = D^m[f](x,u(x),\partial_x u(x),\dots,\partial_x^{k+m}u(x)), \quad \text{for } m \in \mathbb{N},\ x \in \Omega.$$

On $\mathbb{C}_x \times X[\eta]$, the well-definedness of $D[f]$ and the chain rule for the composition also hold.

Theorem 2.9 (chain rule on $\mathbb{C}_x \times X[\eta]$, I). *Let $f(x,z)$ be a holomorphic function on an open set $U \subset \mathbb{C}_x \times X[\eta]$.*

(1) $D[f](x,z)$ converges absolutely and defines a holomorphic function on U.

(2) Let $u(x)$ be a holomorphic function on an open set $\Omega \subset \mathbb{C}_x$. Assume that

$$(x,u(x),\partial_x u(x),\partial_x^2 u(x),\dots) \in U, \quad \text{for } x \in \Omega.$$

Then, the composition

$$g(x) := f(x,u(x),\partial_x u(x),\partial_x^2 u(x),\dots)$$

is holomorphic on Ω and satisfies

$$\partial_x^m g(x) = D^m[f](x,u(x),\partial_x u(x),\partial_x^2 u(x),\dots), \quad \text{for } m \in \mathbb{N},\ x \in \Omega.$$

Theorem 2.10 (chain rule on $\mathbb{C}_x \times X[\eta]$, II). *Let $f(x,z)$ be a holomorphic function on an open set $U \subset \mathbb{C}_x \times X[\eta]$, and $\phi(x,z)$ a holomorphic function on $V \subset \mathbb{C}_x \times X[\eta']$. Assume that*

$$\vec{\phi}(x,z) := (x,\phi(x,z),D[\phi](x,z),D^2[\phi](x,z),\dots) \in U, \quad \text{for any } (x,z) \in V,$$

and that $\vec{\phi}$ *is locally bounded as a map from* V *to* $\mathbb{C}_x \times X[\eta]$*. Then, the composition*

$$g(x,z) := f \circ \vec{\phi}(x,z) = f(x, \phi(x,z), D[\phi](x,z), D^2[\phi](x,z), \dots)$$

is holomorphic on V*, and satisfies*

$$D^m[g](x,z) = D^m[f] \circ \vec{\phi}(x,z), \quad \text{for } m \in \mathbb{N},\ (x,z) \in V.$$

§ 3. Unique solvability of coupling equation

In this section, we study the unique solvability of the initial value problem of the coupling equation

$$\begin{cases} \partial_t \phi + \sum_{m\ge 0} D^m[F](t,x,z_0,z_1,\dots,z_{m+1})\partial_{z_m}\phi = G(t,x,\phi,D[\phi]), \\ \phi(0,x,z) = z_0, \end{cases} \tag{3.1}$$

where $D = \partial_x + \sum_{i\in\mathbb{N}} z_{i+1}\partial_{z_i}$ as in (2.1). This initial value problem is often written as

$$\begin{cases} \partial_t \phi + \sum_{m\ge 0} D^m[F] \cdot \partial_{z_m}\phi = G \circ \vec{\phi}, \\ \phi|_{t=0} = z_0, \end{cases}$$

for short, under the notation

$$\begin{aligned} \vec{\phi}(t,x,z) &:= (t, x, (D^i[\phi](t,x,z))_{i\in\mathbb{N}}) \\ &= (t, x, \phi(t,x,z), D[\phi](t,x,z), D^2[\phi](t,x,z), \dots). \end{aligned}$$

Consider holomorphic functions F and G in a neighborhood of

$$\hat{K}_0^{\mathbb{C}} := \{(t,x,z_0,z_1) \in \mathbb{C}^4 \mid |t| \le r_0, |x| \le R_0, |z_0| \le \rho_0, |z_1| \le \rho_0\}.$$

We fix positive constants r and ρ, and a 1-Lipschitz continuous function $d(x)$ on $\mathbb{C}_x$, satisfying

$$0 < r < r_0, \quad 0 < \rho < \rho_0/2, \tag{3.2}$$

$$0 < d_{\max} := \sup_{x\in\mathbb{C}} d(x) \le 1, \quad d(x) > 0 \Rightarrow |x| < R_0. \tag{3.3}$$

These requirements assert

$$\emptyset \ne \{(t,x,z_0,z_1) \mid |t| < r,\ d(x) > 0,\ |z_0| \le 2\rho, |z_1| \le 2\rho\} \subset \hat{K}_0^{\mathbb{C}}.$$

We also define functions $\xi_j(z,y)$ $(j \in \mathbb{N})$ and $\mu(z,\eta)$ by

$$\xi_j(z,y) := \sum_{i\in\mathbb{N}} z_{j+i}\frac{y^i}{i!}, \tag{3.4}$$

$$\mu(z,\eta) := \rho^{-1}\{\xi_0(|z|,\eta) + \xi_1(|z|,\eta) + \xi_2(|z|,\eta)\}, \tag{3.5}$$

for $\eta \ge 0$, $y \in \mathbb{C}$, $z \in \mathbb{C}^{\mathbb{N}}$. We regard $\mu(z,\eta) = +\infty$ when $\xi_2(|z|,\eta)$ is divergent.

Definition 3.1 (weight functions and domains). We define the weight function $\omega_{d,c,\varepsilon}(t,x,z)$ for $c \geq 1$, $\varepsilon \geq 0$, $t \geq 0$, $x \in \mathbb{C}$, $z \in \mathbb{C}^{\mathbb{N}}$, and the domain $\Omega^{\mathbb{C}}_{d,c,+0}$ by

$$\begin{aligned}\omega_{d,c,\varepsilon}(t,x,z) &:= d(x) - (ct/r+\varepsilon) - \mu(z, ct/r+\varepsilon),\\ \Omega^{\mathbb{C}}_{d,c,+0} &:= \bigcup_{\varepsilon>0}\{(t,x,z) \in \mathbb{C}_t \times \mathbb{C}_x \times \mathbb{C}^{\mathbb{N}}_z \mid \omega_{d,c,\varepsilon}(|t|,x,z) > 0\}.\end{aligned}$$

The set $\Omega^{\mathbb{C}}_{d,c,+0}$ can be covered as

$$\Omega^{\mathbb{C}}_{d,c,+0} = \bigcup_{s>0} U_s, \quad U_s := \Omega^{\mathbb{C}}_{d,c,+0} \cap \{|t|<s\} \times \mathbb{C}_x \times X[cs/r]$$

and each U_s is an open subset in $\mathbb{C}_t \times \mathbb{C}_x \times X[cs/r]$. Using this covering, we can introduce a notion of holomorphy for a function f on $\mathscr{X}^{\mathbb{C}}_{d,c}$, as the holomorphy of $f|_{U_s}$ for any s.

Definition 3.2 (space $\mathscr{X}^{\mathbb{C}}_{d,c}$). We introduce the space $\mathscr{X}^{\mathbb{C}}_{d,c}$ of all holomorphic functions $\phi(t,x,z)$ on $\Omega^{\mathbb{C}}_{d,c,+0}$ satisfying the following inequalities with a suitable constant C:

$$|\phi(t,x,z)| \leq C\rho, \tag{3.6a}$$

$$|D[\phi](t,x,z)| \leq C\rho, \tag{3.6b}$$

$$|D^2[\phi](t,x,z)| \leq \frac{C\rho}{\omega_{d,c,0}(|t|,x,z)^{1/2}}, \tag{3.6c}$$

$$|\partial_{z_m}\phi(t,x,z)| \leq \frac{C}{\omega_{d,c,0}(|t|,x,z)^{1/2}} \cdot \frac{(ct/r)^m}{m!}, \quad m \in \mathbb{N}, \tag{3.6d}$$

$$|\partial_{z_m}D[\phi](t,x,z)| \leq \frac{C}{\omega_{d,c,0}(|t|,x,z)^{1/2}} \sum_{i=0}^{\min\{m,1\}} \frac{(ct/r)^{m-i}}{(m-i)!}, \quad m \in \mathbb{N}, \tag{3.6e}$$

for any $(t,x,z) \in \Omega^{\mathbb{C}}_{d,c,+0}$.

For a holomorphic function ϕ on $\Omega^{\mathbb{C}}_{d,c,+0}$, we denote by $\|\phi\|_{d,c,1}$ the minimum constant $C \geq 0$ such that the inequality (3.6a) holds. Similarly, we also denote by $\|\phi\|_{d,c,2}$, $\|\phi\|_{d,c,3}$, $\|\phi\|_{d,c,4}$, and $\|\phi\|_{d,c,5}$, the minimum constants $C \geq 0$ corresponding to (3.6b), (3.6c), (3.6d) and (3.6e), respectively.

Note that we use Nagumo type estimates of derivatives for functions in $\mathscr{X}^{\mathbb{C}}_{d,c}$, which involve the factor $\omega_{d,c,0}(|t|,x,z)^{-1/2}$ in (3.6c), (3.6d) and (3.6e). Refer for such estimates to Nagumo [3] and Walter [7].

The space $\mathscr{X}^{\mathbb{C}}_{d,c}$ becomes a Banach space with the norm

$$\|\phi\|_{d,c,\mathrm{A}} := \max\{\|\phi\|_{d,c,1}, \|\phi\|_{d,c,2}, \|\phi\|_{d,c,3}, \|\phi\|_{d,c,4}, \|\phi\|_{d,c,5}\}.$$

We also define the semi-norms

$$\begin{aligned}\|\phi\|_{d,c,\text{1-3}} &:= \max\{\|\phi\|_{d,c,1}, \|\phi\|_{d,c,2}, \|\phi\|_{d,c,3}\},\\ \|\phi\|_{d,c,45} &:= \max\{\|\phi\|_{d,c,4}, \|\phi\|_{d,c,5}\}.\end{aligned}$$

and a complete metric space

$$\mathscr{Z}^{\mathbb{C}}_{d,c}(\alpha,\beta) := \{\phi \in \mathscr{Z}^{\mathbb{C}}_{d,c} \mid \|\phi\|_{d,c,\text{1-3}} \le \alpha,\ \|\phi\|_{d,c,45} \le \beta\},$$

for positive constants α and β.

We shall show that the initial value problem (3.1) has a unique solution in $\mathscr{Z}^{\mathbb{C}}_{d,c}(\alpha,\beta)$ for a suitable choice of α and β. In fact, for $\phi \in \mathscr{Z}^{\mathbb{C}}_{d,c}(\alpha,\beta)$ with suitable α and β, (3.1) is equivalent to the integral equation

$$\phi = T[\phi] := z_0 - R_F[\phi] + S_G[\phi],$$

where

$$R_F[\phi] := \int_0^t \sum_{m\ge 0} D^m[F]\cdot \partial_{z_m}\phi\Big|_{t=\tau} d\tau,$$

$$S_G[\phi] := \int_0^t G(t,x,\phi,D[\phi])\Big|_{t=\tau} d\tau,$$

and we have the following result.

Theorem 3.3. *Let r, ρ, $\hat{K}^{\mathbb{C}}_0$, $d(x)$, $d_{\max}$, F, G be as above. Then there exists a constant A depending on F, G, $\hat{K}^{\mathbb{C}}_0$, and ρ, satisfying the following property: We fix α and β arbitrarily as*

$$d_{\max} < \alpha \le 2, \quad \beta \ge 2,$$

and take $c \ge 1$ sufficiently large depending also on α and β. Then, T becomes a contraction map from $\mathscr{Z}^{\mathbb{C}}_{d,c}(\alpha,\beta)$ to itself, with respect to the distance function $(\phi,\phi') \mapsto \|\phi-\phi'\|_{d,c,\mathrm{A}}$. As a conclusion, the initial value problem (3.1) *has a unique solution in $\mathscr{Z}^{\mathbb{C}}_{d,c}(\alpha,\beta)$.*

Let us give a sketch of the proof. We can show the estimates

$$\|z_0\|_{d,c,\text{1-3}} \le d_{\max}, \quad \|z_0\|_{d,c,45} \le 1,$$

with arbitrary $c \ge 1$, which in particular imply $z_0 \in \mathscr{Z}^{\mathbb{C}}_{d,c}(\alpha,\beta)$ with α and β as in the theorem. Then, we use the three propositions below for R_F and S_G, and by choosing c large enough, we can show that T maps $\mathscr{Z}^{\mathbb{C}}_{d,c}(\alpha,\beta)$ into itself and that T is a contraction. Note that the constants L_j, M_j, N_j below are independent of c.

Proposition 3.4. *R_F is well-defined as a linear operator on $\mathscr{Z}^{\mathbb{C}}_{d,c}$, and there exist constants L_0 and L_1 such that*

$$\|R_F[\phi]\|_{d,c,\text{1-3}} \le \frac{r}{c}\cdot L_0 \max\{\|\phi\|_{d,c,\text{1-3}}, \|\phi\|_{d,c,45}\},$$

$$\|R_F[\phi]\|_{d,c,45} \le \frac{r}{c}\cdot L_1 \|\phi\|_{d,c,45},$$

for $\phi \in \mathscr{Z}^{\mathbb{C}}_{d,c}$.

Proposition 3.5. *Let us fix α as $0 < \alpha \le 2$ and take an arbitrary β. Then, S_G is well-defined as a map from $\mathscr{Z}^{\mathbb{C}}_{d,c}(\alpha,\beta)$ to $\mathscr{Z}^{\mathbb{C}}_{d,c}$, and there exists constants M_0 and M_1 such that*

$$\|S_G[\phi]\|_{d,c,1\text{-}3} \le \frac{r}{c} \cdot M_0$$
$$\|S_G[\phi]\|_{d,c,45} \le \frac{r}{c} \cdot M_1 \|\phi\|_{d,c,45},$$

for $\phi \in \mathscr{Z}^{\mathbb{C}}_{d,c}(\alpha,\beta)$.

Proposition 3.6. *Let us fix α as $0 < \alpha \le 2$ and take an arbitrary β. Then, there exist N_0, N_1 and N_2 such that*

$$\|S_G[\phi] - S_G[\phi']\|_{d,c,1\text{-}3} \le \frac{r}{c} \cdot N_0 \|\phi - \phi'\|_{d,c,1\text{-}3},$$
$$\|S_G[\phi] - S_G[\phi']\|_{d,c,45} \le \frac{r}{c} \cdot (N_1 \beta \|\phi - \phi'\|_{d,c,1\text{-}3} + N_2 \|\phi - \phi'\|_{d,c,45}),$$

for ϕ, $\phi' \in \mathscr{Z}^{\mathbb{C}}_{d,c}(\alpha,\beta)$.

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