RIMS Kôkyûroku Bessatsu
B57 (2016), 085–091

A Laplace transform of Laplace hyperfunctions in several variables

By

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Abstract

We construct a Laplace transform of Laplace hyperfunctions in several variables with support in an $\mathbb{R}_+$-conic closed convex cone and it's inverse transform.

§ 1. Introduction

H. Komatsu ([3]-[8]) established the theory of Laplace hyperfunctions in one variable in order to consider the Laplace transform of a hyperfunction. Using the theory effectively, he had succeeded in giving a justification of the Heaviside operational calculus on a wider class of functions. A Laplace hyperfunction in one variable is presented by a difference of boundary values of holomorphic functions of exponential type along the real axis. Recently, N. Honda and the author established a vanishing theorem of cohomology groups on a Stein open subset with values in the sheaf of holomorphic functions of exponential type in the paper [2]. By the theorem, we can construct the sheaf of one dimensional Laplace hyperfunctions introduced by H. Komatsu. Furthremore, in the paper [1], we established an edge of the wedge type theorem for holomorphic functions of exponential type and gave the sheaf of Laplace hyperfunctions in several variables. In this note, we announce the construction of a Laplace transform of Laplace hyperfunctions in several variables and related results without proofs. The detailed proofs will appear in a forthcoming publication.

Received August 7, 2015. Revised October 18, 2015. Accepted October 18, 2015.
2010 Mathematics Subject Classification(s): 32A45, 44A10
Key Words: *Laplace transforms, Laplace hyperfunctions, holomorphic functions of exponential type.*

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§ 2. The Sheaf of Laplace hyperfunctions $\mathcal{B}^{\exp}_{\overline{M}}$ in several variables

In this section, we recall the vanishing theorem of cohomology groups on a Stein open subset with coefficients in holomorphic functions of exponential type and the definition of the sheaf of Laplace hyperfunctions $\mathcal{B}^{\exp}_{\overline{M}}$ on $\overline{M}$. For more details, we refer the reader to [1] and [2].

Let $n \in \mathbb{N}$ and M be an n-dimensional $\mathbb{R}$-vector space ($n \geq 1$) with an inner product, and let E be its complexification. We denote by $\mathbb{D}_E$ the radial compactification of E which is defined by the disjoint union of E and the copy $((E \setminus \{0\})/\mathbb{R}_+)\infty$ of the quotient space $(E \setminus \{0\})/\mathbb{R}_+$. Here $\mathbb{R}_+$ denotes the set of positive real numbers. The radial compactification $\mathbb{D}_E$ of E can be identified with the disjoint union of $\mathbb{C}^n$ and the copy $S^{2n-1}\infty$ of S^{2n-1}, where S^{2n-1} is the real $(2n-1)$-dimensional unit sphere.

Let Ω be an open subset in $\mathbb{D}_E$. A holomorphic function $f(z)$ in $\Omega \cap E$ is said to be of exponential type if $f(z)$ satisfies the following condition: For any compact subset K in Ω, there exist constants $C_K > 0$ and $H_K > 0$ such that

$$(2.1) \qquad |f(z)| \leq C_K e^{H_K |z|} \qquad (z \in K \cap E).$$

Let Z be a subset in $\mathbb{D}_E$. We denote by $\mathrm{clos}^1_\infty(Z)$ the subset in $S^{2n-1}\infty$ defined by

$$(2.2) \qquad z\infty \in \mathrm{clos}^1_\infty(Z) \quad \Leftrightarrow \quad \begin{cases} \text{There exist points } \{z_k\}_k \text{ in } Z \cap E \text{ such that} \\ z_k \to z\infty \text{ in } \mathbb{D}_E \text{ and } |z_{k+1}|/|z_k| \to 1 \text{ as } k \to \infty. \end{cases}$$

Define

$$(2.3) \qquad N^1_\infty(Z) := S^{2n-1}\infty \setminus \mathrm{clos}^1_\infty(E \setminus Z).$$

Definition 2.1. Let Ω be an open subset in $\mathbb{D}_E$. We say that Ω is regular at ∞ if $N^1_\infty(\Omega) = \Omega \cap S^{2n-1}\infty$.

We give both examples of open sets satisfying the regularity condition and otherwise.

Example 2.2 ([2], Example 3.6). Let $\mathbb{D}_\mathbb{C}$ denote the radial compactification of $\mathbb{C}$. For the set $\Omega := \mathbb{D}_\mathbb{C} \setminus \{1, 2, 3, 4, \ldots, +\infty\}$, we have $N^1_\infty(\Omega) = S^1\infty \setminus \{+\infty\}$. Hence Ω is regular at ∞. However, for the set $\Omega := \mathbb{D}_\mathbb{C} \setminus \{1, 2, 4, 8, 16, \ldots, +\infty\}$, Ω is not regular at ∞ because of $N^1_\infty(\Omega) = S^1\infty$.

We have the vanishing theorem of cohomology groups on a Stein open subset for $\mathcal{O}^{\exp}_{\mathbb{D}_E}$.

Theorem 2.3 ([2], Theorem 3.7). *Let Ω be an open subset in $\mathbb{D}_E$. If $\Omega \cap E$ is pseudo-convex in E and Ω is regular at ∞, then we have*

$$(2.4) \qquad \mathrm{H}^k(\Omega, \mathcal{O}^{\exp}_{\mathbb{D}_E}) = 0 \quad (k \neq 0).$$

The following example shows that the theorem does not holds without the regularity condition.

Example 2.4 ([2], Example 3.17). We consider the case $n = 2$, i.e., $E = \mathbb{C}^2$ and $\mathbb{D}_E = \mathbb{D}_{\mathbb{C}^2}$. Let $(1, 0)\infty \in ((E \setminus \{0\})/\mathbb{R}_+)\infty$. Set

$$U := \left\{(z_1,\, z_2) \in E;\, |\arg(z_1)| < \frac{\pi}{4},\, |z_2| < |z_1|\right\},$$
$$\Omega := \left(\overline{U}\right)^{\circ} \setminus \{(1,0)\infty\} \subset \mathbb{D}_E.$$

Note that $\Omega \cap E = U$ is pseudo-convex in E and Ω is not regular at ∞. In this case, we have $\mathrm{H}^1(\Omega,\, \mathcal{O}^{\exp}_{\mathbb{D}_E}) \neq 0$.

We have the edge of the wedge type theorem for $\mathcal{O}^{\exp}_{\mathbb{D}_E}$. Let $\overline{M}$ be the closure of M in $\mathbb{D}_E$.

Theorem 2.5 ([1], Corollary 3.16). *The closed subset $\overline{M} \subset \mathbb{D}_E$ is purly n-codimentional relative to the sheaf $\mathcal{O}^{\exp}_{\mathbb{D}_E}$, i.e.,*

$$\mathscr{H}^k_{\overline{M}}(\mathcal{O}^{\exp}_{\mathbb{D}_E}) = 0 \qquad (k \neq n). \tag{2.5}$$

By Theorem 2.5, we can construct the sheaf of Laplace hyperfunctions on $\overline{M}$.

Definition 2.6. The sheaf of Laplace hyperfunctions on $\overline{M}$ is defined by

$$\mathcal{B}^{\exp}_{\overline{M}} := \mathscr{H}^n_{\overline{M}}(\mathcal{O}^{\exp}_{\mathbb{D}_E}) \underset{\mathbb{Z}_{\overline{M}}}{\otimes} \omega_{\overline{M}}. \tag{2.6}$$

Here $\omega_{\overline{M}}$ is the orientation sheaf $\mathscr{H}^n_{\overline{M}}(\mathbb{Z}_{\mathbb{D}_E})$ and $\mathbb{Z}_{\mathbb{D}_E}$ is the constant sheaf on $\mathbb{D}_E$ having stalk $\mathbb{Z}$.

§ 3. A Laplace transform of $\Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}})$

In this section, we construct a Laplace transform of Laplace hyperfunctions and it's inverse Laplace transform. We first get the representation of all the sections of Laplace hyperfunctions with support in an $\mathbb{R}_+$-conic closed convex cone in $\overline{M}$.

Let Z be a subset in E and $a \in M$. We set

$$Z_a := Z + \{a\} = \{z + a; z \in Z\}. \tag{3.1}$$

For a subset $Z \subset \mathbb{D}_E$, we denote by $N_\infty(Z)$ the subset $S^{2n-1}\infty \setminus \overline{(E \setminus Z)}$ in $S^{2n-1}\infty$. For an open subset $U \subset E$, we define the open subset in $\mathbb{D}_E$

$$\widehat{U} := U \cup N_\infty(U). \tag{3.2}$$

Definition 3.1. Let V be an $\mathbb{R}^+$-conic open cone in M and Ω an subset in $\overline{M}$. Let U be an open subset in $\mathbb{D}_E$. We say that U is a wedge of the type $\Omega \times \sqrt{-1}V$ if, for any open proper subcone $V^{'}$ of V, there exists an open neighborhood W of Ω in $\mathbb{D}_E$ such that

$$(M \times \widehat{\sqrt{-1}V'}) \cap W \subset U. \tag{3.3}$$

We have the following proposition.

Proposition 3.2. *Let K be an $\mathbb{R}_+$-conic closed cone in M and V an linear proper open cone in M, i.e., V is given by the intersection of finite number of half space in M. Then there exist an open neighborhood Ω of $\overline{K}$ in $\overline{M}$ and an open subset U in $\mathbb{D}_E$ such that the following conditions are satisfied:*

1. *U is a wedge of the type $\Omega \times \sqrt{-1}V$.*
2. *U is stein and regular at ∞.*
3. *U is an open neighborhood of $\Omega \setminus \overline{K}$ in $\mathbb{D}_E$.*

Let $a \in M$ and K be an $\mathbb{R}_+$-conic closed convex cone in M. We consider the representation of $\Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}})$. Take vectors $\omega_0, \dots, \omega_n \in S^{n-1}$. For the dual $\omega_j^\circ = \{y \in M; y\omega_j > 0\}$ of ω_j and $\overline{K_a}$, we can take an open neighborhood Ω of $\overline{K_a}$ in $\overline{M}$ and an open subset $U_j \subset \mathbb{D}_E$ of the wedge of the type $\Omega \times \sqrt{-1}\omega_j^\circ$ which satisfy the conditions in Proposition 3.2. We also take a neighborhood U of $\overline{K_a}$ in $\mathbb{D}_E$ which is stein and regular at ∞. Set

$$\mathfrak{U} = \{U, U_0, \dots, U_n\}, \ \ \mathfrak{U}^{'} = \{U_0, \dots, U_n\} \tag{3.4}$$

Then we have the following representations of $\Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}})$.

$$\begin{aligned} \Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}}) &= \mathrm{H}^n(\mathfrak{U} \bmod \mathfrak{U}^{'}, \mathcal{O}^{\exp}_{\mathbb{D}_E}) \\ &= \frac{\mathrm{Ker}\{\bigoplus_{j=0}^n \mathcal{O}^{\exp}_{\mathbb{D}_E}(\bigcap_{l\neq j} U_l) \to \mathcal{O}^{\exp}_{\mathbb{D}_E}(\bigcap_{l=0}^n U_l)\}}{\mathrm{Im}\{\bigoplus_{j\neq k} \mathcal{O}^{\exp}_{\mathbb{D}_E}(\bigcap_{l\neq j,\,k} U_l) \to \bigoplus_{j=0}^n \mathcal{O}^{\exp}_{\mathbb{D}_E}(\bigcap_{l\neq j} U_l)\}}. \end{aligned} \tag{3.5}$$

Let us define the Laplace transform for an element $f = \bigoplus_{j=0}^n F_j$ of the above representation of $\Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}})$. Take a closed cone $L \subset \Omega$ which contains K and fix a point $\omega \in \bigcap_{l\neq j}\omega_l^\circ$. Set

$$D_j := \{x + \sqrt{-1}y \in E \ ; \ x \in L, \, y = \varphi(x)\omega\},$$

where φ is a continuous function from L to $\mathbb{R}_+ \cup \{0\}$. We can assume that φ satisfies the following three conditions:

1. $\varphi(x) = 0$ in ∂L,

2. $\overline{D_j} \cap \overline{K_a} = \emptyset$,

3. $\overline{D}_j \subset U_j$.

Then we define the Laplace transform for $f = \bigoplus_{j=0}^{n} F_j \in \Gamma_{\overline{K_a}}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}})$ by

$$\mathscr{L}(f)(\lambda) := \sum_{j=0}^{n} \sigma_j \int_{D_j} F_j(z) e^{-\lambda z} dz, \tag{3.6}$$

where $\sigma_j := \mathrm{sgn}\ (\ \det(\omega_0, \cdots, \omega_{j-1}, \omega_{j+1}, \cdots, \omega_n)\ )$. Note that the definition of (3.6) does not depend on the choice of L, ω and φ. Let us define the sheaf $\mathcal{O}^{a,\mathrm{inf}}_{\mathbb{D}_E}$. For an open subset Ω in $\mathbb{D}_E$, $\mathcal{O}^{a,\mathrm{inf}}_{\mathbb{D}_E}(\Omega)$ consists of holomorphic functions $f(z)$ on $\Omega \cap E$ for which the following estimate holds: For any compact subset $K \subset \Omega$ and $\epsilon > 0$, there exists $C_{K,\epsilon}$ satisfying

$$|e^{az} f(z)| \le C_{K,\epsilon} e^{\epsilon |z|}, \qquad z \in K \cap E. \tag{3.7}$$

Let $a \in M$ and $K \subset M$ be an $\mathbb{R}_+$-conic closed cone in M. We denote by K° the dual open cone of K in E with respect to $\mathrm{Re}(z\xi)$, i.e.,

$$K^\circ := \{\zeta \in E\ ;\ \mathrm{Re}(z\zeta) > 0 \quad \text{for} \quad z \in K\}. \tag{3.8}$$

We also denote by K°_M} the dual open cone of K in M. Since the function $\mathscr{L}(f)(\lambda)$ defined by (3.6) belongs to $\mathcal{O}^{a,\,\mathrm{inf}}_{\mathbb{D}_E}(N_\infty(K^\circ))$, the Laplace transform gives the following morphism:

$$\mathscr{L}\ :\ \Gamma_{\overline{K}_a}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}}) \longrightarrow \mathcal{O}^{a,\,\mathrm{inf}}_{\mathbb{D}_E}(N_\infty(K^\circ)). \tag{3.9}$$

We see that the morphism does not depend on the representation of $\Gamma_{\overline{K}_a}(\overline{M}, \mathcal{B}^{\exp}_{\overline{M}})$. Hence $\mathscr{L}$ is well-defined. Next we define the inverse Laplace transform.

Definition 3.3. Let S be an open subset in $S^{2n-1}\infty$, and U an open subset in $\mathbb{D}_E$. We say that U has the opening wider than or equal to S at ∞ if

$$S \subset N_\infty(U).$$

The following lemma is important.

Lemma 3.4. *The following conditions are equivalent:*

1. $f \in \mathcal{O}^{a,\,\mathrm{inf}}_{\mathbb{D}_E}(N_\infty(K^\circ))$

2. *There exists an open subset U in E whose opening is wider or equal to $N_\infty(K^\circ)$ such that f is holomorphic on U and, for any compact subset K in $\widehat{U}$, there exists an infra-linear function $\phi_K(s)$ satisfying*

$$|e^{az}f(z)| \le e^{\phi_K(|z|)}, \qquad z \in K \cap E.$$

3. *there exists an infra-linear function $\phi(s)$ and an open subset U in E whose opening is wider or equal to $N_\infty(K^\circ)$ such that f is holomorphic on U with*

$$|e^{az}f(z)| \le e^{\phi(|z|)}, \qquad z \in U.$$

For $f \in \mathcal{O}^{a,\,\inf}_{\mathbb{D}_E}(N_\infty(K^\circ))$, we set

$$f_k(z) := \frac{1}{(2\pi\sqrt{-1})^n}\int_{T_k} f(\lambda)e^{\lambda z}d\lambda. \tag{3.10}$$

Here the path of the integration T_k is given by

$$T_k := \left\{\ \lambda = \xi + \sqrt{-1}\eta \in E\ ;\ \eta \in \Sigma_k, \quad \xi = \psi(|\eta|)\hat{\xi}\ \right\}, \tag{3.11}$$

where $\Sigma_k = \{\eta \in M; \eta = \sum_{j\neq k} t_j\omega_j,\ t_j \ge 0\}$, ψ is an infra-linear function, and $\hat{\xi}$ is a point in K°_M}. Then $f_k(z)$ does not depend on the choice of ψ and $\hat{\xi}$.

It follows from Lemma 3.4 that f_k is the holomorphic function of exponential type on $(\widehat{M + \sqrt{-1}\bigcap_{j\neq k}\omega_j^\circ})$. Hence we get the morphism

$$\mathscr{S}\ :\ \mathcal{O}^{a,\,\inf}_{\mathbb{D}_E}(N_\infty(K^\circ)) \longrightarrow \mathcal{B}^{\exp}_{\overline{M}}(\overline{M}) \tag{3.12}$$

by

$$\mathscr{S}(f) = \bigoplus_{0\le k\le n}\sigma_k f_k.$$

Then we have the following results.

Lemma 3.5. $\operatorname{supp}(\mathscr{S}(f)) \subset \overline{K_a}$ for $f \in \mathcal{O}^{a,\,\inf}_{\mathbb{D}_E}(N_\infty(K^\circ))$.

Finally we give our main theorem.

Theorem 3.6. $\mathscr{S}\circ\mathscr{L} = id_{\Gamma_{\overline{K}_a}(\overline{M},\mathcal{B}^{\exp}_{\overline{M}})}, \qquad \mathscr{L}\circ\mathscr{S} = id_{\mathcal{O}^{a,\,\inf}_{\mathbb{D}_E}(N_\infty(K^\circ))}.$

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