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Multi-microlocalization

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Abstract

The purpose of this paper is to report on the foundations of multi-microlocalization, in particular, to give the fiber formula for the multi-microlocalization functor and estimate of microsupport of a multi-microlocalized object. We also give some applications of these results.

§ 1. Multi-specialization

In this section we recall some results of [3]. We first fix some notations, then we recall the notion of multi-normal deformation and the definition of the functor of multi-specialization with some basic properties.

§ 1.1. Notations

Let X be a real analytic manifold with $\dim X = n$, and let $\chi = \{M_1, \dots, M_\ell\}$ be a family of closed submanifolds in X ($\ell \geq 1$). Throughout the paper all the manifolds are always assumed to be countable at infinity. We set, for $N \in \chi$, $\iota(N) := \bigcap_{N \subsetneq M_j} M_j$. Here $\iota(N) := X$ if there exists no j with $N \subsetneq M_j$. We set, for $N \in \chi$ and $p \in N$,

$$\mathrm{NR}_p(N) := \{M_j \in \chi;\, p \in M_j,\, N \not\subseteq M_j \text{ and } M_j \not\subseteq N\}.$$

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Let us consider the following conditions for χ.

H1 Each $M_j \in \chi$ is connected and the submanifolds are mutually distinct, i.e. $M_j \neq M_{j'}$ for $j \neq j'$.

H2 For any $N \in \chi$ and $p \in N$ with $\mathrm{NR}_p(N) \neq \emptyset$, we have

$$\left(\bigcap_{M_j \in \mathrm{NR}_p(N)} T_p M_j \right) + T_p N = T_p X. \tag{1.1}$$

H3 $M_j \neq \iota(M_j)$ for any $j \in \{1, 2, \dots, \ell\}$.

Note that, if χ satisfies the condition H2, the configuration of two submanifolds must be either 1. or 2. below.

1. Both submanifolds intersect transversely.
2. One of them contains the other.

It follows from Proposition 1.2 [3] that, at every $p \in \cap_{j=1}^{\ell} M_j$, there exist a system of local coordinates $(x_1, x_2, \dots, x_n)$ and subsets $I_1, \dots, I_\ell \subseteq \{1, \dots, n\}$ such that $M_j = \{x_k = 0\ ;\ k \in I_j\}$ for $j = 1, \dots, \ell$. Furthermore, these $I_1, \dots, I_\ell$ satisfy the conditions

$$\begin{aligned} &\text{(i) either } I_j \subsetneqq I_k,\ I_k \subsetneqq I_j \text{ or } I_j \cap I_k = \emptyset \text{ holds for any } j \neq k, \\ &\text{(ii) } \left(\bigcup_{I_k \subsetneqq I_j} I_k \right) \subsetneqq I_j \text{ for any } j. \end{aligned} \tag{1.2}$$

Hence, for any $j \in \{1, 2, \dots, \ell\}$, the set

$$\hat{I}_j := I_j \setminus \left(\bigcup_{I_k \subsetneqq I_j} I_k \right) \tag{1.3}$$

is not empty. For convenience, we set $\hat{I}_0 = I_0 := \{1, \dots, n\} \setminus \left(\bigcup_{j=1}^{\ell} I_j \right)$. Then, in local coordinates, we can write the coordinates $(x_1, \dots, x_n)$ by

$$(x^{(0)}, x^{(1)}, \dots, x^{(\ell)}), \tag{1.4}$$

where $x^{(j)}$ denotes the coordinates $(x_i)_{i \in \hat{I}_j}$ $(j = 0, \dots, \ell)$. We also define, for $j \in \{0, 1, \dots, \ell\}$

$$\hat{J}_j = \{k \in \{1, \dots, \ell\}\ ;\ \hat{I}_j \subseteq I_k\} = \{k \in \{1, \dots, \ell\}\ ;\ I_j \subseteq I_k\}. \tag{1.5}$$

Note that, with this notation, we have $\hat{J}_0 = \emptyset$ and

$$I_i \subseteq I_j \quad \Rightarrow \quad \hat{J}_j \subseteq \hat{J}_i. \tag{1.6}$$

§ 1.2. Multi-normal deformation

In [3] the notion of multi-normal deformation was introduced. Here we consider a slight generalization where we replace the condition H2 with the weaker one. Let $\chi = \{M_1, \dots, M_\ell\}$ be a family of closed submanifolds of X. We say that χ is *simultaneously linearizable* on $M = M_1 \cap \dots \cap M_\ell$ if for every $x \in M$ there exist a neighborhood V of x and a system of local coordinates $(x_1, \dots, x_n)$ there for which we can find subsets I_j's of $\{1, \dots, n\}$ such that each $M_j \cap V$ is defined by equations $x_i = 0$ $(i \in I_j)$. Note that if χ satisfies the condition H2, then it is simultaneously linearizable. Now, through the section, we assume that χ is simultaneously linearizable on M.

First recall the classical construction of [4] of the normal deformation of X along M_1. We denote it by $\widetilde{X}_{M_1}$ and we denote by $t_1 \in \mathbb{R}$ the deformation parameter. Set $\widetilde{\Omega}_{M_1} = \{(x; t_1) \ ; \ t_1 \neq 0\}$ and define $\widetilde{M_2} := \overline{(p_{M_1}|_{\widetilde{\Omega}_{M_1}})^{-1} M_2}$. Then $\widetilde{M_2}$ is a closed smooth submanifold of $\widetilde{X}_{M_1}$. Now we can define the normal deformation along M_1, M_2 as $\widetilde{X}_{M_1,M_2} := (\widetilde{X}_{M_1})^{\sim}_{\widetilde{M}_2}$. Then we can define recursively the normal deformation along χ as

$$\widetilde{X} = \widetilde{X}_{M_1,\dots,M_\ell} := (\widetilde{X}_{M_1,\dots,M_{\ell-1}})^{\sim}_{\widetilde{M}_\ell}.$$

Set $S_\chi = \{t_1, \dots, t_\ell = 0\}$, $M = \bigcap_{i=1}^{\ell} M_i$ and $\Omega_\chi = \{t_1, \dots, t_\ell > 0\}$. Then we have the commutative diagram

$$\begin{array}{ccccc} S_\chi & \xrightarrow{\ s\ } & \widetilde{X} & \xleftarrow{\ i_\Omega\ } & \Omega_\chi \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle \widetilde{p}} & \\ M & \xrightarrow{\ i_M\ } & X. & & \end{array} \tag{1.7}$$

Let us consider the diagram (1.7). In local coordinates let $I_1, \dots, I_\ell \subseteq \{1, \dots, n\}$ such that $M_i = \{x_k = 0 \ ; \ k \in I_i\}$. For $j \in \{0, \dots, \ell\}$ set $t_{\hat{J}_j} = \prod_{k \in \hat{J}_j} t_k$, where $t_1, \dots, t_\ell \in \mathbb{R}$ and $t_{\hat{J}_0} = 1$. Then $p : \widetilde{X} \to X$ is defined by

$$(x^{(0)}, x^{(1)} \dots, x^{(\ell)}; t_1, \dots, t_\ell) \mapsto (t_{\hat{J}_0} x^{(0)}, t_{\hat{J}_1} x^{(1)}, \dots, t_{\hat{J}_\ell} x^{(\ell)}).$$

Definition 1.1. Let Z be a subset of X. The multi-normal cone to Z along χ is the set $C_\chi(Z) = \overline{\widetilde{p}^{-1}(Z)} \cap S_\chi$.

Let us consider the canonical map $T_{M_j}\iota(M_j) \to M_j \hookrightarrow X$ $(j = 1, \dots, \ell)$, and then, we write for short

$$\underset{X, 1\le j\le \ell}{\times} T_{M_j}\iota(M_j) := T_{M_1}\iota(M_1) \underset{X}{\times} T_{M_2}\iota(M_2) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_\ell}\iota(M_\ell).$$

When χ satisfies the conditions H1, H2 and H3 we have $S_\chi \simeq \underset{X, 1\le j\le \ell}{\times} T_{M_j}\iota(M_j)$.

Example 1.2. Let us see two typical examples of multi-normal deformations in the complex case. Let $X = \mathbb{C}^2$ ($\simeq \mathbb{R}^4$ as a real manifold) with coordinates (z_1, z_2).

1. (Majima) Let $\chi = \{M_1, M_2\}$ with $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1\}$, $I_2 = \{2\}$, $J_1 = \{1\}$, $J_2 = \{2\}$ (in $\mathbb{R}^4$, if $z_1 = (x_1, x_2)$ and $z_2 = (x_3, x_4)$ we have $I_1 = \{1, 2\}$, $I_2 = \{3, 4\}$, $J_1 = J_2 = \{1\}$, $J_3 = J_4 = \{2\}$). The map $p : \widetilde{X} \to X$ is defined by

$$(z_1, z_2;\, t_1, t_2) \mapsto (t_1 z_1, t_2 z_2).$$

   Remark that the deformation is real though X is complex. In particular $t_1, t_2 \in \mathbb{R}$. We have $\iota(M_1) = \iota(M_2) = X$ and then the zero section S of $\widetilde{X}$ is isomorphic to $T_{M_1}X \underset{X}{\times} T_{M_2}X$.

2. (Takeuchi) Let $\chi = \{M_1, M_2\}$ with $M_1 = \{0\}$ and $M_2 = \{z_2 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1, 2\}$, $I_2 = \{2\}$, $J_1 = \{1\}$, $J_2 = \{1, 2\}$ (in $\mathbb{R}^4$, if $z_1 = (x_1, x_2)$ and $z_2 = (x_3, x_4)$ we have $I_1 = \{1, 2, 3, 4\}$, $I_2 = \{3, 4\}$, $J_1 = J_2 = \{1\}$, $J_3 = J_4 = \{1, 2\}$). The map $p : \widetilde{X} \to X$ is defined by

$$(z_1, z_2;\, t_1, t_2) \mapsto (t_1 z_1, t_1 t_2 z_2).$$

   We have $\iota(M_1) = M_2$, $\iota(M_2) = X$ and then the zero section S of $\widetilde{X}$ is isomorphic to $T_{M_1}M_2 \underset{X}{\times} T_{M_2}X$.

Example 1.3. Let us see three typical examples of multi-normal deformations in the real case. Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3).

1. (Majima) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1\}$, $I_2 = \{2\}$, $I_3 = \{3\}$, $J_1 = \{1\}$, $J_2 = \{2\}$, $J_3 = \{3\}$. The map $p : \widetilde{X} \to X$ is defined by

$$(x_1, x_2, x_3;\, t_1, t_2, t_3) \mapsto (t_1 x_1, t_2 x_2, t_3 x_3).$$

We have $\iota(M_1) = \iota(M_2) = \iota(M_3) = X$ and then the zero section S of $\widetilde{X}$ is isomorphic to $T_{M_1}X \underset{X}{\times} T_{M_2}X \underset{X}{\times} T_{M_3}X$.

2. (Takeuchi) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1,2,3\}$, $I_2 = \{2,3\}$, $I_3 = \{3\}$, $J_1 = \{1\}$, $J_2 = \{1,2\}$, $J_3 = \{1,2,3\}$. The map $p : \widetilde{X} \to X$ is defined by

$$(x_1, x_2, x_3;\, t_1, t_2, t_3) \mapsto (t_1x_1, t_1t_2x_2, t_1t_2t_3x_3).$$

We have $\iota(M_1) = M_2$, $\iota(M_2) = M_3$, $\iota(M_3) = X$ and then the zero section S of $\widetilde{X}$ is isomorphic to $T_{M_1}M_2 \underset{X}{\times} T_{M_2}M_3 \underset{X}{\times} T_{M_3}X$.

3. (Mixed) Let $\chi = \{M_1, M_2, M_3\}$ with $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1,2,3\}$, $I_2 = \{2\}$, $I_3 = \{3\}$, $J_1 = \{1\}$, $J_2 = \{1,2\}$, $J_3 = \{1,3\}$. The map $p : \widetilde{X} \to X$ is defined by

$$(x_1, x_2, x_3;\, t_1, t_2, t_3) \mapsto (t_1x_1, t_1t_2x_2, t_1t_3x_3).$$

We have $\iota(M_1) = M_2 \cap M_3$, $\iota(M_2) = \iota(M_3) = X$ and then the zero section S is isomorphic to $T_{M_1}(M_2 \cap M_3) \underset{X}{\times} T_{M_2}X \underset{X}{\times} T_{M_3}X$.

Example 1.4. For well understanding, let us give an example of mixed type in $X = \mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4). Let $\chi = \{M_1, M_2, M_3, M_4\}$ with $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$, $M_3 = \{x_3 = 0\}$ and $M_4 = \{x_4 = 0\}$. Then χ satisfies H1, H2 and H3. We have $I_1 = \{1,2,3,4\}$, $I_2 = \{2,3\}$, $I_3 = \{3\}$, $I_4 = \{4\}$ and $J_1 = \{1\}$, $J_2 = \{1,2\}$, $J_3 = \{1,2,3\}$, J_4={1,4}. The map $p : \widetilde{X} \to X$ is defined by

$$(x_1, x_2, x_3;\, t_1, t_2, t_3) \mapsto (t_1x_1, t_1t_2x_2, t_1t_2t_3x_3, t_1t_4x_4).$$

We have $\iota(M_1) = M_2 \cap M_4$, $\iota(M_2) = M_3$, $\iota(M_3) = \iota(M_4) = X$ and then the zero section S is isomorphic to $T_{M_1}(M_2 \cap M_4) \underset{X}{\times} T_{M_2}M_3 \underset{X}{\times} T_{M_3}X \underset{X}{\times} T_{M_4}X$.

When χ satisfies conditions H1, H2 and H3, the zero-section S_χ becomes a vector bundle over M. However, in general, the simultaneously linearizable condition is not enough to assure the existence of a vector bundle structure on S_χ, as the following example shows. The important exceptional case where χ does not satisfy H2 but S_χ has a vector bundle structure is studied in § 3.1.

Example 1.5. Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3), and let $\chi = \{M_1, M_2\}$ be a family of closed submanifolds in X defined by $M_1 = \{x_2 = x_3 = 0\}$ and $M_2 = \{x_1 = x_3 = 0\}$. Then S_χ is locally isomorphic to $\mathbb{R}^3$ with coordinates (ξ_1, ξ_2, ξ_3). Let $f = (f_1, f_2, f_3) : X \to Y$ be a coordinates transformation on X to its copy Y with coordinates (y_1, y_2, y_3) which sends M_1 and M_2 to their copy's defined by the same equations $\{y_2 = y_3 = 0\}$ and $\{y_1 = y_3 = 0\}$ respectively. Then the associated coordinates transformation from S_χ to its copy S_χ with coordinates (η_1, η_2, η_3) is given by

$$\begin{aligned}
\eta_1 &= \frac{\partial f_1}{\partial x_1}(0)\xi_1, \\
\eta_2 &= \frac{\partial f_2}{\partial x_2}(0)\xi_2, \\
\eta_3 &= \frac{\partial f_3}{\partial x_3}(0)\xi_3 + \frac{\partial^2 f_3}{\partial x_1 \partial x_2}(0)\xi_1\xi_2.
\end{aligned}$$

Hence S_χ is not a vector bundle over $M = \{0\}$.

From now on we assume conditions H1, H2 and H3. Let $q \in \bigcap_{1\le j\le \ell} M_j$ and $p_j = (q;\, \xi_j)$ be a point in $T_{M_j}\iota(M_j)$ $(j = 1, 2, \ldots, \ell)$. We set $p = p_1 \underset{X}{\times} \cdots \underset{X}{\times} p_\ell \in \underset{X, 1\le j\le \ell}{\times} T_{M_j}\iota(M_j)$, and $\tilde{p}_j = (q;\, \tilde{\xi}_j) \in T_{M_j}X$ denotes the image of the point p_j by the canonical embedding $T_{M_j}\iota(M_j) \hookrightarrow T_{M_j}X$. We denote by $\mathrm{Cone}_{\chi,j}(p)$ $(j = 1, 2, \ldots, \ell)$ the set of open conic cones in $(T_{M_j}X)_q \simeq \mathbb{R}^{n-\dim M_j}$ that contain the point $\tilde{\xi}_j \in (T_{M_j}X)_q \simeq \mathbb{R}^{n-\dim M_j}$.

Definition 1.6. We say that an open set $G \subset (TX)_q$ is a multi-cone along χ with direction to $p \in \left(\underset{X, 1\le j\le \ell}{\times} T_{M_j}\iota(M_j)\right)_q$ if G is written in the form

$$G = \bigcap_{1\le j\le \ell} \pi_{j,\,q}^{-1}(G_j) \qquad G_j \in \mathrm{Cone}_{\chi,j}(p)$$

where $\pi_{j,\,q} : (TX)_q \to (T_{M_j}X)_q$ is the canonical projection. We denote by $\mathrm{Cone}_\chi(p)$ the set of multi-cones along χ with direction to p.

For any $q \in X$, there exists an isomorphism $\psi : X \simeq (TX)_q$ near q with $\psi(q) = (q;\, 0)$ that satisfies $\psi(M_j) = (TM_j)_q$ for any $j = 1, \ldots, \ell$.

Let Z be a subset of X. When χ satisfies H1, H2 and H3 we also have the following equivalence: $p \notin C_\chi(Z)$ if and only if there exist an open subset $\psi(q) \in U \subset (TX)_q$ and a multi-cone $G \in \mathrm{Cone}_\chi(\psi_*(p))$ such that $\psi(Z) \cap G \cap U = \emptyset$ holds.

Example 1.7. We now give two examples of multi-cones in the complex case. Let $X = \mathbb{C}^2$ with coordinates (z_1, z_2).

1. (Majima) Let $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then $\mathrm{Cone}_\chi(p)$ for $p = (0, 0;\ 1, 1)$ is nothing but the set of multi sectors along $Z_1 \cup Z_2$ with their direction to $(1, 1)$.

2. (Takeuchi) Let $M_1 = \{0\}$ and $M_2 = \{z_2 = 0\}$. For $p = (0, 0;\ 1, 1) \in T_{M_1} M_2 \underset{X}{\times} T_{M_2} X$, it is easy to see that a cofinal set of $\mathrm{Cone}_\chi(p)$ is, for example, given by the family of the sets
$$\{(\eta_1, \eta_2);\ |\eta_1| < \epsilon|\eta_2|,\ \eta_1, \eta_2 \in S\}_{S \ni 1, \epsilon > 0},$$
where S is a sector in $\mathbb{C}$ containing the direction 1.

Example 1.8. We now give three examples of multi-cones in the real case. Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3).

1. (Majima) Let $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. For $p = (0, 0, 0;\ 1, 1, 1) \in T_{M_1} X \underset{X}{\times} T_{M_2} X \underset{X}{\times} T_{M_3} X$, it is easy to see that $\mathrm{Cone}_\chi(p) = \{(\mathbb{R}^+)^3\}$.

2. (Takeuchi) Let $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. For $p = (0, 0, 0;\ 1, 1, 1) \in T_{M_1} M_2 \underset{X}{\times} T_{M_2} M_3 \underset{X}{\times} T_{M_3} X$, it is easy to see that a cofinal set of $\mathrm{Cone}_\chi(p)$ is, for example, given by the family of the sets
$$\{(\xi_1, \xi_2, \xi_3);\ |\xi_2| + |\xi_3| < \epsilon\xi_1,\ |\xi_3| < \epsilon\xi_2,\ \xi_3 > 0\}_{\epsilon > 0}.$$

3. (Mixed) Let $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. For $p = (0, 0, 0;\ 1, 1, 1) \in T_{M_1}(M_2 \cap M_3) \underset{X}{\times} T_{M_2} X \underset{X}{\times} T_{M_3} X$, a cofinal set of $\mathrm{Cone}_\chi(p)$ is, for example, given by the family of the sets
$$\{(\xi_1, \xi_2, \xi_3);\ |\xi_2| + |\xi_3| < \epsilon\xi_1,\ \xi_2 > 0,\ \xi_3 > 0\}_{\epsilon > 0}.$$

Example 1.9. We now consider the case of Example 1.4. Let $X = \mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4). Let $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = x_3 = 0\}$, $M_3 = \{x_3 = 0\}$ and $M_4 = \{x_4 = 0\}$. For $p = (0, 0, 0, 0;\ 1, 1, 1, 1) \in T_{M_1}(M_2 \cap M_4) \underset{X}{\times} T_{M_2} M_3 \underset{X}{\times} T_{M_3} X \underset{X}{\times} T_{M_4} X$, a cofinal set of $\mathrm{Cone}_\chi(p)$ is, for example, given by the family of the sets
$$\{(\xi_1, \xi_2, \xi_3, \xi_4);\ |\xi_2| + |\xi_4| < \epsilon\xi_1,\ |\xi_3| < \epsilon\xi_2,\ \xi_3 > 0,\ \xi_4 > 0\}_{\epsilon > 0}.$$

This definition is also compatible with the restriction to a subfamily of χ. Namely, let $k \leq \ell$ and $K = \{j_1, \dots, j_k\}$ be a subset of $\{1, 2, \dots, \ell\}$. Set $\chi_K = \{M_{j_1}, \dots, M_{j_k}\}$ and $S_K := T_{M_{j_1}} \iota_\chi(M_{j_1}) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_{j_k}} \iota_\chi(M_{j_k}) \underset{X}{\times} M$. Let Z be a subset of X. Then we have

$$C_\chi(Z) \cap S_K = C_{\chi_K}(Z) \cap S_K.$$

Remark that we assume that conditions H1, H2 and H3 are satisfied, in the weak condition of simultaneous linearizability S_χ has no vector bundle structure in general and the definition of S_K does not make sense.

§ 1.3. Multi-specialization

Let k be a field and denote by $\mathrm{Mod}(k_{X_{sa}})$ (resp. $D^b(k_{X_{sa}})$) the category (resp. bounded derived category) of sheaves on the subanalytic site X_{sa}. For the theory of sheaves on subanalytic sites we refer to [5, 6]. For the theory of multi-specialization we refer to [3]. Let χ be a family of submanifolds satisfying H1, H2 and H3.

Definition 1.10. The multi-specialization along χ is the functor

$$\nu_\chi^{sa} \colon D^b(k_{X_{sa}}) \to D^b(k_{S_{\chi sa}}),\ F \mapsto s^{-1} R\Gamma_{\Omega_\chi} p^{-1} F.$$

We can give a description of the sections of the multi-specialization of $F \in D^b(k_{X_{sa}})$: let V be a conic subanalytic open subset of S_χ. Then:

$$\mathrm{H}^j(V; \nu_M^{sa} F) \simeq \varinjlim_U \mathrm{H}^j(U; F),$$

where U ranges through the family of open subanalytic subsets of X such that $C_\chi(X \setminus U) \cap V = \emptyset$. Let $p = (q; \xi) \in \underset{X, 1 \leq j \leq \ell}{\times} T_{M_j} \iota(M_j)$, let $B_\epsilon \subset (TX)_q$ be an open ball of radius $\epsilon > 0$ with its center at the origin and set

$$\mathrm{Cone}_\chi(p, \epsilon) := \{G \cap B_\epsilon;\ G \in \mathrm{Cone}_\chi(p)\}.$$

Applying the functor $\rho^{-1} \colon D^b(k_{S_{\chi sa}}) \to D^b(k_{S_\chi})$ (see [6] for details) we can calculate the fibers at $p \in \underset{X, 1 \leq j \leq \ell}{\times} T_{M_j} \iota(M_j)$ which are given by

$$(H^j \rho^{-1} \nu_\chi^{sa} F)_p \simeq \varinjlim_W \mathrm{H}^j(W; F),$$

where W ranges through the family $\mathrm{Cone}_\chi(p, \epsilon)$ for $\epsilon > 0$.

If there is no risk of confusion, in the rest of the paper we will use the notation

$$\nu_\chi = \rho^{-1}\nu^{sa}_\chi \colon D^b(k_{X_{sa}}) \to D^b(k_{S_\chi}).$$

§ 2. Multi-microlocalization

In this section we introduce the functor of multi-microlocalization as the Fourier-Sato transform of multi-specialization. We then compute its stalks as inductive limits of sections supported on convex subanalytic cones. We refer to [2] for the proofs.

§ 2.1. Definition

Now we are going to apply the Fourier-Sato transform to the multi-specialization. We refer to [4] for the classical Fourier-Sato transform and to [7] for its generalization to subanalytic sheaves. First, we need a general result: Let $\tau_i \colon E_i \to Z$ $(1 \le i \le \ell)$ be vector bundles over Z, and let E_i^* be the dual bundle of E_i. We denote by $\wedge_i$ and $\vee_i$ the Fourier-Sato and the inverse Fourier-Sato transformations on E_i respectively. Moreover we denote by $\wedge_i^*$ and $\vee_i^*$ the Fourier-Sato and the inverse Fourier-Sato transformations on E_i^* respectively. Set $E := E_1 \underset{Z}{\times} \cdots \underset{Z}{\times} E_\ell$ and $E^* := E_1^* \underset{Z}{\times} \cdots \underset{Z}{\times} E_\ell^*$ for short. Let $\tau \colon E \to Z$ be the the canonical projection. Set $P_i' := \{(\eta, \xi) \in E_i \underset{Z}{\times} E_i^*;\ \langle \eta, \xi \rangle \leqslant 0\}$. Further set

$$P' := P_1' \underset{Z}{\times} \cdots \underset{Z}{\times} P_\ell', \qquad P^+ := E \underset{Z}{\times} E^* \setminus P',$$

and denote by $p_1' \colon P' \to E$, $p_2' \colon P' \to E^*$, and $p_1^+ \colon P^+ \to E_i$, $p_2^+ \colon P^+ \to E^*$ the canonical projections respectively. Let F and G be multi-conic objects on E and E^* respectively. Then we set for short $\wedge_E$ (resp. $\vee_E^*$) the composition of the Fourier-Sato transforms $\wedge_i$ (resp. the composition of the inverse Fourier-Sato transforms $\vee_i^*$) on E_i for each $i \in \{1, \dots, \ell\}$.

Let F and G be multi-conic objects on E and E^* respectively. Then $F^{\wedge_E}$ and $G^{\vee_E^*}$ are independent of the order of the Fourier-Sato transformations $\wedge_i$ and inverse the Fourier-Sato transformations $\vee_i^*$ respectively. It follows that

$$G^{\vee_E^*} = Rp'_{1*}p_2'^{!}G.$$

We shall need some notations. For a subset $K = \{i_1, \dots, i_k\} \subseteq \{1, \dots, \ell\}$, set $\chi_K := \{M_i;\ i \in K\}$, $S_i := T_{M_i}\iota(M_i) \underset{X}{\times} M$ $(j = 1, \dots, \ell)$ and

$$S_K := T_{M_{i_1}}\iota(M_{i_1}) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_{i_k}}\iota(M_{i_k}) = S_{i_1} \underset{M}{\times} \cdots \underset{M}{\times} S_{i_k}.$$

Let S_K^* be the dual of S_K:

$$S_K^* := T^*_{M_{i_1}}\iota(M_{i_1}) \underset{X}{\times} \cdots \underset{X}{\times} T^*_{M_{i_k}}\iota(M_{i_k}) = S^*_{i_1} \underset{M}{\times} \cdots \underset{M}{\times} S^*_{i_k}.$$

Given $C_{i_j} \subseteq S_{i_j}$, $j = 1, \dots, k$, we set for short $C_K := C_{i_1} \underset{X}{\times} \cdots \underset{X}{\times} C_{i_k} \subset S_K$. Define $\wedge_K$ as the composition of the Fourier-Sato transformations $\wedge_{i_k}$ on S_{i_k} for each $i_k \in K$.

Let $I, J \subseteq \{1, \dots, \ell\}$ be such that $I \sqcup J = \{1, \dots, \ell\}$. We still denote by $\pi\colon S_I \underset{M}{\times} S_J^* \to M$ the projection. We define the functor $\nu^{\mathrm{sa}}_{\chi_I}\mu^{\mathrm{sa}}_{\chi_J}$ by

$$\nu^{\mathrm{sa}}_{\chi_I}\mu^{\mathrm{sa}}_{\chi_J}\colon D^b(k_{X_{\mathrm{sa}}}) \ni F \mapsto \nu^{\mathrm{sa}}_{\chi}(F)^{\wedge_J} \in D^b(k_{(S_I \underset{M}{\times} S_J^*)_{\mathrm{sa}}}).$$

Composing with the functor ρ^{-1}, we set for short

$$\nu_{\chi_I}\mu_{\chi_J} := \rho^{-1}\nu^{\mathrm{sa}}_{\chi_I}\mu^{\mathrm{sa}}_{\chi_J}\colon D^b(k_{X_{\mathrm{sa}}}) \to D^b(k_{S_I \underset{M}{\times} S_J^*}).$$

When $I = \emptyset$, we obtain the functor of the multi-microlocalization: Set $\wedge := \wedge_{\{1,\dots,\ell\}}$ for short.

Definition 2.1. The multi-microlocalization along χ is the functor

$$\mu^{\mathrm{sa}}_{\chi}\colon D^b(k_{X_{\mathrm{sa}}}) \ni F \mapsto \nu^{\mathrm{sa}}_{\chi}(F)^{\wedge} \in D^b(k_{S^*_{\chi \mathrm{sa}}}).$$

As above, we set for short

$$\mu_{\chi} := \rho^{-1}\mu^{\mathrm{sa}}_{\chi}\colon D^b(k_{X_{\mathrm{sa}}}) \to D^b(k_{S^*_{\chi}}).$$

§ 2.2. Stalks

Let X be a real analytic manifold and consider a family of submanifolds $\chi = \{M_1, \dots, M_\ell\}$ satisfying H1, H2 and H3. Let $S = T_{M_1}\iota(M_1) \underset{X}{\times} \cdots \underset{X}{\times} T_{M_\ell}\iota(M_\ell)$. Locally $p \in S$ is given by $p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)})$, with $\xi^{(k)} \in T_{M_k}\iota(M_k)$. Set $M = \bigcap_{j=1}^{\ell} M_j$. Let $\tau_j : T_{M_j}\iota(M_j) \hookrightarrow T_{M_j}X$ denote the canonical injection and let $\pi_j : S \to T_{M_j}\iota(M_j)$ be the canonical projection.

Set $S^* := T^*_{M_1}\iota(M_1) \underset{X}{\times} \dots \underset{X}{\times} T^*_{M_\ell}\iota(M_\ell)$. Let $V = V_1 \underset{X}{\times} \dots \underset{X}{\times} V_\ell$ be a multi-conic open subanalytic subset in S^*, and let $\pi : S^* \to M$ denote the canonical projection. We set, for short, $V^\circ := V_1^\circ \underset{X}{\times} \dots \underset{X}{\times} V_\ell^\circ$ the multi-polar cone in S.

Now we are going to find a stalk formula for the multi-microlocalization given by a limit of sections with support (locally) contained in closed convex cones. As the problem is local, we may assume that $X = \mathbb{R}^n$ and $q = 0$ with coordinates $(x_1, \dots, x_n)$, and that there exists a subset I_k $(k = 1, 2, \dots, \ell)$ in $\{1, 2, \dots, n\}$ with the conditions (1.2) such that each submanifold M_k is given by $\{x = (x_1, \dots, x_n) \in \mathbb{R}^n;\ x_i = 0\,(i \in I_k)\}$. Recall that $\hat{I}_k$ was defined by (1.3) and that we set $M = \cap_k M_k$ and $n_k = \sharp \hat{I}_k$. Then locally we have

$$X = M \times (N_1 \times N_2 \times \cdots \times N_\ell) = M \times N,$$

where N_k is $\mathbb{R}^{n_k}$ with coordinates $x^{(k)} = (x_i)_{i \in \hat{I}_k}$. Set, for $k \in \{1, \dots, \ell\}$,

$$\begin{aligned} J_{\prec k} &:= \{j \in \{1, \dots, \ell\};\ I_j \subsetneqq I_k\}, \\ J_{\succ k} &:= \{j \in \{1, \dots, \ell\};\ I_j \supsetneqq I_k\}, \\ J_{\not\parallel k} &:= \{j \in \{1, \dots, \ell\};\ I_j \cap I_k = \emptyset\}. \end{aligned} \tag{2.1}$$

Clearly we have

$$k \in J_{\prec j} \Leftrightarrow I_k \subsetneqq I_j \Leftrightarrow j \in J_{\succ k}, \tag{2.2}$$

and, by the conditions H1, H2 and H3, we also have

$$J_{\prec k} \sqcup \{k\} \sqcup J_{\succ k} \sqcup J_{\not\parallel k} = \{1, 2, \dots, \ell\}. \tag{2.3}$$

Let $p = p_1 \times \cdots \times p_\ell = (q; \xi^{(1)}, \dots, \xi^{(\ell)}) \in T^*_{M_1}\iota(M_1) \underset{X}{\times} \dots \underset{X}{\times} T^*_{M_\ell}\iota(M_\ell)$ and consider the following conic subset in N

$$\gamma_k := \left\{ (x^{(j)})_{j=1,\dots,\ell} \in N;\ \begin{array}{ll} x^{(j)} = 0 & (j \in J_{\prec k} \sqcup J_{\not\parallel k}), \\ x^{(j)} \in \mathbb{R}^{n_j} & (j \in J_{\succ k}), \\ \langle x^{(j)}, \xi^{(k)} \rangle > 0 & (j = k) \end{array} \right\}. \tag{2.4}$$

Note that, if $\xi^{(k)} = 0$, then γ_k is empty.

Example 2.2. We now compute γ_k of (2.4) on the complex case in the following two typical situations. Let $X = \mathbb{C}^2$ with coordinates (z_1, z_2).

1. (Majima) Let $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then

$$\gamma_1 = \{(z_1, 0);\ \mathrm{Re}\,\langle z_1, \eta_1\rangle > 0\},$$
$$\gamma_2 = \{(0, z_2);\ \mathrm{Re}\,\langle z_2, \eta_2\rangle > 0\}.$$

2. (Takeuchi) Let $M_1 = \{0\}$ and $M_2 = \{z_2 = 0\}$. Then

$$\gamma_1 = \{(z_1, 0);\ \mathrm{Re}\,\langle z_1, \eta_1\rangle > 0\},$$
$$\gamma_2 = \{(z_1, z_2);\ \mathrm{Re}\,\langle z_2, \eta_2\rangle > 0\}.$$

Example 2.3. We now compute γ_k of (2.4) on the real case in three typical situations. Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3).

1. (Majima) Let $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then

$$\gamma_1 = \{(x_1, 0, 0);\ \langle x_1, \xi_1 \rangle > 0\},$$
$$\gamma_2 = \{(0, x_2, 0);\ \langle x_2, \xi_2 \rangle > 0\},$$
$$\gamma_3 = \{(0, 0, x_3);\ \langle x_3, \xi_3 \rangle > 0\}.$$

2. (Takeuchi) Let $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then

$$\gamma_1 = \{(x_1, 0, 0);\ \langle x_1, \xi_1 \rangle > 0\},$$
$$\gamma_2 = \{(x_1, x_2, 0);\ \langle x_2, \xi_2 \rangle > 0\},$$
$$\gamma_3 = \{(x_1, x_2, x_3);\ \langle x_3, \xi_3 \rangle > 0\}.$$

3. (Mixed) Let $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then

$$\gamma_1 = \{(x_1, 0, 0);\ \langle x_1, \xi_1 \rangle > 0\},$$
$$\gamma_2 = \{(x_1, x_2, 0);\ \langle x_2, \xi_2 \rangle > 0\},$$
$$\gamma_3 = \{(x_1, 0, x_3);\ \langle x_3, \xi_3 \rangle > 0\}.$$

Theorem 2.4. *Let $p = p_1 \times \cdots \times p_\ell = (q;\ \xi^{(1)}, \dots, \xi^{(\ell)}) \in S^*$, and let $F \in D^b(k_{X_{sa}})$. Then we have*

$$H^k(\mu_\chi F)_p \simeq \varinjlim_{G,U} \mathrm{H}^k_G(U; F). \tag{2.5}$$

Here U is an open subanalytic neighborhood of q in X and G is a closed subanalytic subset in the form $M \times \left(\sum_{k=1}^{\ell} G_k\right)$ with G_k being a closed subanalytic convex cone in N satisfying $G_k \setminus \{0\} \subset \gamma_k$, where γ_k is defined in (2.4).

Now let us consider the mixed cases between specialization and microlocalization. Let $I, J \subseteq \{1, \dots, \ell\}$ be such that $I \sqcup J = \{1, \dots, \ell\}$, and let $p = p_1 \times \cdots \times p_\ell = (q;\ \xi^{(1)}, \dots, \xi^{(\ell)}) = (q;\ \xi_I, \xi_J) \in S_I \underset{X}{\times} S_J^*$. Locally we may identify S_J with its dual. Set for short $\nu_{\chi_I}\mu_{\chi_J} := \rho^{-1}\nu^{sa}_{\chi_I}\mu^{sa}_{\chi_J}$.

As in Theorem 2.4 we can find a family which (locally) consists of convex cones defining the stalk formula in the mixed case.

Theorem 2.5. *Let* $p = p_1 \times \cdots \times p_\ell = (q;\, \xi^{(1)}, \ldots, \xi^{(\ell)}) \in S_I \underset{X}{\times} S_J^*$*, and let* $F \in D^b(k_{X_{sa}})$*. Then we have*

$$H^k(\nu_{\chi_I}\mu_{\chi_J}F)_p \simeq \varinjlim_{G, W_\epsilon} \mathrm{H}^k_G(W_\epsilon; F). \tag{2.6}$$

Here $W_\epsilon = W \cap B_\epsilon$*, with* $W \in \mathrm{Cone}_\chi(q;\, \xi_I, 0_J)$*,* B_ϵ *is an open ball of radius* $\epsilon > 0$ *containing* q *and a closed subanalytic subset* $G = M \times \left(\sum_{k=1}^{\ell} G_k\right)$ *with* G_k *being a closed subanalytic convex cone in* N *satisfying* $G_k \setminus \{0\} \subset \gamma_k$*, where* γ_k *is defined in* (2.4)*.*

§ 3. Multi-microlocalization and microsupport

In this section we give an estimate of the microsupport of the multi-microlocalization. The main point is to find a suitable ambient space: this is done (via Hamiltonian isomorphism) by identifying T^*S_χ with the normal deformation of T^*X with respect to a suitable family of submanifolds χ^*. We refer to [2] for the proofs.

§ 3.1. Geometry

Let X be a real analytic manifold and consider a family of submanifolds $\chi = \{M_1, \ldots, M_\ell\}$ satisfying H1, H2 and H3. We consider the conormal bundle T^*X with local coordinates $(x^{(0)}, x^{(1)}, \ldots, x^{(\ell)};\, \xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(\ell)})$, where $x^{(j)} = (x_{j_1}, \ldots, x_{j_p})$ with $\hat{I}_j = \{j_1, \ldots, j_p\}$ etc. We use the notations in § 2.1; for example, we set $S_i := T_{M_i}\iota(M_i) \underset{X}{\times} M$. Let $I, J \subseteq \{1, \ldots, \ell\}$ be such that $I \sqcup J = \{1, \ldots, \ell\}$. Recall that

$$S_\chi = S_1 \underset{X}{\times} \cdots \underset{X}{\times} S_\ell,$$

$$S_\chi^* = S_1^* \underset{X}{\times} \cdots \underset{X}{\times} S_\ell^*,$$

$$S_I \underset{M}{\times} S_J^* = \big(\underset{M, i\in I}{\times} S_i\big) \underset{M}{\times} \big(\underset{M, j\in J}{\times} S_j^*\big)\,.$$

Then we consider a mapping

$$\begin{aligned} H_{IJ} \colon T^*S_\chi \ni{} & (x^{(0)}, x^{(1)}, \ldots, x^{(\ell)};\, \xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(\ell)}) \\ & \mapsto (x^{(0)}, (x^{(i)})_{i\in I}, (\xi^{(j)})_{j\in J};\, \eta^{(0)}, (\xi^{(i)})_{i\in I}, (-x^{(j)})_{j\in J}) \in T^*(S_I \underset{M}{\times} S_J^*). \end{aligned}$$

Note that H_{IJ} is induced by the Hamiltonian isomorphisms $T^*S_J \xrightarrow{\sim} T^*S_J^*$ and it gives a bundle isomorphism over M; that is, H_{IJ} does not depend on the choice of local coordinates.

Hence, using Proposition 5.5.5 of [4] repeatedly, we obtain:

Proposition 3.1. *Let $I, J \subseteq \{1, \dots, \ell\}$ be such that $I \sqcup J = \{1, \dots, \ell\}$. Then, under the identification given by H_{IJ}, for any $F \in D^b(k_X)$ it follows that*

$$\begin{array}{ccc} T^*S_\chi & = & T^*(S_I \underset{M}{\times} S_J^*) \\ \cup & & \cup \\ \mathrm{SS}(\nu_\chi(F)) & = & \mathrm{SS}(\nu_{\chi_I}\mu_{\chi_J}(F)). \end{array}$$

In particular, it follows that

$$\begin{array}{ccc} T^*S_\chi & = & T^*S_\chi^* \\ \cup & & \cup \\ \mathrm{SS}(\nu_\chi(F)) & = & \mathrm{SS}(\mu_\chi(F)). \end{array}$$

Next, we study the relation between the normal deformations of T^*X with respect to $\chi^* := \{T^*_{M_1}X, \dots, T^*_{M_\ell}X\}$ and of X with respect to χ. We denote by $\widetilde{T^*X}_{\chi^*} := \widetilde{T^*X}_{T^*_{M_1}X,\dots,T^*_{M_\ell}X}$ the normal deformation of T^*X with respect to χ^* and by S_{χ^*} its zero-section. Set $x := (x^{(0)}, x^{(1)}, \dots, x^{(\ell)})$, $\xi := (\xi^{(0)}, \xi^{(1)}, \dots, \xi^{(\ell)})$ and $t := (t_1, \dots, t_\ell)$. We have a mapping

$$\widetilde{T^*X}_{\chi^*} \ni (x;\, \xi;\, t) \mapsto (\mu_x(x;\, t);\, \mu_\xi(\xi;\, t)) \in T^*X$$

defined by

$$\begin{aligned} \mu_x(x;\, t) &:= (t_{\hat{J}_0}x^{(0)}, t_{\hat{J}_1}x^{(1)}, \dots, t_{\hat{J}_\ell}x^{(\ell)}), \\ \mu_\xi(\xi;\, t) &:= (t_{\hat{J}_0^c}\xi^{(0)}, t_{\hat{J}_1^c}\xi^{(1)}, \dots, t_{\hat{J}_\ell^c}\xi^{(\ell)}), \end{aligned}$$

where $\hat{J}_j^c := \{1, \dots, \ell\} \setminus \hat{J}_j$ $(j = 0, 1, \dots, \ell)$. In particular $\hat{J}_0^c = \{1, \dots, \ell\}$ since $\hat{J}_0 = \emptyset$. In particular $t_{\hat{J}_0} = 1$ and $t_{\hat{J}_0^c} = t_1 \cdots t_\ell$.

Theorem 3.2. *As vector bundles, there exist the following canonical isomorphism*:

$$S_{\chi^*} \simeq T^*S_\chi \simeq T^*S_\chi^*.$$

Example 3.3. Let $X = \mathbb{C}^2$ with coordinates (z_1, z_2) and consider T^*X with coordinates $(z;\, \eta) = (z_1, z_2;\, \eta_1, \eta_2)$. Set $t = (t_1, t_2) \in (\mathbb{R}^+)^2$.

1. (Majima) Let $M_1 = \{z_1 = 0\}$ and $M_2 = \{z_2 = 0\}$. Then $\chi^* = \{T^*_{M_1}X, T^*_{M_2}X\}$ and we have a map

$$\widetilde{T^*X} \to T^*X,$$
$$(z;\, \eta;\, t) \mapsto (\mu_z(z;\, t);\, \mu_\eta(\eta;\, t)),$$

which is defined by

$$\mu_z(z;\, t) = (t_1 z_1, t_2 z_2),$$
$$\mu_\eta(\eta;\, t) = (t_2 \eta_1, t_1 \eta_2).$$

By Theorem 3.2, we have $S^*_\chi \simeq T^*(T_{M_1}X \underset{X}{\times} T_{M_2}X) \simeq T^*(T^*_{M_1}X \underset{X}{\times} T^*_{M_2}X)$.

2. (Takeuchi) Let $M_1 = \{0\}$ and $M_2 = \{z_2 = 0\}$. Then $\chi^* = \{T^*_{M_1}X, T^*_{M_2}X\}$ and we have a map

$$\widetilde{T^*X} \to T^*X,$$
$$(z;\, \eta;\, t) \mapsto (\mu_z(z;\, t);\, \mu_\eta(\eta;\, t)),$$

which is defined by

$$\mu_z(z;\, t) = (t_1 z_1, t_1 t_2 z_2),$$
$$\mu_\eta(\eta;\, t) = (t_2 \eta_1, \eta_2).$$

By Theorem 3.2, we have $S^*_\chi \simeq T^*(T_{M_1}M_2 \underset{X}{\times} T_{M_2}X) \simeq T^*(T^*_{M_1}M_2 \underset{X}{\times} T^*_{M_2}X)$.

Example 3.4. Let $X = \mathbb{R}^3$ with coordinates (x_1, x_2, x_3) and consider T^*X with coordinates $(x;\, \xi) = (x_1, x_2, x_3;\, \xi_1, \xi_2, \xi_3)$. Set $t = (t_1, t_2, t_3) \in (\mathbb{R}^+)^3$.

1. (Majima) Let $M_1 = \{x_1 = 0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then $\chi^* = \{T^*_{M_1}X, T^*_{M_2}X, T^*_{M_3}X\}$ and we have a map

$$\widetilde{T^*X} \to T^*X,$$
$$(x;\, \xi;\, t) \mapsto (\mu_x(x;\, t);\, \mu_\xi(\xi;\, t)),$$

which is defined by

$$\mu_x(x;\, t) = (t_1 x_1, t_2 x_2, t_3 x_3),$$
$$\mu_\xi(\xi;\, t) = (t_2 t_3 \xi_1, t_1 t_3 \xi_2, t_2 t_3 \xi_3).$$

By Theorem 3.2, we have $S^*_\chi \simeq T^*(T_{M_1}X \underset{X}{\times} T_{M_2}X \underset{X}{\times} T_{M_3}X) \simeq T^*(T^*_{M_1}X \underset{X}{\times} T^*_{M_2}X \underset{X}{\times} T^*_{M_3}X)$.

2. (Takeuchi) Let $M_1 = \{0\}$, $M_2 = \{x_2 = x_3 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then $\chi^* = \{T^*_{M_1}X, T^*_{M_2}X, T^*_{M_3}X\}$ and we have a map

$$\widetilde{T^*X} \to T^*X,$$
$$(x;\, \xi;\, t) \mapsto (\mu_x(x;\, t);\, \mu_\xi(\xi;\, t)),$$

which is defined by

$$\mu_x(x;\, t) = (t_1x_1, t_1t_2x_2, t_1t_2t_3x_3),$$
$$\mu_\xi(\xi;\, t) = (t_2t_3\xi_1, t_3\xi_2, \xi_3).$$

By Theorem 3.2, we have $S^*_\chi \simeq T^*(T_{M_1}M_2 \underset{X}{\times} T_{M_2}M_3 \underset{X}{\times} T_{M_3}X) \simeq T^*(T^*_{M_1}M_2 \underset{X}{\times} T^*_{M_2}M_3 \underset{X}{\times} T^*_{M_3}X)$.

3. (Mixed) Let $M_1 = \{0\}$, $M_2 = \{x_2 = 0\}$ and $M_3 = \{x_3 = 0\}$. Then $\chi^* = \{T^*_{M_1}X, T^*_{M_2}X, T^*_{M_3}X\}$ and we have a map

$$\widetilde{T^*X} \to T^*X,$$
$$(x;\, \xi;\, t) \mapsto (\mu_x(x;\, t);\, \mu_\xi(\xi;\, t)),$$

which is defined by

$$\mu_x(x;\, t) = (t_1x_1, t_1t_2x_2, t_1t_3x_3),$$
$$\mu_\xi(\xi;\, t) = (t_2t_3\xi_1, t_3\xi_2, t_2\xi_3).$$

By Theorem 3.2, we have $S^*_\chi \simeq T^*(T_{M_1}(M_2 \cap M_3) \underset{X}{\times} T_{M_2}X \underset{X}{\times} T_{M_3}X) \simeq T^*(T^*_{M_1}(M_2 \cap M_3) \underset{X}{\times} T^*_{M_2}X \underset{X}{\times} T^*_{M_3}X)$.

§ 3.2. Estimate of microsupport

In this section we shall prove an estimate for the microsupport of the multi-specialization and multi-microlocalization of a sheaf on X. We refer to [4] for the theory of microsupport of sheaves.

Since the problem is local, we may assume that $X = \mathbb{R}^n$ with coordinates $(x_1, \dots, x_n)$. Let I_k $(k = 1, 2, \dots, \ell)$ in $\{1, 2, \dots, n\}$ such that each submanifold M_k is given by

$$\{x = (x_1, \dots, x_n) \in \mathbb{R}^n;\, x_i = 0\,(i \in I_k)\}\,.$$

Theorem 3.5. *Let $F \in D^b(k_X)$ and take a point*

$$p_0 = (x_0^{(0)}, x_0^{(1)}, \dots, x_0^{(\ell)}; \xi_0^{(0)}, \xi_0^{(1)}, \dots, \xi_0^{(\ell)}) \in T^* S_\chi.$$

Assume that $p_0 \in \mathrm{SS}(\nu_\chi(F))$. Then there exist sequences

$$\{(c_{1,k}, \dots, c_{\ell,k})\}_{k=1}^{\infty} \subset (\mathbb{R}^+)^\ell,$$
$$\{(x_k^{(0)}, x_k^{(1)}, \dots, x_k^{(\ell)}; \xi_k^{(0)}, \xi_k^{(1)}, \dots, \xi_k^{(\ell)})\}_{k=1}^{\infty} \subset \mathrm{SS}(F),$$

such that

$$\begin{cases} \lim\limits_{k\to\infty} c_{j,k} = \infty, \qquad (j = 1, \dots, \ell), \\ \lim\limits_{k\to\infty} (x_k^{(0)}, x_k^{(1)} c_{\hat{J}_1,k}, \dots, x_k^{(\ell)} c_{\hat{J}_\ell,k}; \xi_k^{(0)} c_k, \xi_k^{(1)} c_{\hat{J}_1^c,k}, \dots, \xi_k^{(\ell)} c_{\hat{J}_\ell^c,k}) \\ \qquad = (x_0^{(0)}, x_0^{(1)}, \dots, x_0^{(\ell)}; \xi_0^{(0)}, \xi_0^{(1)}, \dots, \xi_0^{(\ell)}), \end{cases}$$

where $c_k := \prod_{j=1}^{\ell} c_{j,k}$, $\hat{J}_j^c := \{1, \dots, \ell\} \setminus \hat{J}_j$ and $c_{J,k} := \prod_{j\in J} c_{j,k}$ for any $J \subseteq \{1, \dots, \ell\}$.

Then, an estimate of the microsupport of multi-specialization and multi-microlocalization of sheaves on X follows from Theorem 3.5:

Theorem 3.6. *Let $F \in D^b(k_X)$. Then*

$$\mathrm{SS}(\nu_\chi(F)) = \mathrm{SS}(\mu_\chi(F)) \subseteq C_{\chi^*}(\mathrm{SS}(F)).$$

§ 4. Microfunctions along χ

In this section, thanks to the vanishing results of [2] we introduce multi-microlocalized objects along χ which are natural extensions of sheaves of microfunctions and holomorphic ones.

§ 4.1. The result for some family of real analytic submanifolds

Let $X = \mathbb{C}^n$ with coordinates $(z_1 = x_1 + \sqrt{-1}y_1, \dots, z_n = x_n + \sqrt{-1}y_n)$ and $\zeta_i = \xi_i + \sqrt{-1}\eta_i$ $(i = 1, \dots, n)$ the dual variable of $z_i = x_i + \sqrt{-1}y_i$. Let I_j $(j = 1, 2, \dots, \ell)$ be a subset of $\{1, 2, \dots, n\}$ which satisfies the conditions (1.2), and set $I_0 = \{1, \dots, n\} \setminus \left(\bigcup_{j=1}^{\ell} I_j\right)$. Let $I_\mathbb{R}$ be a subset of $I := \bigcup_{1\le j\le \ell} I_j$ and $I_\mathbb{C} := I \setminus I_\mathbb{R}$. Define, for $i \in I$, the function $q_i(z)$ in X by

$$q_i(z) := \begin{cases} z_i & (i \in I_\mathbb{C}), \\ \sqrt{-1}\,\mathrm{Im}\, z_i & (i \in I_\mathbb{R}). \end{cases}$$

Then we define the closed real analytic submanifolds

$$N_j := \{z \in X;\, q_i(z) = 0,\, i \in I_j\} \qquad (j = 1, \dots, \ell),$$

and set

$$\chi := \{N_1, \dots, N_\ell\}, \qquad N = N_1 \cap \dots \cap N_\ell.$$

In what follows, we regard the function q_i as the complex coordinate variable z_i if $i \in I_{\mathbb{C}}$ and as the imaginary coordinate variable $\sqrt{-1}y_i$ if $i \in I_{\mathbb{R}}$. In the same way, p_i is regraded as the dual variable of q_i, that is, p_i denotes ζ_i if $i \in I_{\mathbb{C}}$ and $\sqrt{-1}\eta_i$ if $i \in I_{\mathbb{R}}$. As usual convention, we write by $q^{(j)}$ (resp. $p^{(j)}$) the coordinates q_i's (resp. p_i's) with $i \in \hat{I}_j$. Under these conventions, the coordinates of S^*_χ are given by

$$(q^{(0)};\, p^{(1)},\, \dots,\, p^{(\ell)}),$$

where $q^{(0)}$ denotes the set of the coordinate variables z_i's ($i \in I_0$) and x_i's ($i \in I_{\mathbb{R}}$). Let $\theta_* = (q_*; p_*) = (q_*^{(0)};\, p_*^{(1)},\, \dots,\, p_*^{(\ell)}) \in S^*_\chi$. Recall the definition of $J^*(\theta_*)$, that is,

$$\begin{aligned} J^*(\theta_*) &:= \{j \in \{1, \dots, \ell\};\, p_*^{(\alpha)} = 0 \text{ for all } \alpha \in J_{\preceq j}\} \\ &= \{j \in \{1, \dots, \ell\};\, p_*^{(\alpha)} = 0 \text{ for all } \alpha \text{ with } N_j \subset N_\alpha\}. \end{aligned}$$

We set

$$I^*(\theta_*) := \bigcup_{j \in J^*(\theta_*)} \hat{I}_j \subset \{1, \dots, n\}. \tag{4.1}$$

Then we define the integer $N(\theta_*)$ by

$$N(\theta_*) = \#I + \#(I^*(\theta_*) \cap I_{\mathbb{C}}), \tag{4.2}$$

where $\#$ denotes the number of elements in a set. Note that $\#I$ is equal to $\operatorname{Codim}_{\mathbb{C}} N$, i.e., the complex codimension of the maximal complex linear subspace contained in N.

Theorem 4.1. *We have*

$$\mathrm{H}^k(\mu_\chi(\mathcal{O}^{\mathrm{w}}_{X_{sa}}))_{\theta_*} = 0 \qquad (k \neq N(\theta_*)).$$

We also have the similar results for $\mathcal{O}^t_X$ and $\mathcal{O}_X$

Theorem 4.2. *We have*

$$H^k(\mu_\chi(\mathcal{F}))_{\theta_*} = 0 \qquad (k \neq \operatorname{codim}_{\mathbb{C}} N = \#I),$$

where $\mathcal{F}$ is either $\mathcal{O}^t_{X_{sa}}$ or $\mathcal{O}_X$.

§ 4.2. The typical examples

As the results given in the previous subsection has been considered in a fairly general situation, we here describe the corresponding results for typical cases.

We first consider the corresponding result for families of complex submanifolds, i.e., $I = I_{\mathbb{C}}$. Let X be a complex manifold and $\chi = \{Z_1, \dots, Z_\ell\}$ a family of closed complex submanifolds of X which satisfies the conditions H1, H2 and H3. Set $Z = Z_1 \cap \cdots \cap Z_\ell$. Let $p = (q;\, \zeta) = (q;\, \zeta^{(1)}, \dots, \zeta^{(\ell)}) \in S^*_\chi$. Remember that the subset $J^*(p)$ of $\{1, \dots, \ell\}$ was defined by

$$J^*(p) := \{j \in \{1, \dots, \ell\};\, \zeta^{(\alpha)} = 0 \text{ for all } \alpha \text{ with } Z_j \subset Z_\alpha\}.$$

We also define $\hat{J}^*(p)$ by the subset of $J^*(p)$ that consists of the minimal elements with respect to the order relation $k \prec j \iff Z_k \subsetneq Z_j$ for $k, j \in J^*(p)$. Now we define the integer $N(p)$ by

$$N(p) = \operatorname{codim}_{\mathbb{C}} Z + \sum_{j \in \hat{J}^*(p)} \operatorname{codim}_{\mathbb{C}} Z_j. \tag{4.3}$$

Then the following result immediately comes from Theorem 4.1.

Corollary 4.3. *We have*

$$H^k(\mu_\chi(\mathcal{O}^{\mathrm{w}}_{X_{sa}}))_p = 0 \quad (k \neq N(p)).$$

We also have

$$H^k(\mu_\chi(\mathcal{F}))_p = 0 \quad (k \neq \operatorname{codim}_{\mathbb{C}} Z),$$

where $\mathcal{F}$ is either $\mathcal{O}^t_X$ or $\mathcal{O}_X$.

Definition 4.4. The sheaf of holomorphic microfunctions along χ in S^*_χ is defined by

$$\mathcal{C}^{\mathbb{R}}_\chi := \mu_\chi(\mathcal{O}_X) \underset{\mathbb{Z}_{S^*_\chi}}{\otimes} or_{S^*_\chi}[\operatorname{codim}_{\mathbb{C}} Z], \tag{4.4}$$

where $or_{S^*_\chi}$ denotes the orientation sheaf of S^*_χ. We also define $\mathcal{C}^{\mathbb{R},f}_\chi$ and $\mathcal{C}^{\mathbb{R},\mathrm{w}}_\chi$ by replacing $\mathcal{O}_X$ in the above definition with $\mathcal{O}^t_X$ and $\mathcal{O}^{\mathrm{w}}_X$ respectively.

Note that $\mathcal{C}_\chi^{\mathbb{R}}$ and $\mathcal{C}_\chi^{\mathbb{R},f}$ are really sheaves on S_χ^*. We also note that $\mathcal{C}_\chi^{\mathbb{R},\mathrm{w}}$ is a complex. It is, however, concentrated in degree 0 outside the zero section, i.e., $\{(z;\zeta^{(1)},\dots,\zeta^{(\ell)})\in S_\chi^*;\ \zeta^{(j)}\neq 0\}$.

Next we consider the corresponding result for the case $I = I_{\mathbb{R}}$. Let M be a connected real analytic manifold and X its complexification. Let $\Theta_1, \dots, \Theta_\ell$ be real analytic vector subbundles of TM which are involutive, that is, $[\theta_1, \theta_2] \in \Theta_k$ for any vector fields $\theta_1, \theta_2 \in \Theta_k$. We denote by $\Theta_k^{\mathbb{C}} \subset TX$ the complex vector subbundle over X that is a complexification of Θ_k near M. Now we introduce the conditions for Θ_k's which are counterparts of the ones H1, H2 and H3. Set, for $1 \le k \le \ell$,

$$\mathrm{NR}(k) := \{j \in \{1,\dots,\ell\};\ \Theta_j \not\subseteq \Theta_k, \Theta_k \not\subseteq \Theta_j\}$$

Then we assume that, for any $q \in M$ and any k with $\mathrm{NR}(k) \neq \emptyset$,

$$(TM)_q = (\Theta_k)_q + \left(\bigcap_{j\in \mathrm{NR}(k)} (\Theta_j)_q\right).$$

We also assume that, for simplicity, Θ_k's are mutually distinct, i.e., $\Theta_{k_1} \neq \Theta_{k_2}$ if $k_1 \neq k_2$.

Let $N_{M,j} \subset X$ $(j = 1,\dots,\ell)$ be the union of the complex integral submanifolds of the involutive complex vector bundle $\Theta_j^{\mathbb{C}} \subset TX$ passing through each point $q \in M$, that is,

$$N_{M,j} := \bigcup_{q\in M} \mathcal{L}(\Theta_j^{\mathbb{C}}, q)$$

where $\mathcal{L}(\Theta_j^{\mathbb{C}}, q)$ denotes the complex integral submanifold of Θ_j passing through the point q. Set

$$\chi := \{N_{M,1},\dots,N_{M,\ell}\}, \quad N_M := N_{M,1} \cap \cdots \cap N_{M,\ell} \subset X.$$

Corollary 4.5. *Let $p \in S_\chi^*$. Then we have*

$$H^k(\mu_\chi(\mathcal{F}))_p = 0 \quad (k \neq \mathrm{codim}_{\mathbb{R}} N_M),$$

where $\mathcal{F}$ is either $\mathcal{O}^{\mathrm{w}}_{X_{sa}}$, $\mathcal{O}^{t}_{X_{sa}}$ or $\mathcal{O}_X$.

Definition 4.6. The sheaf of microfunctions along χ with holomorphic parameters is defined by

$$\mathcal{C}_{N_M,\chi} := \mu_\chi(\mathcal{O}_X) \underset{\mathbb{Z}_{S_\chi^*}}{\otimes} or_{S_\chi^*}[\mathrm{codim}_{\mathbb{R}} N_M], \tag{4.5}$$

where $or_{S^*_\chi}$ denotes the orientation sheaf of S^*_χ. We also define $\mathcal{C}^f_{N_M,\chi}$ and $\mathcal{C}^{\mathrm{w}}_{N_M,\chi}$ by replacing $\mathcal{O}_X$ in the above definition with $\mathcal{O}^t_X$ and $\mathcal{O}^{\mathrm{w}}_X$ respectively.

Note that these are really sheaves in S^*_χ, that is, they are concentrated in degree 0 everywhere.

§ 5. Applications to $\mathcal{D}$-modules

In this section, we consider applications of the multi-microlocalization to $\mathcal{D}$-module theory. We refer to [2] for the proofs.

§ 5.1. Uchida's Triangle

First, recall the notations of § 2.1; for example, let $\tau_i \colon E_i \to Z$ $(1 \le i \le \ell)$ be a vector bundle over Z, and let E_i^* be the dual bundle of E_i.

Theorem 5.1 (cf. [12]). *Let F be a multi-conic object on E. Then there exists the natural isomorphism*

$$\tau^! R\tau_! F \simeq Rp_{1*}\, p_2^!(F^{\wedge_E}),$$

and the natural morphism $F \to \tau^! R\tau_! F$ is embedded to the following distinguished triangle:

$$F \to \tau^! R\tau_! F \to Rp^+_{1*} p_2^{+!}(F^{\wedge_E}) \xrightarrow{+1} .$$

Therefore, we obtain the following:

Theorem 5.2. *Let X be a real analytic manifold, and assume that the family $\chi = \{M_i\}_{i=1}^{\ell}$ of submanifolds in X satisfies conditions* H1, H2 *and* H3. *Set $M := \bigcap_{i=1}^{\ell} M_i$. Then, for any $F \in D^b(k_{X_{\mathrm{sa}}})$, there exists the following distinguished triangle*:

(5.1) $$\nu_\chi(F) \to \tau^{-1} R\Gamma_M(F) \otimes \omega^{\otimes -1}_{M/X} \to Rp^+_{1*}(p^+_2)^{-1}\mu_\chi(F) \otimes \omega^{\otimes -1}_{M/X} \xrightarrow{+1} .$$

By Theorem 3.6, under the identifications $T^*S^*_\chi = T^*S_\chi = S_{\chi^*}$, we have

$$\mathrm{SS}(\mu_\chi(F)) = \mathrm{SS}(\nu_\chi(F)) \subset C_{\chi^*}(\mathrm{SS}(F)).$$

In particular we obtain $\operatorname{supp} \mu_\chi(F) \subset S^*_\chi \cap C_{\chi^*}(\mathrm{SS}(F))$. Thus we obtain:

Corollary 5.3. *If $\dot{S}^*_\chi \cap C_{\chi^*}(\mathrm{SS}(F)) = \emptyset$, then $\nu_\chi(F) \simeq \tau^{-1} R\Gamma_M(F) \otimes \omega^{\otimes -1}_{M/X}$.*

§ 5.2. Solutions of $\mathcal{D}$-modules

Let X be a complex manifold, and let $\chi = \{Y_i\}_{i=1}^{\ell}$ be a family of closed complex submanifolds of X which satisfies the conditions H1, H2 and H3. Set $Y := \bigcap_{i=1}^{\ell} Y_i$. As usual, let $\mathcal{D}_X$ be the sheaf of holomorphic linear differential operators on X. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module, and let $\mathrm{Ch}\,\mathcal{M}$ denote the characteristic variety of $\mathcal{M}$. Then, for $F = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$, it is known that $\mathrm{SS}(F) = \mathrm{Ch}\,\mathcal{M}$. From (5.1), we have

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{O}_X)) &\to \tau^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_Y(\mathcal{O}_X)) \otimes \omega_{Y/X}^{\otimes -1} \\ &\to Rp_{1*}^{+}(p_2^{+})^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{O}_X)) \otimes \omega_{Y/X}^{\otimes -1} \xrightarrow{+1} . \end{aligned}$$

Let $f\colon Y \hookrightarrow X$ be the canonical embedding. We define the *inverse image* of $\mathcal{M}$ by $Df^*\mathcal{M} := \mathcal{O}_Y \overset{L}{\underset{f^{-1}\mathcal{O}_X}{\otimes}} f^{-1}\mathcal{M}$.

Theorem 5.4. *Assume that Y is non-characteristic for $\mathcal{M}$. Then*

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{O}_X)) &\simeq \tau^{-1} R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{O}_Y) \\ &\simeq \tau^{-1} f^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X). \end{aligned}$$

Let M be a real analytic manifold, and let $\chi = \{N_i\}_{i=1}^{\ell}$ be a family of closed real analytic submanifolds of M which satisfies the conditions H1, H2 and H3. Set $N := \bigcap_{i=1}^{\ell} N_i$. We consider the multi-normal deformation $\widetilde{M}_\chi$ along χ. Let X be the complexification of M, and Y the complexification of N in X. Let $\iota\colon M \hookrightarrow X$ the canonical embedding. Let $\mathcal{B}_M$ be the sheaf of hyperfunctions on M. Then by (5.1) we obtain

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{B}_M)) &\to \tau^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, R\Gamma_N(\mathcal{B}_M)) \otimes \omega_{N/M}^{\otimes -1} \\ &\to Rp_{1*}^{+}(p_2^{+})^{-1} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mu_\chi(\mathcal{B}_M)) \otimes \omega_{N/M}^{\otimes -1} \xrightarrow{+1} . \end{aligned}$$

For any conic subset $A \subset T^*X$ we can define $\iota^{\#}(A) := T^*M \cap C_{T_M^*X}(A)$ ([4, Definition 6.2.3]). Note that $(x_0;\xi_0) \in \iota^{\#}(A)$ if and only if there exists a sequence $\{(x_\nu + \sqrt{-1}\,y_\nu; \xi_\nu + \sqrt{-1}\,\eta_\nu)\}_{\nu=1}^{\infty} \subset A$ such that

$$\lim_{\nu\to\infty} (x_\nu + \sqrt{-1}\,y_\nu; \xi_\nu) = (x_0;\xi_0), \quad \lim_{\nu\to\infty} |y_\nu|\,|\eta_\nu| = 0.$$

Theorem 5.5. *Assume that $N \hookrightarrow M$ is hyperbolic for $\mathcal{M}$; that is, $\dot{T}_N^*M \cap \iota^{\#}(\mathrm{Ch}\,\mathcal{M}) = \emptyset$. Then*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{B}_M)) \simeq \tau^{-1} R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{B}_N).$$

We impose the following conditions:

(1) $\varLambda \subset \dot{T}^*X$ is a $\mathbb{C}^\times$-conic closed regular involutory complex submanifold,

(2) $\mathcal{M}$ has regular singularities along $\varLambda$,

(3) $\dot{T}^*_N M \cap \iota^\#(\varLambda) = \emptyset$.

Let $\mathcal{D}b_M$ and $\mathcal{C}^\infty_M$ be the sheaves of distributions and $\mathcal{C}^\infty$-functions on M.

Theorem 5.6. *Assume Conditions (1), (2), (3). Then*

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{D}b_M)) \xrightarrow{\sim} \tau^{-1} R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{D}b_N).$$

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \nu_\chi(\mathcal{C}^\infty_M)) \xrightarrow{\sim} \tau^{-1} R\mathcal{H}om_{\mathcal{D}_Y}(Df^*\mathcal{M}, \mathcal{C}^\infty_N).$$

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