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Singularities in a compressible perfect fluid

By

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Abstract

Considering a compressible perfect fluid, we study how the singularities propagate in a complex domain. We explain the structure of the singularities, and the properties of the divergence and the rotation.

§ 1. Introduction

We consider a 3-D compressible perfect fluid in a complex domain. Let $x = (x_0, x') = (x'', x_3) = (x_0, x_1, x_2, x_3) \in \mathbf{C}^4$. We denote the velocity of the fluid by $u(x) = (u_1(x), u_2(x), u_3(x))$ and the density by $\rho(x)$. We assume that the pressure p is determined by the density ρ due to some physical law. Therefore we assume $p = p(\rho(x))$. Denoting $Lf(x) = \partial_{x_0} f + u_1\partial_{x_1} f + u_2\partial_{x_2} f + u_3\partial_{x_3} f$, we consider the following Cauchy problem for the Euler system:

$$
\begin{cases} L\rho = -\rho \operatorname{div} u, & \rho(0, x') = \rho^0(x'), \\ Lu = -\dfrac{p'(\rho)}{\rho}\operatorname{grad}\rho, & u(0, x') = u^0(x'), \end{cases} \tag{1.1}
$$

where $u^0(x') = (u_1^0(x'), u_2^0(x'), u_3^0(x'))$. Let $\Omega \subset \mathbf{C}^4$ be a small neighborhood of the origin, and let $\Omega^0 = \Omega \cap \{x_0 = 0\}$, $Z = \Omega^0 \cap \{x_3 = 0\}$. We assume that the initial values $\rho^0, u_1^0, u_2^0, u_3^0$ are holomorphic on the universal covering space $\mathcal{R}(\Omega^0 \setminus Z)$ of $\Omega^0 \setminus Z$, and that they have some regularity up to Z.

We define $\mathcal{O}(\mathcal{R}(\Omega^0 \setminus Z)) = \{$holomorphic functions on $\mathcal{R}(\Omega^0 \setminus Z)\}$, and

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$$\begin{aligned}\mathcal{O}^{j-q}(\mathcal{R}(\Omega^0\setminus Z)) = \{&f(x)\in\mathcal{O}(\mathcal{R}(\Omega^0\setminus Z));\\ &\partial_x^\alpha f(x) \text{ is bounded if } |\alpha|\le j-1,\\ &x_3^q\partial_x^\alpha f(x) \text{ is bounded if } |\alpha| = j\}\end{aligned}$$

for $j\in\mathbf{N}=\{1,2,3,\cdots\},\ 0<q<1$. We remark that if $f(x')\in\mathcal{O}^{1-q}(\mathcal{R}(\Omega^0\setminus Z))$, then we can naturally define the trace $f(x_1,x_2,0)$. We assume the following:

A1 We have $\rho^0,u_1^0,u_2^0,u_3^0\in\mathcal{O}^{3-q}(\mathcal{R}(\Omega^0\setminus Z))$.

A2 $p(\rho)$ is holomorphic at $\rho=\rho(0)$, and we have $\rho(0)p'(\rho(0))\neq 0$.

In [5], we reported the idea of the following theory: The singularities of the solution appear on three complex hypersurfaces $Z_{-1},Z_0,Z_1\subset\Omega$, which start from Z at $x_0=0$. The propagation velocity of Z_0 coincides with the flow velocity of the fluid, and those of $Z_{\pm1}$ coincide with (flow velocity) $\pm$ (sonic velocity). In the present article we report a more precise result, and in particular we explain the structure of the singularity set in a complex domain.

Remark. Note that [2, 3] studied the case of an incompressible perfect fluid, and proved that the singularities propagate at the flow velocity. In [4] we considered the case for irrotational perfect fluid (i.e. the case of rot $u=0$), and proved that the singularities propagate with the sonic velocity $\sqrt{p'(\rho)}$ forwards and backwards. J. Y. Chemin [1] considered a compressible perfect fluid in a real space, and proved that if the initial values are smooth outside of the origin, then the singularities of the solution propagate along $Y\cup Y'$. Here Y is the orbit of the fluid issuing from the origin, and Y' is a cone spreading around Y at the sonic velocity.

§ 2. Main result

Let $\nabla^j f(x)=(\partial^\alpha f(x);|\alpha|\le j)$. We define $v(x)=(v_0,v_1,v_2,v_3)=(\rho,u_1,u_2,u_3)$. We denote this vector by v and not by u, to avoid confusion with the usual notation $u(x)=(u_1,u_2,u_3)$. From (1.1) we have

$$\begin{cases}L(L^2-p'(v_0)\Delta)v=Q(\nabla^2 v)=(Q_0,Q_1,Q_2,Q_3),\\ \partial_{x_0}^i v(0,x')=v^i(x'), \qquad 0\le i\le 2.\end{cases}\tag{2.1}$$

Here we have $v^i(x')=(v_0^i,v_1^i,v_2^i,v_3^i)$, whose components are naturally determined and belong to $\mathcal{O}^{3-i-q}(\mathcal{R}(\Omega^0\setminus Z))$.

If we know the solution $v(x)$, then we can determine the characteristic functions $\kappa_j(x)$ for $j=-1,0,1$ by the following characteristic equations:

$$\begin{cases}L\kappa_j(x)=j\sqrt{p'(v_0)\{(\partial_{x_1}\kappa_j(x))^2+(\partial_{x_2}\kappa_j(x))^2+(\partial_{x_3}\kappa_j(x))^2\}},\\ \kappa_j(0,x')=x_3.\end{cases}\tag{2.2}$$

We define $Z_j = \{\kappa_j(x) = 0\}$.

Therefore the propagation velocity of Z_0 coincides with the flow velocity. To the contrary, $L^2 - p'(v_0)\Delta$ is the wave operator of the sound, and thus the propagation velocities of $Z_{\pm 1}$ coincide with the sonic velocity ($= \sqrt{p'(v_0)}$ relative to the flow movement).

We make the following approximation. Since the initial values have singularities along $\{x_3 = 0\}$, we can expect

$$\begin{aligned} &L &&\sim L' \stackrel{\text{def}}{=} \partial_{x_0} + u_3(0)\partial_{x_3}, \\ &p'(\rho)\Delta &&\sim a^2\partial_{x_3}^2 \qquad\qquad (a = \sqrt{p'(\rho(0))}). \end{aligned}$$

This means that we can expect

$$\begin{aligned} &L(L^2 - p'(v_0)\Delta)v = \Lambda + Q'(\nabla^3 v), \\ &\Lambda = \Lambda_1\Lambda_0\Lambda_{-1}, \\ &\Lambda_k = \partial_{x_0} + (u_3(0) - ka)\partial_{x_3}, \qquad -1 \le k \le 1. \end{aligned}$$

Here $Q'(\nabla^3 v)$ is a nonlinear function of $\nabla^3 v$, but it is a small perturbation in some sense.

Assume that $(\rho(x), u(x))$ is a solution to (1.1) and $\kappa_j(x)$ is a solution to (2.2). If we can expect $L(L^2 - p'(v_0)\Delta) \sim \Lambda$, then each characteristic Z_j should be close to that of Λ_j, (i.e. $\{x \in \mathbf{C}^4;\ x_3 - (u_3(0) - ja)x_0 = 0\}$). Therefore we can expect

$$Z_j = \{\kappa_j = 0\} \subset \bar{U}_j(R) = \{x \in \mathbf{C}^4;\ |x_3 - (u_3(0) - ja)x_0| \le R|x_0|\}.$$

We assume that $0 < R << 1$, and thus we have $\bar{U}_j(R) \cap \bar{U}_k(R) = \{x_0 = x_3 = 0\}$ for $j \ne k$.

Let $\Omega_1(R, r) = \{x \in \mathbf{C}^4;\ |x| < r\} \setminus \left(\bar{U}_{-1}(R) \bigcup \bar{U}_0(R) \bigcup \bar{U}_1(R)\right)$ and let $\pi : \mathcal{R}(\Omega_1(R, r)) \longrightarrow \Omega_1(R, r)$ be the canonical projection. Roughly speaking, we can show the following:

(1) We can solve the Cauchy problem in $\mathcal{R}(\Omega_1(R, r))$.

(2) For each $k \in \{-1, 0, 1\}$ and for each $\tilde{x} \in \mathcal{R}(\Omega_1(R, r))$ near $\pi^{-1}(\bar{U}_k(R))$, we determine the characteristic hypersurface $Z_k \subset \pi^{-1}(\bar{U}_k(R))$ near $\tilde{x}$, and solve the equation outside of Z_k.

Since this set Z_k depends on $\tilde{x}$, we may denote $Z_k = Z_k(\tilde{x})$. Let $\tilde{x}^1, \tilde{x}^2 \in \pi^{-1}(U_k(R))$ satisfy $\pi(\tilde{x}^1) = \pi(\tilde{x}^2) = x \in \Omega_1(R, r)$. Then it may happen that $Z_k(\tilde{x}^1) \ne Z_k(\tilde{x}^2)$.

Let us define

$$\Xi = \{x \in \mathbf{C}^4;\ x_3 - (u_1(0) - a\theta)x_0 = 0, \text{for some } \theta \in [-1, 1]\}.$$

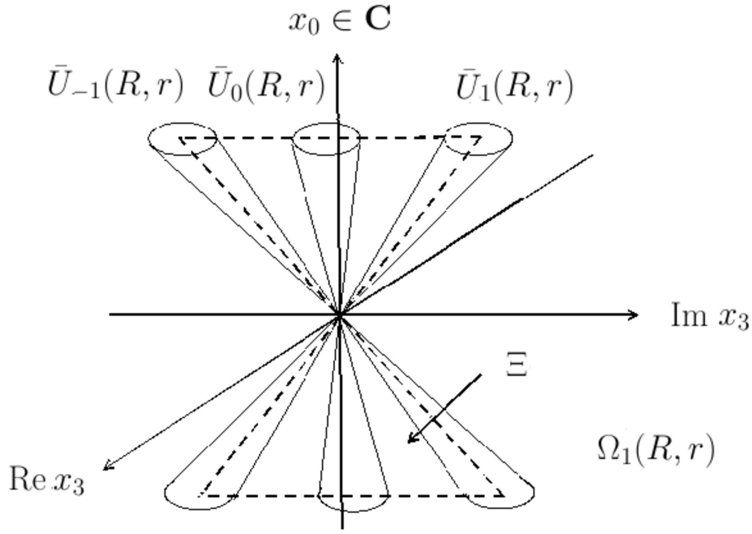

Figure 1. $\bar{U}_j(R,r)$ and Ξ

Let $\Gamma \subset \Omega_1(R,r)$ be a continuous curve starting at $x^0 = (0,0,0,r/2) \in \Omega^0 \setminus Z$ and ending at an arbitrary point $x \in \Omega_1(R,r)$. We denote the homotopy equivalence class of Γ in Ω_1 by $[\Gamma]$. By definition, $\mathcal{R}(\Omega_1(R,r))$ is the set of these homotopy equivalence classes, and the natural projection is defined by $\pi: \ \mathcal{R}(\Omega_1) \ni [\Gamma] \longmapsto x \in \Omega_1(R,r)$. We denote this point $[\Gamma] \in \mathcal{R}(\Omega_1(R,r))$ also by $\tilde{x}$ or simply by x, if confusion is not likely. For each $n \in \mathbf{N}$ we denote by $\mathcal{R}_n(\Omega_1(R,r))$ the set of the homotopy equivalence classes $[\Gamma]$ for which we can choose Γ in such a way that $\Gamma \cap \Xi$ consists of at most n points (These points should be counted as points in $\mathcal{R}(\Omega_1(R,r))$). Therefore we have $\mathcal{R}_0(\Omega_1(R,r)) \subset \mathcal{R}_1(\Omega_1(R,r)) \subset \mathcal{R}_2(\Omega_1(R,r)) \subset \mathcal{R}_n(\Omega_1(R,r)) \nearrow \mathcal{R}(\Omega_1(R,r))$.

Definition. Let $n \in \mathbf{N} = \{1,2,3,\cdots\}$. We say that a function $f(x)$ is of type n, if there exist some $R, r > 0$ such that $f(x)$ is holomorphic on $\mathcal{R}_n(\Omega_1(R,r))$.

We can prove that the solution is of type n for any n.

We denote $U_k(R,r) = \{x \in \mathbf{C}^4; \ |x| < r, |x_3 - (u_3(0) - ka)x_0| < R|x_0|\}$. We next consider how we can extend the solution inside of each $U_k(2R,r)$. Assume that $b(x'')$ is a holomorphic function on $\mathcal{R}_n(U_k(2R,r))$ satisfying $|b(x'')| < R|x_0|$. Since $b(x'')$ is independent of x_3, in fact it is holomorphic on the universal covering space of $\{x'' \in \mathbf{C}^3; \ |x''| < r, \ x_0 \neq 0\}$. We denote $Z(b_0) = \{\tilde{x} \in \mathcal{R}_n(U_k(2R,r)); \ x_3 = b_0(x'')\}$. Note that the projection of $Z(b_0)$ on $\mathbf{C}^4$ is contained in $U_k(R,r)$. Let $\tilde{x}^1 \in \mathcal{R}_n(\Omega_1(R,r)) \cap \pi^{-1}(U_k(2R,r))$. Note that each connected component of $\mathcal{R}(\Omega_1(R,r)) \cap \pi^{-1}(U_k(2R,r))$

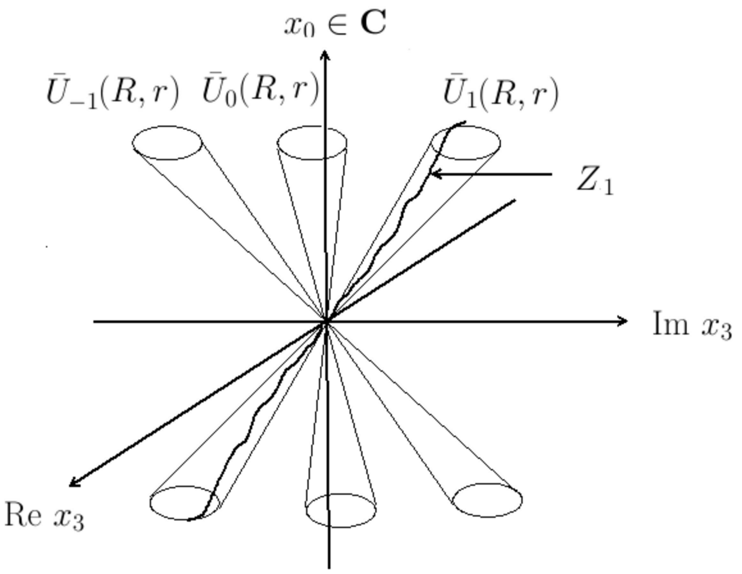

Figure 2. Singularity set

is homeomorphic to $\mathcal{R}(U'_k(R,r))$, where

$$\begin{aligned} U'_k(R,r) &= \Omega_1(R,r) \cap U_k(2R,r) \\ &= \{x \in \mathbf{C}^4;\ |x| < r,\ R|x_0| < |x_3 - (u_3(0) - ka)x_0| < 2R|x_0|\}. \end{aligned}$$

We denote the connected component of $\mathcal{R}(\Omega_1(R,r)) \cap \pi^{-1}(U_k(2R,r))$ containing $\tilde{x}^1$ by $\mathcal{R}(U'_k(R,r,\tilde{x}^1))$. Then we have

$$\mathcal{R}(U'_k(R,r,\tilde{x}^1)) \subset \mathcal{R}(\mathcal{R}(U_k(2R,r)) \setminus Z(b_0)).$$

We can identify

$$\begin{array}{ccc} \mathcal{R}(U_k(2R,r)) & \longrightarrow & U_k(2R,r) \times \mathbf{R} \\ \Psi & & \Psi \\ \tilde{x} & \longmapsto & (x, \arg x_0) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{R}(\mathcal{R}(U_k(2R,r,\tilde{x}^1)) \setminus Z(b_0)) & \longrightarrow & U_k(2R,r) \times \mathbf{R}^2 \\ \Psi & & \Psi \\ \tilde{x} & \longmapsto & (x, \arg x_0, \arg(x_3 - b_0))) \end{array}.$$

(See Figure 3).

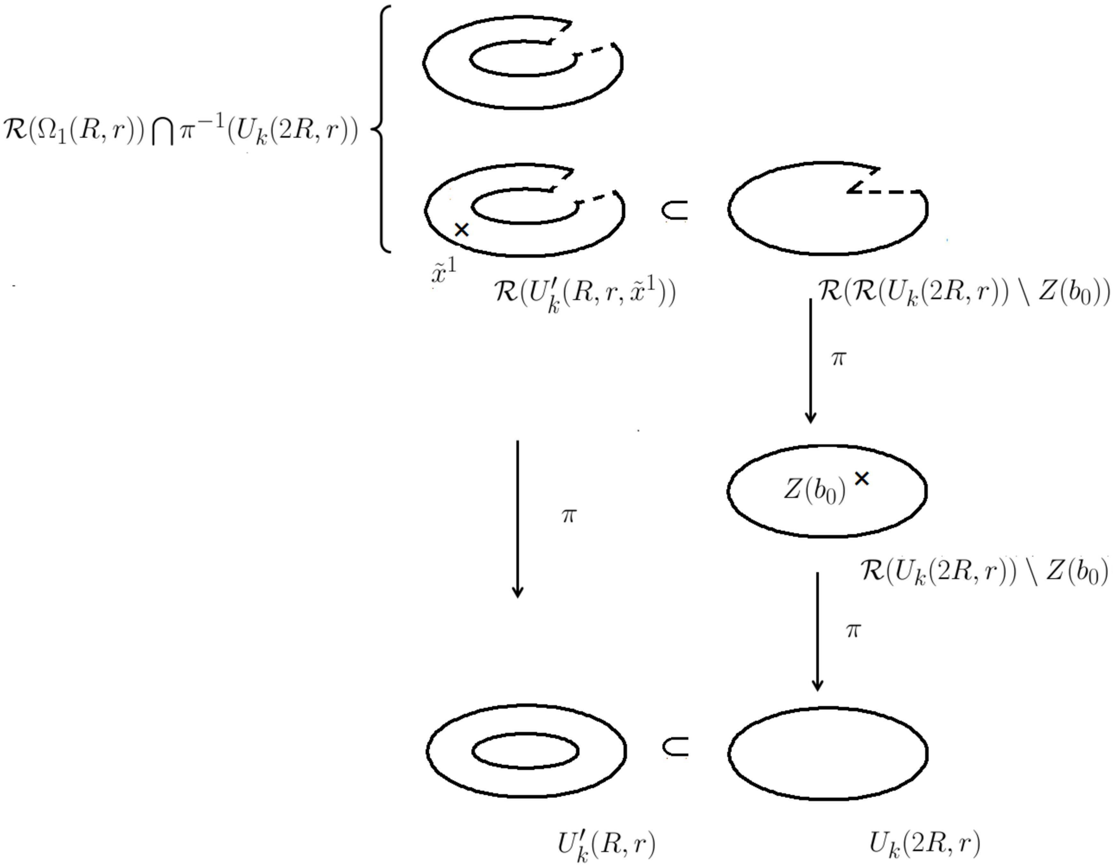

Figure 3. Structure of the universal covering space

If $n \in \mathbf{N}$, we define

$$
\begin{aligned}
&\mathcal{R}_n(U_k(R,r,\tilde{x}^1)\setminus Z(b_0)) \\
&= \{(x, \arg x_0, \arg(x_3 - b_0))) \in U_k(2R,r,\tilde{x}^1)\times \mathbf{R}^2; \\
&\qquad \Big|\arg\Big(\frac{x_0}{x_3 - b_0(x)}\Big) - \arg\Big(\frac{x_0^1}{x_1^1 - b_0(x^1)}\Big)\Big| < n\pi\}.
\end{aligned}
$$

Definition. Let $n \in \mathbf{N} = \{1,2,3,\cdots\}$, $k \in \{-1,0,1\}$. We assume that $f(x)$ is a function of type n. We say that a function $f(x)$ is of type (n,k), if there exist some $R,r>0$ and the following conditions are satisfied. For any $\tilde{x}^1 \in \mathcal{R}(\Omega_1(R,r))\cap \pi^{-1}(U_k(2R,r))$, there exists a holomorphic function $b(x'')$ on $\mathcal{R}_n(U_k(2R,r,\tilde{x}^1))$ satisfying $|b(x'')| < R|x_0|$. Furthermore, we can analytically continue the branch of $f(x)$ at $\tilde{x}^1$ to $\mathcal{R}_n(\mathcal{R}(U_k(2R,r))\setminus Z(b_0))$. We call $Z(b_0)$ the singularity set corresponding to $\tilde{x}^1$. Here the constants R,r may depend on n.

Remark. Roughly speaking, we want to say that there are three singularity sets Z_{-1}, Z_0, Z_1 outside of which the solution is holomorphic. However, the singularity set is defined in correspondence to $\tilde{x}^1 \in \mathcal{R}(\Omega_1(R,r))$, not on the base space. Therefore we first indicate a point $\tilde{x}^1 \in \mathcal{R}_n(\Omega_1(R,r)) \cap \pi^{-1}(U_k(2R,r))$, and clarify the singularity structure near this point.

Definition. Let $n \in \mathbf{N} = \{1,2,3,\cdots\}$, $k \in \{-1,0,1\}$, $0 < q < 1$, $m \in \mathbf{N}$. We assume that $f(x)$ is a function of type (n,k). We say that $f(x)$ is of type $(n,k,m-q)$ if it satisfies the following additional condition for the above constants R, r and an arbitrary point $\tilde{x}^1 \in \mathcal{R}_n(\Omega_1(R,r)) \cap \pi^{-1}(U_k(2R,r))$: We can extend the branch of $f(x)$ at $\tilde{x}^1$ to $\mathcal{R}(\mathcal{R}(U_k(2R,r,\tilde{x}^1)) \setminus Z(b_0))$, and we have

$$|\partial_x^\alpha f(x)| \leq C|x_3 - b(x'')|^{-(|\alpha|-m+q)_+} \tag{2.3}$$

for some $C >$ on $\mathcal{R}(\mathcal{R}(U_k(2R,r,\tilde{x}^1)) \setminus Z(b_0))$ if $|\alpha| \leq m$.

Remark. If $f(x)$ is of type $(n,k,1-q)$, we can define the trace of $f(x)|_{Z(b_0)}$ on the singularity set $Z(b_0)$ as before.

Theorem 2.1. *For each $n \in \mathbf{N}$ and $k \in \{-1,0,1\}$, there exists a holomorphic solution (ρ, u_1, u_2, u_3) of (1.1) and κ_k of (2.2) of type $(n,k,3-q)$. Furthermore, κ_k does not vanish at any point in its domain of definition, and $\kappa_k(x)|_{Z(b_0)} = 0$. In this sense way may write $Z_k = Z(b_0)$.*

We next study the regularity order of $\operatorname{rot} u$ and $\operatorname{div} u$. For the sake of simplicity, let us consider an example. Let $Z'_j = \{x \in \mathbf{C}^4;\ x_3 - (u_3(0) - ja)x_0 = 0\}$ for $-1 \leq j \leq 1$, and consider a function

$$f(x) = (x_3 - (u_3(0)+a)x_0)^{3-q} + \sum_{j=0,1} (x_3 - (u_3(0) - ja)x_0)^{2-q}.$$

This function has three singularity sets Z'_{-1}, Z'_0, Z'_1. It is $(2-q)$-Hölder continuous along Z'_0 and Z'_1, but is $(3-q)$-Hölder continuous along Z'_{-1}. However, we cannot directly say that $f(x)$ is of type $(n,k,3-q)$ even for $k=-1$. In fact, $\partial_{x_3}^2 f$ contains unbounded terms $(x_3 - u_3(0)x_0)^{-q}$ and $(x_3 - (u_3(0)-a)x_0)^{-q}$, although we are considering only the singularity along Z'_{-1}. Multiplying by an additional factor x_0, we can say that $x_0 f(x)$ is of type $(n,k,3-q)$ for $k=-1$. Note that this additional factor x_0 does not have an essential meaning for the singularity along Z'_{-1}. Using this expression, we have the following result.

Theorem 2.2. (i) If $k \neq 0$, then each component of $x_0 \operatorname{rot} u$ is of type $(n,k,3-q)$. (ii) Let $0 \leq j \leq 3$. If $k = 0$, then $x_0 \operatorname{div} u$ is of type $(n,k,3-q)$, and $x_0\rho$ is of type $(n,k,4-q)$.

This means the following fact. Since the solution u is of type $(n,k,3-q)$, $\operatorname{div} u$ and the components of $\operatorname{rot} u$ are trivially of type $(n,k,2-q)$. However, if $k \neq 0$, then the

Hölder exponent of components of $\operatorname{rot} u$ along Z_k increases by one, and thus they have weak singularities there. In other words, the rotation has a strong singularity along Z_0, and weak singularity along $Z_{\pm 1}$. Its singularity is likely to propagate at the flow velocity. By the way, it is known that in an incompressible perfect fluid, the singularities propagate only at the flow velocity (See [2, 3]). One may think that the rotation and the divergence are the sources of the singularities in a fluid, and in an incompressible fluid the rotation is the only source of singularities in this sense . Therefore our result is in accordance with that of [2, 3]. On the other hand, if $k = 0$, then the Hölder exponent of $\operatorname{div} u$ along Z_k increases by one, and thus they have weak singularities there. In other words, the divergence has a strong singularity along $Z_{\pm 1}$, and weak singularity along Z_0. Of course the density has a similar property. Its singularity is likely to propagate at the sonic velocity. By the way, it is known that in an irrotational perfect fluid, the singularities propagate only at the sonic velocity (See [4]), which is in accordance with the present result.

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