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Computing structures of holonomic D-modules associated with a simple line singularity

By

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§ 1. Introduction

The purpose of this paper is to describe a method to analyze the structure of holonomic D-modules associated with a hypersurface with non-isolated simple line singularities. We show in particular, by using two examples, the proposed method effectively determines the monodromy structure, on the singular locus of a given hypersurface, of the local system induced by the relevant holonomic D-module.

Let D_X be the sheaf of rings of holomorphic partial differential operators on a complex manifold X and $D_X[s]$ the sheaf of rings $D_X[s] = D_X \bigotimes_{\mathbb{C}} \mathbb{C}[s]$, where s is an indeterminate. Let f be a holomorphic function on X and let J_f be the Jacobian ideal generated by the partial derivatives of f.

In 1970's, the following three ideal were introduced in the theory of b-functions.

$$\mathrm{Ann}_{D_X[s]}f^s,\ \mathrm{Ann}_{D_X[s]}f^s + D_X[s]f,\ \mathrm{Ann}_{D_X[s]}f^s + D_X[s]J_f + D_X[s]f,$$

where $\mathrm{Ann}_{D_X[s]}f^s$ is the annihilator of f^s in $D_X[s]$. The annihilator $\mathrm{Ann}_{D_X[s]}f^s$ and the associated $D_X[s]$-module $D_X[s]/\mathrm{Ann}_{D_X[s]}f^s$ were introduced and investigated by M. Kashiwara to prove the existence of b-functions and the rationality of their roots. M. Kashiwara also showed in the same paper [3] that the b-function $b_f(s)$ of f can be defined as the minimal polynomial of the action of s on the $D_X[s]$-module $D_X[s]/(\mathrm{Ann}_{D_X[s]}f^s + D_X[s]f)$, defined by the second ideal presented above. The last ideal and the associated $D_X[s]$-module $D_X[s]/(\mathrm{Ann}_{D_X[s]}f^s + D_X[s]J_f + D_X[s]f)$ were effectively utilized in [16] by T. Yano to compute b-functions for many cases. In 1997,

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T. Oaku considered b-functions in the context of computational algebraic analysis and introduced an algorithm for computing annihilator $\mathrm{Ann}_{D_X[s]}f^s$. Furthermore, using the ideal $\mathrm{Ann}_{D_X[s]}f^s + D_X[s]f$, he succeeded to derive an algorithm for computing b-functions.

The authors of the present paper and T. Oaku [12] have recently examined, for $\beta \in \mathbb{C}$, holonomic D_X-modules

$$D_X[s]/(\mathrm{Ann}_{D_X[s]}f^s + D_X[s]J_f + D_X[s]f + D_X[s](s-\beta))$$

associated with a hypersurface with simple line singularities. They have investigated in particular the monodromy structure of the local systems, on the singular locus Σ of the hypersurface, induced by the relevant holonomic D_X-modules associated with 14 type simple line singularities given in [1].

In this paper, we focus our attention to two cases, the transverse A_2 singularity and the transverse E_6 singularity and illustrate the method to analyze the structure of holonomic D_X-modules in question. A key ingredient of the proposed method is the concept of local cohomology supported on the singular locus.

§ 2. Preparation

In this section we recall some basic results on holonomic D-modules and simple line singularities relevant to our study.

§ 2.1. Holonomic D-modules

Let X be a complex manifold of dimension n and D_X the sheaf of rings on X of holomorphic partial differential operators.

Definition 2.1 ([2]). Let M be a holonomic D_X-module on X. A stratification $X = \bigsqcup_\alpha S_\alpha$ of X is said to be regular with respect to M if it satisfies

(i) the Whitney conditions (a), (b)

(ii) the singular support of M is contained in $\bigcup T^*_{S_\alpha}X$, the union of the conormal bundle of strata.

Let Y be a complex submanifold of X of codimension d. Let $B_{Y|X}$ denote the left D_X-module of algebraic local cohomology $\mathcal{H}^d_{[Y]}(\mathcal{O}_X)$.

Theorem 2.2 ([2]). *Let M be a holonomic D_X-module whose support is contained in a submanifold Y and whose singular support is contained in T^*_YX. Then M is locally isomorphic to the direct sum of finite copies of $B_{Y|X}$.*

Let f be a holomorphic function on X and let J_f be the Jacobian ideal of f. Let $b_f(s)$ be the b-function of f and let $\tilde{b}_f(s)$ denote the reduced b-function of f defined to be $\tilde{b}_f(s) = b_f(s)/(s+1)$.

Theorem 2.3 ([16]). *Let S be the hypersurface $\{x \in X \mid f(x) = 0\}$ and let Σ denote the singular locus of the hypersurface S. Then, for a root β of the reduced b-function of f, the D_X-module defined by*

$$D_X[s]/(\mathrm{Ann}_{D_X[s]} f^s + D_X[s]J_f + D_x[s]f + D_X(s-\beta))$$

is a holonomic D_X-module whose support is contained in the singular set Σ.

§ 2.2. Simple line singularities

Now let f be a defining function of a hypersurface S with simple line singularities introduced by T. de Jong ([1]). Then the singular locus Σ of S is a complex line. According to [1], the singular locus Σ is stratified by two strata $\Sigma_1 = \Sigma - \{O\}$ and $\Sigma_0 = \{O\}$. Classical results, recalled in the preceding subsection, on holonomic D_X-modules and on the theory of b-functions yield that the support of the holonomic D_X-module

$$M_\beta = D_X[s]/(\mathrm{Ann}_{D_X[s]} f^s + D_X[s]J_f + D_x[s]f + D_X(s-\beta)),$$

for a root β of the reduced b-function of f, is contained in Σ. If the stratification $\Sigma = \Sigma_1 \sqcup \Sigma_0$ is regular with respect to M_β, then, for a point $Q \in \Sigma_1$, the holonomic D_X-module M_β is *locally* isomorphic to the direct sum of the finite copies of $B_{\Sigma_1|X}$.

In order to understand the monodromy structure of the local system on Σ_1 induced by the holonomic D_X-module M_β, it is natural to consider the *multivaluedness* of algebraic local cohomology solutions of the system M_β on the stratum Σ_1.

§ 3. A_{2-1} type

Let us consider A_{2-1} type simple line singularity defined by $f(x, y, z) = xy^3 + z^2$. Let $S = \{(x, y, z) \in X \mid f(x, y, z) = 0\}$, where X is a neighborhood of the origin O in $\mathbb{C}^3$. Note that f is a weighted homogeneous polynomial. Define the weight vector w_f of f by $\frac{1}{4}(1, 1, 2)$ so that the weighted degree of f is equal to 1. Since the Jacobian ideal J_f generated by the partial derivatives $\partial_x f = \partial f/\partial x$, $\partial_y f = \partial f/\partial y$, $\partial_z f = \partial f/\partial z$ is $(y^3, 3xy^2, 2z)$, the singular locus of S is $\Sigma = \{(x, y, z) \mid y = z = 0\}$, a complex line. The singular locus Σ is stratified by two strata $\Sigma = \Sigma_0 \sqcup \Sigma_1$, where $\Sigma_1 = \Sigma \setminus \{O\}$ and $\Sigma_0 = \{O\}$. Note that since $\Sigma_1 \cong \mathbb{C} - \{O\}$, the fundamental group of the stratum Σ_1 is non-trivial.

By executing an algorithm derived in [11] by T. Oaku, we get the following partial differential operators as a set of generators of the annihilator $\mathrm{Ann}_{D_X[s]} f^s$:

$$\begin{cases} P_1 = 2z\partial_y - 3xy^2\partial_z, & P_2 = 2z\partial_x - y^3\partial_z, \\ P_3 = -3x\partial_x + y\partial_y, & P_4 = 2y\partial_y + 3z\partial_z - 6s \end{cases}$$

with $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$ and $\partial_z = \partial/\partial z$. Let E denote the Euler operator

$$E = \frac{1}{4}x\partial_x + \frac{1}{4}y\partial_y + \frac{1}{2}z\partial_z.$$

Then $2P_4 - P_3 = 12(E - s)$ holds. Since $P_i \in D_X[s]J_f \cap \mathrm{Ann}_{D_X[s]} f^s$ $(i = 1, 2)$, we have

$$I = \mathrm{Ann}_{D_X[s]} f^s + D_X[s]J_f + D_X[s]f = D_X[s]\left(P_3,\, E - s,\, \partial_x f,\, \partial_y f,\, \partial_z f\right). \tag{3.1}$$

As previously described in §2, the holonomic D-module M_β defines a local system on Σ_1 for a root β of the reduced b-function $\tilde{b}_f(s)$. In §3.1, the monodromy structure for the local system is decided by the algebraic local cohomology solutions annihilated by J_f, P_3 and $E - s$. In §3.2, we calculate algebraic local cohomology solutions supported on Σ_0.

§ 3.1. Algebraic local cohomology solutions supported on Σ_1

Let $\mathcal{H}^2_{[\Sigma_1]}(\mathcal{O}_X)$ be the sheaf of algebraic local cohomology supported on Σ_1, where $\mathcal{O}_X$ is the sheaf on X of holomorphic functions. Set

$$H_{\Sigma_1} = \left\{ \psi \in \mathcal{H}^2_{[\Sigma_1]}(\mathcal{O}_X) \;\middle|\; J_f\psi = 0 \right\}.$$

Then any germ at a point $Q \in \Sigma_1$ of the sheaf H_{Σ_1} can be represented as a linear combination

$$h_1(x)\begin{bmatrix} 1 \\ yz \end{bmatrix} + h_2(x)\begin{bmatrix} 1 \\ y^2z \end{bmatrix},$$

where $[\ ]$ denotes the Grothendieck symbol and $h_1(x), h_2(x)$ are germs at Q of holomorphic functions on Σ_1. Taking the representation of a local section of H_{Σ_1} into account, we explicitly compute algebraic local cohomology classes ψ that satisfy $J_f\psi = P_3\psi = (E - s)\psi = 0$ as follows.

(i) Put $\psi_1 = h_1(x)\begin{bmatrix} 1 \\ yz \end{bmatrix}$. Then, P_3 acts on ψ_1 as $P_3\psi_1 = -(3xh_1' + h_1)\begin{bmatrix} 1 \\ yz \end{bmatrix}$ with $h_1' = \dfrac{dh_1}{dx}$. By $3xh_1' + h_1 = 0$, $h_1(x)$ is decided as const $x^{-\frac{1}{3}}$. It follows from $(E - s)\psi_1 = -(\frac{1}{12} + \frac{1}{4} + \frac{1}{2} + s)\psi_1$ that $s = -\frac{5}{6}$. Thus, we have

$$\psi_1 = \text{const}\ x^{-\frac{1}{3}}\begin{bmatrix} 1 \\ yz \end{bmatrix}, \qquad s = -\frac{5}{6}.$$

(ii) Put $\psi_2 = h_2(x) \begin{bmatrix} 1 \\ y^2 z \end{bmatrix}$. In the same manner as (i), we have

$$\psi_2 = \text{const } x^{-\frac{2}{3}} \begin{bmatrix} 1 \\ y^2 z \end{bmatrix}, \qquad s = -\frac{7}{6}.$$

Note that the monodromy structure on Σ_1 is shown by the multi-valuedness of ψ_1 and ψ_2 with respect to the variable x ([15]).

Remark 3.1. Let $r_x(y, z) = f(x, y, z) = xy^3 + z^2$ for $x \neq 0$. Here y and z are variables and $x(\neq 0)$ corresponding to a point $(x, 0, 0) \in \Sigma_1$ is regarded as parameter. Then, r_x is a weighted homogeneous polynomial in y, z with respect to the weight vector $w_{r_x} = \frac{1}{6}(2, 3)$. The weighted degree of the constructed solution with respect to w_f and also w_{r_x} is equal to the value of s respectively, namely, the exponent $\lambda = -\frac{1}{3}$ of ψ_1 satisfies

$$\frac{1}{4} \times \lambda + \frac{1}{4} \times (-1) + \frac{2}{4} \times (-1) = \frac{2}{6} \times (-1) + \frac{3}{6} \times (-1) = -\frac{5}{6}$$

and the exponent $\lambda = -\frac{2}{3}$ of ψ_2 satisfies

$$\frac{1}{4} \times \lambda + \frac{1}{4} \times (-2) + \frac{2}{4} \times (-1) = \frac{2}{6} \times (-2) + \frac{3}{6} \times (-1) = -\frac{7}{6}.$$

Remark 3.2. Since the factor $s + 1$ of the b-function of r_x given by

$$b_{r_x}(s) = (s + 1)(6s + 5)(6s + 7)$$

comes from the non-singular part, the reduced local b-function on the stratumΣ_1 of f is equal to $(6s + 5)(6s + 7)$. Therefore, what we have computed implies in particular that the weighted degree of solutions are compatible with the roots of local b-function on the stratum Σ_1 of the defining function f.

§ 3.2. Algebraic local cohomology solutions supported on Σ_0

Any algebraic local cohomology class in $\mathcal{H}^3_{[O]}(\mathcal{O}_X)$ supported at the origin can be written in the form

$$\sum a_{i,j,k} \begin{bmatrix} 1 \\ x^i y^j z^k \end{bmatrix}.$$

Let us construct a solution φ in $\mathcal{H}^3_{[O]}(\mathcal{O}_X)$ satisfying $J_f \varphi = P_3 \varphi = (E - s)\varphi = 0$. Recall $J_f = (\partial_x f, \partial_y f, \partial_z f) = (y^3, 3xy^2, 2z)$. Any solution φ in $\mathcal{H}^3_{[O]}(\mathcal{O}_X)$ for $z\varphi = 0$ is expressed in the form

$$\varphi = \sum a_{i,j,1} \begin{bmatrix} 1 \\ x^i y^j z \end{bmatrix}. \tag{3.2}$$

By putting (3.2) into $y^3\varphi = 0$ and $xy^2\varphi = 0$, we see
(3.3)
$$\left\{\varphi \in \mathcal{H}^3_{[O]}(\mathcal{O}_X) \mid J_f\varphi = 0\right\} = \mathrm{Span}\left\{\begin{bmatrix} 1 \\ x^iyz \end{bmatrix} (i \geq 1),\ \begin{bmatrix} 1 \\ x^iy^2z \end{bmatrix} (i \geq 1),\ \begin{bmatrix} 1 \\ xy^3z \end{bmatrix}\right\}.$$

Since $P_3 \begin{bmatrix} 1 \\ x^iy^jz \end{bmatrix} = (3i-j)\begin{bmatrix} 1 \\ x^iy^jz \end{bmatrix}$, the relation $j = 3i$ holds for $\begin{bmatrix} 1 \\ x^iy^jz \end{bmatrix}$. Therefore $\varphi = \mathrm{const}\begin{bmatrix} 1 \\ xy^3z \end{bmatrix}$. Finally, s is decided by $(E-s)\varphi = -\left(\frac{1}{4}+\frac{3}{4}+\frac{2}{4}+s\right)\varphi$.

Summing up, the algebraic local cohomology solution supported on Σ_0 is the following.
$$\varphi = \mathrm{const}\begin{bmatrix} 1 \\ xy^3z \end{bmatrix}, \qquad s = -\frac{3}{2}.$$
Notice that the weighted degree of φ is equal to the value of s.

As a consequence, we have

$$M_{-\frac{5}{6}} = D_X\left(x^{-\frac{1}{3}}\begin{bmatrix} 1 \\ yz \end{bmatrix}\right), \quad M_{-\frac{7}{6}} = D_X\left(x^{-\frac{2}{3}}\begin{bmatrix} 1 \\ y^2z \end{bmatrix}\right) \tag{3.4}$$

and
$$M_{-\frac{3}{2}} = D_X\left(\begin{bmatrix} 1 \\ xy^3z \end{bmatrix}\right).$$

Moreover, we see

$$\mathrm{Ch}(M_{-\frac{5}{6}}) = \mathrm{Ch}(M_{-\frac{7}{6}}) = T^*_{\Sigma_1}X \cup T^*_{\Sigma_0}X \quad \text{and} \quad \mathrm{Ch}(M_{-\frac{3}{2}}) = T^*_{\Sigma_0}X, \tag{3.5}$$

where $\mathrm{Ch}(M_\beta)$ denotes the characteristic variety of M_β.

Remark 3.3. Since the global b-function $b_f(s)$ of f is $b_f(s) = (s+1)(2s+3)(6s+5)(6s+7)$ and that of r_x is $b_{r_x}(s) = (s+1)(6s+5)(6s+7)$, $s = -\frac{3}{2}$ is a root of the local b-function on the stratum Σ_0 of f. Therefore, the result which says that holonomic system $M_{-\frac{3}{2}}$ corresponding to the root $-\frac{3}{2}$ is supported on the stratum Σ_0 is consistent with this fact.

Note that the b-function $b_f(s)$ presented above is computed by using an algorithm implemented by M. Noro ([10]).

In the rest of this section, we propose here an alternative method to compute algebraic local cohomology solutions, that utilizes the homogeneity of solutions. We start from the fact that $2s+3$ is a factor of the local b-function on the stratum Σ_0 of f. The combination of (i, j, k) satisfying $-\dfrac{1}{4}(i+j+2k) = -\dfrac{3}{2}$ is given by

$$(i,\, j,\, k) = (1,\, 1,\, 2),\, (1,\, 3,\, 1),\, (2,\, 2,\, 1),\, (3,\, 1,\, 1).$$

Since the index (i, j, k) of φ satisfying $J_f\varphi = P_i\varphi = 0$ $(1 \le i \le 4)$ is only $(1, 3, 1)$, we immediately get the desired solution φ.

§ 4. E_{6-3} type

In this section, we consider the case of E_{6-3} type simple line singularity. Let $f(x, y, z) = y^4 + xz^3 + y^2z^2$ and let $S = \{(x, y, z) \in X \mid f(x, y, z) = 0\}$, where X is a neighborhood of the origin O in $\mathbb{C}^3$. Define $w_f = \frac{1}{4}(1, 1, 1)$, then the weighted degree of f is equal to 1. As the Jacobian ideal $J_f = (\partial_x f, \partial_y f, \partial_z f)$ of f is $(z^3, 4y^3 + 2yz^2, 3xz^2 + 2y^2z)$, the singular locus of S is given by $\Sigma = \{(x, y, z) \mid y = z = 0\} \subset S$ and Σ is stratified by two strata $\Sigma = \Sigma_0 \sqcup \Sigma_1$, where $\Sigma_1 = \Sigma \setminus \{O\}$ and $\Sigma_0 = \{O\}$.

According to the Oaku's algorithm [11] implemented in a computer algebra system Risa/Asir ([9]), a set of generators of the annihilator $\mathrm{Ann}_{D_X[s]} f^s$ can be computed as follows :

$$\begin{cases} P_1 = 2y(2y^2+z^2)\partial_x - z^3\partial_y, \\ P_2 = -z(3xz+2y^2)\partial_y + 2y(2y^2+z^2)\partial_z, \\ P_3 = -(3xz+2y^2)\partial_x + z^2\partial_z, \\ P_4 = (9x^2+2y^2)\partial_x - yz\partial_y - (3xz-2y^2)\partial_z, \\ P_5 = -2y(3x-z)\partial_x - z^2\partial_y + 2yz\partial_z, \\ P_6 = x\partial_x + y\partial_y + z\partial_z - 4s. \end{cases}$$

Note that the operators P_i's are of weighted homogeneous and $P_6 = 4(E - s)$, where E is the Euler operator. It follows from $P_i \in D_X[s]J_f \cap \mathrm{Ann}_{D_X[s]} f^s$ $(i = 1, 2)$ that

$$\mathrm{Ann}_{D_X[s]} f^s + D_X[s]f + D_X[s]J_f = D_X[s]\,(P_3,\, P_4,\, P_5,\, P_6,\, \partial_x f,\, \partial_y f,\, \partial_z f)\,.$$

§ 4.1. Algebraic local cohomology solutions supported on Σ_1

Set $H_{\Sigma_1} = \left\{\psi \in \mathcal{H}^2_{[\Sigma_1]}(\mathcal{O}_X) \mid J_f\psi = 0\right\}$. Then it is easy to see, by using a method described in [6] if necessary, that for a point $Q \in \Sigma_1$, any germ at Q of the sheaf H_{Σ_1} is represented as a linear combination of the form $\sum_{i=1}^{6} h_i(x)\sigma_i$, where $h_i(x)$'s are germs at Q of holomorphic functions on Σ_1 and algebraic local cohomology classes σ_i's are defined by

$$
\begin{aligned}
&\sigma_1 = \begin{bmatrix} 1 \\ yz \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 1 \\ y^2z \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 \\ yz^2 \end{bmatrix}, \quad \sigma_4 = \begin{bmatrix} 1 \\ y^3z \end{bmatrix}, \\
&\sigma_5 = \begin{bmatrix} 1 \\ y^2z^2 \end{bmatrix}, \quad \sigma_6 = \begin{bmatrix} 1 \\ yz^3 \end{bmatrix} - \frac{3}{2}x \begin{bmatrix} 1 \\ y^3z^2 \end{bmatrix}.
\end{aligned} \tag{4.1}
$$

The following are bases of algebraic local cohomology solutions in question supported on Σ_1.

$$
\begin{aligned}
\psi_1 &= x^{-\frac{1}{3}} \begin{bmatrix} 1 \\ yz \end{bmatrix}, \quad s = -\frac{7}{12}, \\
\psi_2 &= x^{-\frac{1}{3}} \begin{bmatrix} 1 \\ y^2z \end{bmatrix}, \quad s = -\frac{5}{6}, \\
\psi_3 &= x^{-\frac{2}{3}} \begin{bmatrix} 1 \\ yz^2 \end{bmatrix} + \frac{1}{12}x^{-\frac{5}{3}} \begin{bmatrix} 1 \\ yz \end{bmatrix}, \quad s = -\frac{11}{12}, \\
\psi_4 &= x^{-\frac{1}{3}} \begin{bmatrix} 1 \\ y^3z \end{bmatrix} - \frac{1}{3}x^{-\frac{4}{3}} \begin{bmatrix} 1 \\ yz^2 \end{bmatrix} - \frac{1}{18}x^{-\frac{7}{3}} \begin{bmatrix} 1 \\ yz \end{bmatrix}, \quad s = -\frac{13}{12}, \\
\psi_5 &= x^{-\frac{2}{3}} \begin{bmatrix} 1 \\ y^2z^2 \end{bmatrix} + \frac{1}{6}x^{-\frac{5}{3}} \begin{bmatrix} 1 \\ y^2z \end{bmatrix}, \quad s = -\frac{7}{6}, \\
\psi_6 &= x^{-\frac{5}{3}} \left(\begin{bmatrix} 1 \\ yz^3 \end{bmatrix} - \frac{3}{2}x \begin{bmatrix} 1 \\ y^3z^2 \end{bmatrix} \right) - \frac{3}{8}x^{-\frac{5}{3}} \begin{bmatrix} 1 \\ y^3z \end{bmatrix} \\
&\quad + \frac{5}{24}x^{-\frac{8}{3}} \begin{bmatrix} 1 \\ yz^2 \end{bmatrix} + \frac{7}{144}x^{-\frac{11}{3}} \begin{bmatrix} 1 \\ yz \end{bmatrix}, \quad s = -\frac{17}{12}.
\end{aligned} \tag{4.2}
$$

Notably, the monodromy structure on the stratum Σ_1 encoded originally in the ideal $\mathrm{Ann}_{D_X[s]}f^s$ is revealed by computing local cohomology solutions of the relevant holonomic systems.

Remark 4.1. Let $r_x(y, z) = y^4 + xz^3 + y^2z^2$ denote the function of two variable y, z, where $x(\neq 0)$ is regarded as the parameter that corresponds to a point $(x, y, z) = (x, 0, 0) \in \Sigma_1$. Then, $r_x(y, z) = 0$ has an isolated singular point at $(y, z) = (0, 0)$ for each x with $x \neq 0$. The polynomial r_x is not of weighted homogeneous whereas one can easily see, by using a method described in [7, 14] for instance, that r_x is a quasi-homogeneous function. Hence, the b-function of r_x is given by

$$
b_{r_x}(s) = (s+1)(6s+5)(6s+7)(12s+7)(12s+11)(12s+13)(12s+17). \tag{4.3}
$$

Therefore, the reduced local b-function on the stratum Σ_1 of f is equal to

$$
(6s+5)(6s+7)(12s+7)(12s+11)(12s+13)(12s+17).
$$

Thus the results (4.2) presented above is also consistent with the local b-function.

We show the method to obtain the result (4.2) by giving the details of computation. Now set

$$\Lambda_{j,Q} = \left\{ \psi \;\middle|\; \psi = \sum_{k=1}^{j} h_k(x)\sigma_k,\ h_k \in \mathcal{O}_{\Sigma_1,Q} \right\}, \quad j = 1, 2, \ldots, 6. \tag{4.4}$$

Then $\Lambda_{1,Q} \subset \Lambda_{2,Q} \subset \cdots \subset \Lambda_{6,Q}$ holds. It is easy to verify by direct computation that each $\Lambda_{j,Q}$ is stable under the action of $P_i,\ i = 1, 2, ..., 6$, namely, $P_i\Lambda_{j,Q} \subset \Lambda_{j,Q}$ holds for any i and j. Our basic strategy is to find local cohomology solutions from each $\Lambda_{j,Q}$, $j = 1, 2, ..., 6$.

Local cohomology solutions ψ_1 and ψ_2 are easily decided as (i) and (ii) in the previous subsection. Let us compute ψ_i $(3 \le i \le 6)$.

(a) Put $\tau = h_3(x)\sigma_3$. We have

$$P_3\tau = -(3xh_3' + 2h_3)\sigma_1, \quad P_4\tau = 3x(3xh_3' + 2h_3)\sigma_3 + h_3\sigma_1, \quad P_5\tau = 0. \tag{4.5}$$

By $3xh_3' + 2h_3 = 0$, h_3 is $x^{-\frac{2}{3}}$. Then the weighted degree of τ is $-\frac{11}{12}$. We try to get ψ_3 by considering the homogeneity. Noticing $h_3\sigma_1$ in the right-hand side of equation for $P_4\tau$, we set $\psi_3 = x^{\frac{2}{3}}\sigma_3 + cx^{\lambda}\sigma_1$. Here c and λ are decided as follows. Since the weighted degree $\frac{1}{4}\lambda - \frac{2}{4}$ of $x^{\lambda}\sigma_1$ is equal to $-\frac{11}{12}$, we get $\lambda = -\frac{5}{3}$. Then $P_4\psi_3 = (1 - 12c)x^{-\frac{2}{3}}\sigma_1,\ P_i\psi_3 = 0\,(i \ne 4)$. Hence $c = \frac{1}{12}$ and $s = -\frac{11}{12}$ and ψ_3 in (4.2) is verified.

(b) Putting $\tau = h_4(x)\sigma_4$, we have

$$P_3\tau = -2h_4'\sigma_1,\ P_4\tau = 3x(3xh_4' + h_4)\sigma_4 - 2h_4\sigma_3 + 2h_4'\sigma_1,\ P_5\tau = -2(3xh_4' + h_4)\sigma_2. \tag{4.6}$$

By $3xh_4' + h_4 = 0$, we have $h_4 = x^{-\frac{1}{3}}$ and

$$P_3\tau = \frac{2}{3}x^{-\frac{4}{3}}\sigma_1, \quad P_4\tau = -2x^{-\frac{1}{3}}\sigma_3 - \frac{2}{3}x^{-\frac{4}{3}}\sigma_1, \quad P_5\tau = 0. \tag{4.7}$$

The weighted degree of τ is $-\frac{13}{12}$. For $\varrho_3 = x^{-\frac{4}{3}}\sigma_3$, a simple computation shows

$$P_3\varrho_3 = 2x^{-\frac{4}{3}}\sigma_1, \quad P_4\varrho_3 = -6x^{-\frac{1}{3}}\sigma_3 + x^{-\frac{4}{3}}\sigma_1, \quad P_5\varrho_3 = 0. \tag{4.8}$$

Comparing (4.7) and (4.8), we set $\psi_4 = \tau - \frac{1}{3}\varrho_3 + cx^{-\frac{7}{3}}\sigma_1$ which is equal to $x^{-\frac{1}{3}}\sigma_4 - \frac{1}{3}x^{-\frac{4}{3}}\sigma_3 + cx^{-\frac{7}{3}}\sigma_1$. Then we have $P_4\psi_4 = -(1+18c)x^{-\frac{4}{3}}\sigma_1,\ P_i\psi_4 = 0\,(i \ne 4)$. Hence $c = -\frac{1}{18}$ and $s = -\frac{13}{12}$. Therefore ψ_4 is checked.

(c) Put $\tau = h_5(x)\sigma_5$. Then τ satisfies

$$(4.9)\qquad \begin{cases} P_3\tau = -(3xh_5' + 2h_5)\sigma_2, \\ P_4\tau = 3x(3xh_5' + 2h_5)\sigma_5 + 2h_5\sigma_2, \\ P_5\tau = -2(3xh_5' + 2h_5)\sigma_3 + 2h_5'\sigma_1. \end{cases}$$

From the above, $h_5 = x^{-\frac{2}{3}}$. The weighted degree of τ is $-\frac{7}{6}$. For $\varrho_2 = x^{-\frac{5}{3}}\sigma_2$, we have

$$(4.10)\qquad P_3\varrho_2 = 0, \quad P_4\varrho_2 = -12x^{-\frac{2}{3}}\sigma_2, \quad P_5\varrho_2 = 8x^{-\frac{5}{3}}\sigma_1.$$

The form of ψ_5 follows from (4.9) and (4.10).

(d) Putting $\tau = h_6(x)\sigma_6$, we get

$$(4.11)\qquad \begin{cases} P_3\tau = \dfrac{3}{2}x(3xh_6' + 5h_6)\sigma_4, \\ P_4\tau = -\dfrac{9}{2}x(3xh_6' + 5h_6)\sigma_6 - \dfrac{9}{2}xh_6\sigma_4 - (3xh_6' + 2h_6)\sigma_3, \\ P_5\tau = 3x(3xh_6' + 5h_6)\sigma_5 - (3xh_6' + 2h_6)\sigma_2. \end{cases}$$

Let us decide h_6 so that $3xh_6' + 5h_6 = 0$ holds , i.e., $h_6 = x^{-\frac{5}{3}}$. Then we have

$$(4.12)\qquad P_3\tau = 0, \quad P_4\tau = -\frac{9}{2}x^{-\frac{2}{3}}\sigma_4 + 3x^{-\frac{5}{3}}\sigma_3, \quad P_5\tau = 3x^{-\frac{5}{3}}\sigma_2.$$

The weighted degree of τ is $-\frac{17}{12}$. For $\varrho_4 = x^{-\frac{5}{3}}\sigma_4$, we see

(4.13)
$$P_3\varrho_4 = \frac{10}{3}x^{-\frac{8}{3}}\sigma_1, \quad P_4\varrho_4 = -12x^{-\frac{2}{3}}\sigma_4 - 2x^{-\frac{5}{3}}\sigma_3 - \frac{10}{3}x^{-\frac{8}{3}}\sigma_1, \quad P_5\varrho_4 = 8x^{-\frac{5}{3}}\sigma_2.$$

Comparing (4.12) and (4.13), we put

$$\psi = \tau - \frac{3}{8}\varrho_4 = x^{-\frac{5}{3}}\sigma_6 - \frac{3}{8}x^{-\frac{5}{3}}\sigma_4.$$

Then we have

$$(4.14)\qquad P_3\psi = -\frac{5}{4}x^{-\frac{8}{3}}\sigma_1, \quad P_4\psi = \frac{15}{4}x^{-\frac{5}{3}}\sigma_3 + \frac{5}{4}x^{-\frac{8}{3}}\sigma_1, \quad P_5\psi = 0.$$

Next, for $\varrho_3 = x^{-\frac{8}{3}}\sigma_3$, we get

$$(4.15)\qquad P_3\varrho_3 = 6x^{-\frac{8}{3}}\sigma_1, \quad P_4\varrho_3 = -18x^{-\frac{5}{3}}\sigma_3 + x^{-\frac{8}{3}}\sigma_1, \quad P_5\varrho_3 = 0.$$

Comparing (4.14) and (4.15), we set

$$\psi_6 = x^{-\frac{5}{3}}\sigma_6 - \frac{3}{8}x^{-\frac{5}{3}}\sigma_4 + \frac{5}{24}x^{-\frac{8}{3}}\sigma_3 + cx^{-\frac{11}{3}}\sigma_1.$$

Then we have

$$P_4\psi_6 = \left(\frac{35}{24} - 30c\right) x^{-\frac{8}{3}}\sigma_1, \quad P_i\psi_6 = 0 \ (i \neq 4). \tag{4.16}$$

This implies $c = \dfrac{7}{144}$ and $s = -\dfrac{17}{12}$, which completes the computation of ψ_6.

Therefore the result (4.2) is obtained.

§ 4.2. Algebraic local cohomology solutions supported on Σ_0

Considering a form of algebraic local cohomology class supported on origin, we put

$$\varphi = \sum a_{i,\,j,\,k} \begin{bmatrix} 1 \\ x^i y^j z^k \end{bmatrix}.$$

Let us compute φ annihilated by J_f and P_i's. By $J_f\varphi = 0$, i.e., $z^3\varphi = (4y^3 + 2yz^2)\varphi = (3xz^2 + 2y^2z)\varphi = 0$, the form of φ is specified as follows.

$$\begin{aligned} \varphi = & \sum_{i\geq 1}\sum_{j=1}^{3} a_{i,\,j,\,1} \begin{bmatrix} 1 \\ x^i y^j z \end{bmatrix} + \sum_{i\geq 1}\sum_{j=1}^{3} a_{i,\,j,\,2} \begin{bmatrix} 1 \\ x^i y^j z^2 \end{bmatrix} \\ & + \sum_{i\geq 1} a_{i,\,1,\,3} \begin{bmatrix} 1 \\ x^i y z^3 \end{bmatrix} + a_{1,\,2,\,3} \begin{bmatrix} 1 \\ x y^2 z^3 \end{bmatrix} + a_{1,\,4,\,1} \begin{bmatrix} 1 \\ x y^4 z \end{bmatrix} \end{aligned} \tag{4.17}$$

with the conditions

$$3a_{i+1,1,3} + 2a_{i,3,2} = 0 \ (i \geq 1) \quad \text{and} \quad a_{1,2,3} + 2a_{1,4,1} = 0. \tag{4.18}$$

Next, we seek φ annihilated by P_i's. For φ of the form (4.17), we have

$$\begin{aligned} P_5\varphi = & \sum_{i\geq 1}\sum_{j=1}^{2} 2(3i-1)a_{i,\,j+1,\,1} \begin{bmatrix} 1 \\ x^i y^j z \end{bmatrix} - \sum_{i\geq 2}\sum_{j=1}^{2} 2(i-1)a_{i-1,\,j+1,\,2} \begin{bmatrix} 1 \\ x^i y^j z \end{bmatrix} \\ & + \sum_{i\geq 1}\sum_{j=1}^{2} 2(3i-2)a_{i,\,j+1,\,2} \begin{bmatrix} 1 \\ x^i y^j z^2 \end{bmatrix} + \sum_{i\geq 1} a_{i,\,1,\,3} \begin{bmatrix} 1 \\ x^i y^2 z \end{bmatrix} \\ & + 2(a_{1,\,2,\,3} + 2a_{1,\,4,\,1}) \begin{bmatrix} 1 \\ x y^3 z \end{bmatrix} - 2a_{1,\,2,\,3} \begin{bmatrix} 1 \\ x^2 y z^2 \end{bmatrix}. \end{aligned} \tag{4.19}$$

By (4.18) and the right-hand side of (4.19), φ satisfying $P_5\varphi = 0$ can be written in the form

$$\begin{aligned}\varphi = &\sum_{i\geq 1} a_{i,\,1,\,1} \begin{bmatrix} 1 \\ x^i yz \end{bmatrix} + \sum_{i\geq 1} a_{i,\,1,\,2} \begin{bmatrix} 1 \\ x^i yz^2 \end{bmatrix} \\ &+ a_{1,\,1,\,3} \begin{bmatrix} 1 \\ xyz^3 \end{bmatrix} + a_{1,\,2,\,3} \begin{bmatrix} 1 \\ xy^2z^3 \end{bmatrix} + a_{1,\,3,\,1} \begin{bmatrix} 1 \\ xy^3z \end{bmatrix} \\ &+ a_{1,\,4,\,1} \begin{bmatrix} 1 \\ xy^4z \end{bmatrix} + a_{2,\,2,\,2} \begin{bmatrix} 1 \\ x^2y^2z^2 \end{bmatrix} + a_{3,\,2,\,1} \begin{bmatrix} 1 \\ x^3y^2z \end{bmatrix},\end{aligned} \tag{4.20}$$

where

$$\begin{aligned} &a_{1,\,1,\,3} + 4a_{1,\,3,\,1} = 0, \quad a_{1,\,2,\,3} + 2a_{1,\,4,\,1} = 0, \\ &a_{1,\,2,\,3} - 4a_{2,\,2,\,2} = 0, \quad a_{2,\,2,\,2} - 4a_{3,\,2,\,1} = 0. \end{aligned} \tag{4.21}$$

Similarly, for φ of the form (4.20), we have

$$P_3\varphi = \sum_{i\geq 1} (3i-2) a_{i,\,1,\,2} \begin{bmatrix} 1 \\ x^i yz \end{bmatrix} + 2a_{1,\,3,\,1} \begin{bmatrix} 1 \\ x^2yz \end{bmatrix}. \tag{4.22}$$

Hence the following conditions are needed for $P_3\varphi = 0$.

$$a_{i,\,1,\,2} = 0 \ (i \neq 2) \quad \text{and} \quad a_{1,\,3,\,1} + 2a_{2,\,1,\,2} = 0. \tag{4.23}$$

Finally, we calculate

$$P_4\varphi = -\sum_{i\geq 1} 3(3i+2) a_{i+1,\,1,\,1} \begin{bmatrix} 1 \\ x^i yz \end{bmatrix} - (2a_{1,\,3,\,1} - a_{2,\,1,\,2}) \begin{bmatrix} 1 \\ x^2yz \end{bmatrix}. \tag{4.24}$$

Therefore the coefficients must satisfy

$$a_{i,\,1,\,1} = 0 \ (i \neq 1,\, 3) \quad \text{and} \quad 2a_{1,\,3,\,1} - a_{2,\,1,\,2} + 24a_{3,\,1,\,1} = 0. \tag{4.25}$$

To sum up, the form of φ is specified as

$$\begin{aligned}\varphi = a_{1,\,1,\,1} &\begin{bmatrix} 1 \\ xyz \end{bmatrix} + a_{1,\,1,\,3} \begin{bmatrix} 1 \\ xyz^3 \end{bmatrix} + a_{1,\,2,\,3} \begin{bmatrix} 1 \\ xy^2z^3 \end{bmatrix} \\ &+ a_{1,\,3,\,1} \begin{bmatrix} 1 \\ xy^3z \end{bmatrix} + a_{1,\,4,\,1} \begin{bmatrix} 1 \\ xy^4z \end{bmatrix} + a_{2,\,1,\,2} \begin{bmatrix} 1 \\ x^2yz^2 \end{bmatrix} \\ &+ a_{2,\,2,\,2} \begin{bmatrix} 1 \\ x^2y^2z^2 \end{bmatrix} + a_{3,\,1,\,1} \begin{bmatrix} 1 \\ x^3yz \end{bmatrix} + a_{3,\,2,\,1} \begin{bmatrix} 1 \\ x^3y^2z \end{bmatrix}\end{aligned}$$

with the conditions

$$a_{1,2,3} = -2a_{1,4,1}, \quad a_{2,2,2} = -\frac{1}{2}a_{1,4,1}, \quad a_{3,2,1} = -\frac{1}{8}a_{1,4,1},$$
$$a_{1,3,1} = -\frac{1}{4}a_{1,1,3}, \quad a_{2,1,2} = \frac{1}{8}a_{1,1,3}, \quad a_{3,1,1} = \frac{5}{192}a_{1,1,3}. \tag{4.26}$$

Then we have $P_1\varphi = P_2\varphi = 0$ and
(4.27)

$$P_6\varphi = -(4s+3)a_{1,1,1}\begin{bmatrix} 1 \\ xyz \end{bmatrix}$$
$$-(4s+5)\left(a_{1,1,3}\begin{bmatrix} 1 \\ xyz^3 \end{bmatrix} + a_{1,3,1}\begin{bmatrix} 1 \\ xy^3z \end{bmatrix} + a_{2,1,2}\begin{bmatrix} 1 \\ x^2yz^2 \end{bmatrix} + a_{3,1,1}\begin{bmatrix} 1 \\ x^3yz \end{bmatrix}\right)$$
$$-(4s+6)\left(a_{1,2,3}\begin{bmatrix} 1 \\ xy^2z^3 \end{bmatrix} + a_{1,4,1}\begin{bmatrix} 1 \\ xy^4z \end{bmatrix} + a_{2,2,2}\begin{bmatrix} 1 \\ x^2y^2z^2 \end{bmatrix} + a_{3,2,1}\begin{bmatrix} 1 \\ x^3y^2z \end{bmatrix}\right).$$

By (4.27), we have linearly independent three algebraic local cohomology solutions supported on Σ_0;

$$\varphi_1 = \begin{bmatrix} 1 \\ xyz \end{bmatrix}, \quad s = -\frac{3}{4},$$
$$\varphi_2 = \begin{bmatrix} 1 \\ xyz^3 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ xy^3z \end{bmatrix} + \frac{1}{8}\begin{bmatrix} 1 \\ x^2yz^2 \end{bmatrix} + \frac{5}{192}\begin{bmatrix} 1 \\ x^3yz \end{bmatrix}, \quad s = -\frac{5}{4},$$
$$\varphi_3 = -2\begin{bmatrix} 1 \\ xy^2z^3 \end{bmatrix} + \begin{bmatrix} 1 \\ xy^4z \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ x^2y^2z^2 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} 1 \\ x^3y^2z \end{bmatrix}, \quad s = -\frac{3}{2}.$$

As a consequence, we have

$$M_{-\frac{3}{4}} = D_X\varphi_1, \; M_{-\frac{5}{4}} = D_X\varphi_2, \; M_{-\frac{3}{2}} = D_X\varphi_3.$$

Note that all these three holonomic systems are simple as D-Module.

In 2010, K. Nishiyama and M. Noro ([8]) devised algorithms to compute local b-functions and stratifications associated with local b-functions. By using their algorithms implemented in Risa/Asir, we get $(4s+3)(4s+5)(2s+3)$ as a factor of the local b-function on the stratum Σ_0 in question of f. Thus, our results of computation are also consistent with the local b-function on the stratum Σ_0 of f

In the rest of this section, we compute algebraic local cohomology solutions φ_1, φ_2 and φ_3 by using alternative method, already mentioned in the preceding section, that utilizes the homogeneity.

Recall $w_f = \frac{1}{4}(1, 1, 1)$.

Case 1: Let us consider the case of $s = -\frac{3}{4}$. The weighted degree of $\left[\frac{1}{x^i y^j z^k}\right]$ is $-\frac{3}{4}$ only when $(i,\ j,\ k) = (1,\ 1,\ 1)$. From this, φ_1 is verified.

Case 2: We consider the case of $s = -\frac{5}{4}$. The combination of $(i,\ j,\ k)$ satisfying $-\frac{1}{4}(i+j+k) = -\frac{5}{4}$ is given by

$$(i,\ j,\ k) = (1,\ 1,\ 3),\ (1,\ 2,\ 2),\ (1,\ 3,\ 1),\ (2,\ 1,\ 2),\ (2,\ 2,\ 1),\ (3,\ 1,\ 1).$$

Note that the algebraic local cohomology classes associated with above are annihilated by J_f. Set

$$\varphi = a\left[\frac{1}{xyz^3}\right] + b\left[\frac{1}{xy^2z^2}\right] + c\left[\frac{1}{xy^3z}\right] + d\left[\frac{1}{x^2yz^2}\right] + e\left[\frac{1}{x^2y^2z}\right] + f\left[\frac{1}{x^3yz}\right]$$

with indeterminate coefficients $a,\ b,\ c,\ d,\ e,\ f$. A simple computation gives

(4.28)

$$\begin{aligned}
P_3\varphi &= b\left[\frac{1}{xy^2z}\right] + 2(c+2d)\left[\frac{1}{x^2yz}\right],\\
P_4\varphi &= (a-2c-12d)\left[\frac{1}{xyz^2}\right] + (2b-15e)\left[\frac{1}{xy^2z}\right] - (2c-d+24f)\left[\frac{1}{x^2yz}\right],\\
P_5\varphi &= 2b\left[\frac{1}{xyz^2}\right] + (a+4c)\left[\frac{1}{xy^2z}\right] - 2(b-5e)\left[\frac{1}{x^2yz}\right].
\end{aligned}$$

The relation of $a,\ b,\ c,\ d,\ e,\ f$ are determined by the above.

Case 3: We finally consider the case of $s = -\frac{3}{2}$. Since the combination of $(i,\ j,\ k)$ satisfying $-\frac{1}{4}(i+j+k) = -\frac{3}{2}$ is given by

$$\begin{gathered}
(1,\ 1,\ 4),\ (1,\ 2,\ 3),\ (1,\ 3,\ 2),\ (1,\ 4,\ 1),\ (2,\ 1,\ 3),\\
(2,\ 2,\ 2),\ (2,\ 3,\ 1),\ (3,\ 1,\ 2),\ (3,\ 2,\ 1),\ (4,\ 1,\ 1).
\end{gathered}$$

Let H_Φ be the vector space generated by the set of $\left[\frac{1}{x^i y^j z^k}\right]$ whose index $(i,\ j,\ k)$ is in the combination above. Then, if we set

$$H_{\Phi_{J_f}} = \{\varphi \in H_\Phi \mid J_f\varphi = 0\},$$

the following set is a basis of $H_{\Phi_{J_f}}$.

$$
\begin{aligned}
&\left[\frac{1}{xyz^4}\right],\ \left[\frac{1}{x^2y^2z^2}\right],\ \left[\frac{1}{x^2y^3z}\right],\ \left[\frac{1}{x^3yz^2}\right],\ \left[\frac{1}{x^3y^2z}\right],\ \left[\frac{1}{x^4yz}\right],\\
&-2\left[\frac{1}{xy^2z^3}\right]+\left[\frac{1}{xy^4z}\right],\quad \left[\frac{1}{xy^3z^2}\right]-\frac{2}{3}\left[\frac{1}{x^2yz^3}\right].
\end{aligned}
\tag{4.29}
$$

Therefore we set

(4.30)

$$
\begin{aligned}
\varphi = a\left[\frac{1}{xyz^4}\right] &+ b\left(-2\left[\frac{1}{xy^2z^3}\right]+\left[\frac{1}{xy^4z}\right]\right)+c\left(\left[\frac{1}{xy^3z^2}\right]-\frac{2}{3}\left[\frac{1}{x^2yz^3}\right]\right)\\
&+d\left[\frac{1}{x^2y^2z^2}\right]+e\left[\frac{1}{x^2y^3z}\right]+f\left[\frac{1}{x^3yz^2}\right]+g\left[\frac{1}{x^3y^2z}\right]+h\left[\frac{1}{x^4yz}\right].
\end{aligned}
$$

Here a, b, c, d, e, f, g, h are undetermined coefficients. A direct computation gives

$$
\begin{aligned}
P_3\varphi &= -a\left[\frac{1}{xyz^3}\right]+c\left[\frac{1}{xy^3z}\right]+2(b+2d)\left[\frac{1}{x^2y^2z}\right]+(4e+7f)\left[\frac{1}{x^3yz}\right],\\
P_4\varphi &= (a+2c)\left[\frac{1}{xyz^3}\right]-6(b+2d)\left[\frac{1}{xy^2z^2}\right]+3(c+e)\left[\frac{1}{xy^3z}\right]\\
&\quad -(\frac{8}{3}c+2e+21f)\left[\frac{1}{x^2yz^2}\right]-2(b-d+12g)\left[\frac{1}{x^2y^2z}\right]\\
&\quad -(4e-f+33h)\left[\frac{1}{x^3yz}\right],\\
P_5\varphi &= (a+2c)\left[\frac{1}{xy^2z^2}\right]+4(b+2d)\left[\frac{1}{x^2yz^2}\right]-(\frac{8}{3}c-10e)\left[\frac{1}{x^2y^2z}\right]\\
&\quad -4(d-4g)\left[\frac{1}{x^3yz}\right].
\end{aligned}
\tag{4.31}
$$

Therefore local cohomology solution φ_3 is determined by (4.31).

Remark 4.2. In the case of weighted homogeneous singularities, if we know, in advance, $\beta \in \mathbb{C}$ such that $\mathrm{Supp}(M_\beta)$ is Σ_0, we can efficiently calculate M_β by the above method.

References

[1] de Jong, T., Some classes of line singularities, *Math. Z.* **198** (1988), 493–517.

[2] Kashiwara, M., On the maximally overdetermined system of linear differential equations, I, *Publ. Res. Inst. Math. Sci.* **10** (1974-1975), 563–579.

[3] Kashiwara, M., B-functions and holonomic systems, *Invent. math.* **38** (1976), 33–53.

[4] Kashiwara, M., On the holonomic systems of linear differential equations, II, *Invent. math.* **49** (1978), 121–135.

[5] Kashiwara, M. and Kawai, T., On the characteristic variety of a holonomic system with regular singularities, *Advances in Math.* **34** (1979), 163–184.

[6] Nabeshima, K. and Tajima, S., On efficient algorithms for computing parametric local cohomology classes associated with semi-quasihomogeneous singularities and standard bases, *Proc. Internat. Symp. on Symbolic and Algebraic Computation ACM*, (2014), 351–358.

[7] Nabeshima, K. and Tajima, S., An algorithm for computing Tjurina stratifications of μ-constant deformations using algebraic local cohomology, *Lecture Notes in Computer Science* **8592** (2014), 523–530.

[8] Nishiyama, K. and Noro, M., Stratification associated with local b-functions, *J. Symbolic Computation* **45** (2010), 462–480.

[9] Noro, M. and Takeshima, T., Risa/Asir - a computer algebra system, *Proc. Internat. Symp. on Symbolic and Algebraic Computation*, P. S. Wang ed. ACM, New York (1992), 387–396.

[10] Noro, M., An efficient modular algorithm for computing the global b-function, in A.M. Cohen, X.S. Gao and N. Takayama (Eds), *Mathematical Software, Proceedings of the First International Congress of Mathematical Software, Beijing, World Scientific* (2002), 147–162.

[11] Oaku, T., Algorithms for the b-function and D-modules associated to a polynomial, *J. Pure and Applied Algebra* **117** & **118** (1997), 495–518.

[12] Oaku, T., Tajima, S. and Umeta, Y., in preparation.

[13] Siersma, D., Singularities with critical locus a 1-dimensional complete intersection and transversal type A_1, *Topology and its Appl.* **27** (1987), 51–73.

[14] Tajima, S., Parametric local cohomology classes and Tjurina stratifications for μ-constant deformations of quasi-homogeneous singularities, *in Several Topics on Real and Complex Singularities, World Scientific* (2014), 189–200.

[15] Tajima, S., On b-functions and algebraic local cohomology classes attached to hypersurfaces with line singularities, *RIMS Kôkyûroku Bessatsu* **52** (2014), 175–191.

[16] Yano. T., On the theory of b-functions, *Pub. Res. Inst. Math. Sci.* **14** (1978), 111–202.