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Computing structures of holonomic D-modules associated with a simple line singularity (Several aspects of microlocal analysis)

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Computing structures of holonomic D-modules associated with a simple line singularity

By

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§1. Introduction

The purpose of this paper is to describe a method to analyze the structure of holonomic D-modules associated with a hypersurface with non-isolated simple line singularities. We show in particular, by using two examples, the proposed method effectively determines the monodromy structure, on the singular locus of a given hypersurface, of the local system induced by the relevant holonomic D-module.

Let $D_X$ be the sheaf of rings of holomorphic partial differential operators on a complex manifold $X$ and $D_X[s]$ the sheaf of rings $D_X[s] = D_X \otimes_{\mathbb{C}} \mathbb{C}[s]$, where $s$ is an indeterminate. Let $f$ be a holomorphic function on $X$ and let $J_f$ be the Jacobian ideal generated by the partial derivatives of $f$.

In 1970's, the following three ideal were introduced in the theory of $b$-functions.

$$\text{Ann}_{D_X[s]} f^s, \text{Ann}_{D_X[s]} f^s + D_X[s]f, \text{Ann}_{D_X[s]} f^s + D_X[s]J_f + D_X[s]f,$$

where $\text{Ann}_{D_X[s]} f^s$ is the annihilator of $f^s$ in $D_X[s]$. The annihilator $\text{Ann}_{D_X[s]} f^s$ and the associated $D_X[s]$-module $D_X[s]/\text{Ann}_{D_X[s]} f^s$ were introduced and investigated by M. Kashiwara to prove the existence of $b$-functions and the rationality of their roots. M. Kashiwara also showed in the same paper [3] that the $b$-function $b_f(s)$ of $f$ can be defined as the minimal polynomial of the action of $s$ on the $D_X[s]$-module $D_X[s]/(\text{Ann}_{D_X[s]} f^s + D_X[s]f)$, defined by the second ideal presented above. The last ideal and the associated $D_X[s]$-module $D_X[s]/(\text{Ann}_{D_X[s]} f^s + D_X[s]J_f + D_X[s]f)$ were effectively utilized in [16] by T. Yano to compute $b$-functions for many cases. In 1997,
T. Oaku considered b-functions in the context of computational algebraic analysis and introduced an algorithm for computing annihilator $\text{Ann}_{D_X[s]} f^s$. Furthermore, using the ideal $\text{Ann}_{D_X[s]} f^s + D_X[s]f$, he succeeded to derive an algorithm for computing b-functions.

The authors of the present paper and T. Oaku [12] have recently examined, for $\beta \in \mathbb{C}$, holonomic $D_X$-modules

$$D_X[s]/(\text{Ann}_{D_X[s]} f^s + D_X[s]J_f + D_X[s]f + D_X[s](s - \beta))$$

associated with a hypersurface with simple line singularities. They have investigated in particular the monodromy structure of the local systems, on the singular locus $\Sigma$ of the hypersurface, induced by the relevant holonomic $D_X$-modules associated with 14 type simple line singularities given in [1].

In this paper, we focus our attention to two cases, the transverse $A_2$ singularity and the transverse $E_6$ singularity and illustrate the method to analyze the structure of holonomic $D_X$-modules in question. A key ingredient of the proposed method is the concept of local cohomology supported on the singular locus.

§ 2. Preparation

In this section we recall some basic results on holonomic D-modules and simple line singularities relevant to our study.

§ 2.1. Holonomic D-modules

Let $X$ be a complex manifold of dimension $n$ and $D_X$ the sheaf of rings on $X$ of holomorphic partial differential operators.

**Definition 2.1** ([2]). Let $M$ be a holonomic $D_X$-module on $X$. A stratification $X = \bigsqcup_{\alpha} S_\alpha$ of $X$ is said to be regular with respect to $M$ if it satisfies

(i) the Whitney conditions (a), (b)

(ii) the singular support of $M$ is contained in $\bigcup T^*_S X$, the union of the conormal bundle of strata.

Let $Y$ be a complex submanifold of $X$ of codimension $d$. Let $B_{Y|X}$ denote the left $D_X$-module of algebraic local cohomology $\mathcal{H}^d_{[Y]}(O_X)$.

**Theorem 2.2** ([2]). Let $M$ be a holonomic $D_X$-module whose support is contained in a submanifold $Y$ and whose singular support is contained in $T^*_Y X$. Then $M$ is locally isomorphic to the direct sum of finite copies of $B_{Y|X}$. 
Let $f$ be a holomorphic function on $X$ and let $J_f$ be the Jacobian ideal of $f$. Let $b_f(s)$ be the b-function of $f$ and let $\tilde{b}_f(s)$ denote the reduced b-function of $f$ defined to be $\tilde{b}_f(s) = b_f(s)/(s + 1)$.

**Theorem 2.3** ([16]). Let $S$ be the hypersurface $\{x \in X \mid f(x) = 0\}$ and let $\Sigma$ denote the singular locus of the hypersurface $S$. Then, for a root $\beta$ of the reduced b-function of $f$, the $D_X$-module defined by

$$D_X[s]/(\text{Ann}_{D_X[s]} f^s + D_X[s] J_f + D_x[s] f + D_X(s - \beta))$$

is a holonomic $D_X$-module whose support is contained in the singular set $\Sigma$.

§ 2.2. **Simple line singularities**

Now let $f$ be a defining function of a hypersurface $S$ with simple line singularities introduced by T. de Jong ([1]). Then the singular locus $\Sigma$ of $S$ is a complex line. According to [1], the singular locus $\Sigma$ is stratified by two strata $\Sigma_1 = \Sigma - \{O\}$ and $\Sigma_0 = \{O\}$. Classical results, recalled in the preceding subsection, on holonomic $D_X$-modules and on the theory of b-functions yield that the support of the holonomic $D_X$-module

$$M_\beta = D_X[s]/(\text{Ann}_{D_X[s]} f^s + D_X[s] J_f + D_x[s] f + D_X(s - \beta)),$$

for a root $\beta$ of the reduced b-function of $f$, is contained in $\Sigma$. If the stratification $\Sigma = \Sigma_1 \cup \Sigma_0$ is regular with respect to $M_\beta$, then, for a point $Q \in \Sigma_1$, the holonomic $D_X$-module $M_\beta$ is locally isomorphic to the direct sum of the finite copies of $B_{\Sigma_1|X}$.

In order to understand the monodromy structure of the local system on $\Sigma_1$ induced by the holonomic $D_X$-module $M_\beta$, it is natural to consider the *multivaluedness* of algebraic local cohomology solutions of the system $M_\beta$ on the stratum $\Sigma_1$.

§ 3. **$A_{2-1}$ type**

Let us consider $A_{2-1}$ type simple line singularity defined by $f(x, y, z) = xy^3 + z^2$. Let $S = \{(x, y, z) \in X \mid f(x, y, z) = 0\}$, where $X$ is a neighborhood of the origin $O$ in $\mathbb{C}^3$. Note that $f$ is a weighted homogeneous polynomial. Define the weight vector $w_f$ of $f$ by $\frac{1}{3}(1, 1, 2)$ so that the weighted degree of $f$ is equal to 1. Since the Jacobian ideal $J_f$ generated by the partial derivatives $\partial_x f = \partial f/\partial x$, $\partial_y f = \partial f/\partial y$, $\partial_z f = \partial f/\partial z$ is $(y^3, 3xy^2, 2z)$, the singular locus of $S$ is $\Sigma = \{(x, y, z) \mid y = z = 0\}$, a complex line. The singular locus $\Sigma$ is stratified by two strata $\Sigma = \Sigma_0 \cup \Sigma_1$, where $\Sigma_1 = \Sigma \setminus \{O\}$ and $\Sigma_0 = \{O\}$. Note that since $\Sigma_1 \cong \mathbb{C} - \{O\}$, the fundamental group of the stratum $\Sigma_1$ is non-trivial.
By executing an algorithm derived in [11] by T. Oaku, we get the following partial differential operators as a set of generators of the annihilator $\text{Ann}_{D_X[s]}f^s$:

$$\begin{align*}
P_1 &= 2z \partial_y - 3xy^2 \partial_z, & P_2 &= 2z \partial_x - y^3 \partial_z, \\
P_3 &= -3x \partial_x + y \partial_y, & P_4 &= 2y \partial_y + 3z \partial_z - 6s
\end{align*}$$

with $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$ and $\partial_z = \partial/\partial z$. Let $E$ denote the Euler operator

$$E = \frac{1}{4}x \partial_x + \frac{1}{4}y \partial_y + \frac{1}{2}z \partial_z.$$ 

Then $2P_4 - P_3 = 12(E - s)$ holds. Since $P_i \in D_X[s]J_f \cap \text{Ann}_{D_X[s]}f^s$ ($i = 1, 2$), we have

$$(3.1) \quad I = \text{Ann}_{D_X[s]}f^s + D_X[s]J_f + D_X[s]f = D_X[s] (P_3, E - s, \partial_x f, \partial_y f, \partial_z f).$$

As previously described in §2, the holonomic $D$-module $M_\beta$ defines a local system on $\Sigma_1$ for a root $\beta$ of the reduced $b$-function $\tilde{b}_f(s)$. In §3.1, the monodromy structure for the local system is decided by the algebraic local cohomology solutions annihilated by $J_f$, $P_3$ and $E - s$. In §3.2, we calculate algebraic local cohomology solutions supported on $\Sigma_0$.

### §3.1. Algebraic local cohomology solutions supported on $\Sigma_1$

Let $\mathcal{H}^2_{[\Sigma_1]}(\mathcal{O}_X)$ be the sheaf of algebraic local cohomology supported on $\Sigma_1$, where $\mathcal{O}_X$ is the sheaf on $X$ of holomorphic functions. Set

$$H_{\Sigma_1} = \left\{ \psi \in \mathcal{H}^2_{[\Sigma_1]}(\mathcal{O}_X) \mid J_f \psi = 0 \right\}.$$ 

Then any germ at a point $Q \in \Sigma_1$ of the sheaf $H_{\Sigma_1}$ can be represented as a linear combination

$$h_1(x) \begin{bmatrix} 1 \\ yz \end{bmatrix} + h_2(x) \begin{bmatrix} 1 \\ y^2z \end{bmatrix},$$

where $[\ ]$ denotes the Grothendieck symbol and $h_1(x), h_2(x)$ are germs at $Q$ of holomorphic functions on $\Sigma_1$. Taking the representation of a local section of $H_{\Sigma_1}$ into account, we explicitly compute algebraic local cohomology classes $\psi$ that satisfy $J_f \psi = P_3 \psi = (E - s) \psi = 0$ as follows.

(i) Put $\psi_1 = h_1(x) \begin{bmatrix} 1 \\ yz \end{bmatrix}$. Then, $P_3$ acts on $\psi_1$ as $P_3 \psi_1 = -(3xh'_1 + h_1) \begin{bmatrix} 1 \\ yz \end{bmatrix}$ with $h'_1 = \frac{dh_1}{dx}$. By $3xh'_1 + h_1 = 0$, $h_1(x)$ is decided as $\text{const } x^{-\frac{1}{3}}$. It follows from $(E - s)\psi_1 = -(\frac{1}{12} + \frac{1}{4} + \frac{1}{2} + s)\psi_1$ that $s = -\frac{5}{6}$. Thus, we have

$$\psi_1 = \text{const } x^{-\frac{1}{3}} \begin{bmatrix} 1 \\ yz \end{bmatrix}, \quad s = -\frac{5}{6}.$$
(ii) Put $\psi_2 = h_2(x) \left[ \frac{1}{y^2 z} \right]$. In the same manner as (i), we have

$$\psi_2 = \text{const } x^{-\frac{2}{3}} \left[ \frac{1}{y^2 z} \right], \quad s = -\frac{7}{6}.$$  

Note that the monodromy structure on $\Sigma_1$ is shown by the multi-valuedness of $\psi_1$ and $\psi_2$ with respect to the variable $x$ ([15]).

**Remark 3.1.** Let $r_x(y, z) = f(x, y, z) = xy^3 + z^2$ for $x \neq 0$. Here $y$ and $z$ are variables and $x(\neq 0)$ corresponding to a point $(x, 0, 0) \in \Sigma_1$ is regarded as parameter. Then, $r_x$ is a weighted homogeneous polynomial in $y, z$ with respect to the weight vector $w_{r_x} = \frac{1}{6}(2, 3)$. The weighted degree of the constructed solution with respect to $w_f$ and also $w_{r_x} = \frac{1}{6}(2, 3)$. The weighted degree of the constructed solution with respect to $w_f$ and also $w_{r_x}$ is equal to the value of $s$ respectively, namely, the exponent $\lambda = -\frac{1}{3}$ of $\psi_1$ satisfies

$$\frac{1}{4} \times \lambda + \frac{1}{4} \times (-1) + \frac{2}{4} \times (-1) = \frac{2}{6} \times (-1) + \frac{3}{6} \times (-1) = -\frac{5}{6}$$

and the exponent $\lambda = -\frac{2}{3}$ of $\psi_2$ satisfies

$$\frac{1}{4} \times \lambda + \frac{1}{4} \times (-2) + \frac{2}{4} \times (-1) = \frac{2}{6} \times (-2) + \frac{3}{6} \times (-1) = -\frac{7}{6}.$$  

**Remark 3.2.** Since the factor $s + 1$ of the b-function of $r_x$ given by

$$b_{r_x}(s) = (s + 1)(6s + 5)(6s + 7)$$

comes from the non-singular part, the reduced local b-function on the stratum $\Sigma_1$ of $f$ is equal to $(6s + 5)(6s + 7)$. Therefore, what we have computed implies in particular that the weighted degree of solutions are compatible with the roots of local b-function on the stratum $\Sigma_1$ of the defining function $f$.

§ 3.2. Algebraic local cohomology solutions supported on $\Sigma_0$

Any algebraic local cohomology class in $\mathcal{H}^3_{(O)}(\mathcal{O}_X)$ supported at the origin can be written in the form

$$\sum a_{i,j,k} \left[ \frac{1}{x^i y^j z^k} \right].$$

Let us construct a solution $\varphi$ in $\mathcal{H}^3_{(O)}(\mathcal{O}_X)$ satisfying $J_f \varphi = P_3 \varphi = (E - s) \varphi = 0$. Recall $J_f = (\partial_x f, \partial_y f, \partial_z f) = (y^3, 3xy^2, 2z)$. Any solution $\varphi$ in $\mathcal{H}^3_{(O)}(\mathcal{O}_X)$ for $z \varphi = 0$ is expressed in the form

$$\varphi = \sum a_{i,j,1} \left[ \frac{1}{x^i y^j z} \right].$$ (3.2)
By putting (3.2) into $y^3 \varphi = 0$ and $xy^2 \varphi = 0$, we see

(3.3)\[
\{ \varphi \in \mathcal{H}^3_{[O]}(\mathcal{O}_X) \mid J_f \varphi = 0 \} = \text{Span}\left\{ \left[ \frac{1}{x^i y z} \right] (i \geq 1), \left[ \frac{1}{x^i y^2 z} \right] (i \geq 1), \left[ \frac{1}{x y^3 z} \right] \right\}.
\]

Since $P_3 \left[ \frac{1}{x^i y^j z} \right] = (3i - j) \left[ \frac{1}{x^i y^j z} \right]$, the relation $j = 3i$ holds for $\left[ \frac{1}{x^i y^j z} \right]$. Therefore $\varphi = \text{const} \left[ \frac{1}{x y^3 z} \right]$. Finally, $s$ is decided by $(E - s) \varphi = -\left( \frac{1}{4} + \frac{3}{4} + \frac{2}{4} + s \right) \varphi$.

Summing up, the algebraic local cohomology solution supported on $\Sigma_0$ is the following.

$$\varphi = \text{const} \left[ \frac{1}{x y^3 z} \right], \quad s = -\frac{3}{2}.$$  

Notice that the weighted degree of $\varphi$ is equal to the value of $s$.

As a consequence, we have

(3.4)\[
M_{-\frac{5}{6}} = D_X \left( x^{-\frac{1}{3}} \left[ \frac{1}{y z} \right] \right), \quad M_{-\frac{7}{6}} = D_X \left( x^{-\frac{2}{3}} \left[ \frac{1}{y^2 z} \right] \right)
\]

and

$$M_{-\frac{3}{2}} = D_X \left( \left[ \frac{1}{x y^3 z} \right] \right).$$

Moreover, we see

(3.5)\[
\text{Ch}(M_{-\frac{5}{6}}) = \text{Ch}(M_{-\frac{7}{6}}) = T^{*}_{\Sigma_1} X \cup T^{*}_{\Sigma_0} X \quad \text{and} \quad \text{Ch}(M_{-\frac{3}{2}}) = T^{*}_{\Sigma_0} X,
\]

where $\text{Ch}(M_\beta)$ denotes the characteristic variety of $M_\beta$.

**Remark 3.3.** Since the global b-function $b_f(s)$ of $f$ is $b_f(s) = (s + 1)(2s + 3)(6s + 5)(6s + 7)$ and that of $r_x$ is $b_{r_x}(s) = (s + 1)(6s + 5)(6s + 7)$, $s = -\frac{3}{2}$ is a root of the local b-function on the stratum $\Sigma_0$ of $f$. Therefore, the result which says that holonomic system $M_{-\frac{3}{2}}$ corresponding to the root $-\frac{3}{2}$ is supported on the stratum $\Sigma_0$ is consistent with this fact.

Note that the b-function $b_f(s)$ presented above is computed by using an algorithm implemented by M. Noro ([10]).

In the rest of this section, we propose here an alternative method to compute algebraic local cohomology solutions, that utilizes the homogeneity of solutions. We
start from the fact that $2s + 3$ is a factor of the local b-function on the stratum $\Sigma_0$ of $f$. The combination of $(i, j, k)$ satisfying $-\frac{1}{4}(i + j + 2k) = -\frac{3}{2}$ is given by

$$(i, j, k) = (1, 1, 2), (1, 3, 1), (2, 2, 1), (3, 1, 1).$$

Since the index $(i, j, k)$ of $\varphi$ satisfying $J_f \varphi = P_i \varphi = 0$ ($1 \leq i \leq 4$) is only $(1, 3, 1)$, we immediately get the desired solution $\varphi$.

§ 4. $E_{6-3}$ type

In this section, we consider the case of $E_{6-3}$ type simple line singularity. Let $f(x, y, z) = y^4 + xz^3 + y^2z^2$ and let $S = \{(x, y, z) \in X \mid f(x, y, z) = 0\}$, where $X$ is a neighborhood of the origin $O$ in $\mathbb{C}^3$. Define $w_f = \left(\frac{1}{4}, 1, 1\right)$, then the weighted degree of $f$ is equal to 1. As the Jacobian ideal $J_f = (\partial_x f, \partial_y f, \partial_z f)$ of $f$ is $(z^3, 4y^3 + 2yz^2, 3xz^2 + 2y^2z)$, the singular locus of $S$ is given by $\Sigma = \{(x, y, z) \mid y = z = 0\} \subset S$ and $\Sigma$ is stratified by two strata $\Sigma = \Sigma_0 \sqcup \Sigma_1$, where $\Sigma_1 = \Sigma \setminus \{O\}$ and $\Sigma_0 = \{O\}$.

According to the Oaku’s algorithm [11] implemented in a computer algebra system Risa/Asir ([9]), a set of generators of the annihilator $\text{Ann}_{D_X[s]}f^s$ can be computed as follows:

$$\left\{ \begin{array}{l}
P_1 = 2y(2y^2 + z^2)\partial_x - z^3\partial_y, \\
P_2 = -z(3xz + 2y^2)\partial_y + 2y(2y^2 + z^2)\partial_z, \\
P_3 = -(3xz + 2y^2)\partial_x + z^2\partial_z, \\
P_4 = (9x^2 + 2y^2)\partial_x - yz\partial_y - (3xz - 2y^2)\partial_z, \\
P_5 = -2y(3x - z)\partial_x - z^2\partial_y + 2yz\partial_z, \\
P_6 = x\partial_x + y\partial_y + z\partial_z - 4s.
\end{array} \right.$$ 

Note that the operators $P_i$’s are of weighted homogeneous and $P_6 = 4(E - s)$, where $E$ is the Euler operator. It follows from $P_i \in D_X[s]J_f \cap \text{Ann}_{D_X[s]}f^s$ ($i = 1, 2$) that

$$\text{Ann}_{D_X[s]}f^s + D_X[s]f + D_X[s]J_f = D_X[s] (P_3, P_4, P_5, P_6, \partial_x f, \partial_y f, \partial_z f).$$

§ 4.1. Algebraic local cohomology solutions supported on $\Sigma_1$

Set $H_{\Sigma_1} = \left\{ \psi \in \mathcal{H}_{[\Sigma_1]}^0(\mathcal{O}_X) \mid J_f \psi = 0 \right\}$. Then it is easy to see, by using a method described in [6] if necessary, that for a point $Q \in \Sigma_1$, any germ at $Q$ of the sheaf $H_{\Sigma_1}$ is represented as a linear combination of the form $\sum_{i=1}^{6} h_i(x)\sigma_i$, where $h_i(x)$’s are germs at $Q$ of holomorphic functions on $\Sigma_1$ and algebraic local cohomology classes $\sigma_i$’s are
defined by

\[ \sigma_1 = \begin{bmatrix} 1 \\ yz \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 1 \\ y^2z \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 \\ yz^2 \end{bmatrix}, \quad \sigma_4 = \begin{bmatrix} 1 \\ y^3z \end{bmatrix}, \]

(4.1)

\[ \sigma_5 = \begin{bmatrix} 1 \\ y^2z^2 \end{bmatrix}, \quad \sigma_6 = \begin{bmatrix} 1 \\ yz^3 \end{bmatrix} - \frac{3}{2}x \begin{bmatrix} 1 \\ y^3z^2 \end{bmatrix}. \]

The following are bases of algebraic local cohomology solutions in question supported on \( \Sigma_1 \).

\[ \psi_1 = x^{-\frac{1}{3}} \begin{bmatrix} 1 \\ yz \end{bmatrix}, \quad s = -\frac{7}{12}, \]

\[ \psi_2 = x^{-\frac{1}{3}} \begin{bmatrix} 1 \\ y^2z \end{bmatrix}, \quad s = -\frac{5}{6}, \]

(4.2)

\[ \psi_3 = x^{-\frac{2}{3}} \begin{bmatrix} 1 \\ yz^2 \end{bmatrix} + \frac{1}{12}x^{-\frac{5}{3}} \begin{bmatrix} 1 \\ yz \end{bmatrix}, \quad s = -\frac{11}{12}, \]

\[ \psi_4 = x^{-\frac{1}{3}} \left( \begin{bmatrix} 1 \\ y^3z \end{bmatrix} - \frac{3}{2}x \begin{bmatrix} 1 \\ y^3z^2 \end{bmatrix} \right) - \frac{3}{8}x^{-\frac{5}{3}} \begin{bmatrix} 1 \\ y^3z \end{bmatrix} \]


\[ + \frac{5}{24}x^{-\frac{8}{3}} \begin{bmatrix} 1 \\ yz^2 \end{bmatrix} + \frac{7}{144}x^{-\frac{11}{3}} \begin{bmatrix} 1 \\ yz \end{bmatrix}, \quad s = -\frac{17}{12}. \]

Notably, the monodromy structure on the stratum \( \Sigma_1 \) encoded originally in the ideal \( \text{Ann}_{D_X[s]} f^s \) is revealed by computing local cohomology solutions of the relevant holonomic systems.

**Remark 4.1.** Let \( r_x(y, z) = y^4 + xz^3 + y^2z^2 \) denote the function of two variable \( y, z \), where \( x(\neq 0) \) is regarded as the parameter that corresponds to a point \( (x, y, z) = (x, 0, 0) \in \Sigma_1 \). Then, \( r_x(y, z) = 0 \) has an isolated singular point at \( (y, z) = (0, 0) \) for each \( x \) with \( x \neq 0 \). The polynomial \( r_x \) is not of weighted homogeneous whereas one can easily see, by using a method described in [7, 14] for instance, that \( r_x \) is a quasi-homogeneous function. Hence, the b-function of \( r_x \) is given by

(4.3) \[ b_{r_x}(s) = (s + 1)(6s + 5)(6s + 7)(12s + 7)(12s + 11)(12s + 13)(12s + 17). \]

Therefore, the reduced local b-function on the stratum \( \Sigma_1 \) of \( f \) is equal to

\[ (6s + 5)(6s + 7)(12s + 7)(12s + 11)(12s + 13)(12s + 17). \]
Thus the results (4.2) presented above is also consistent with the local b-function.

We show the method to obtain the result (4.2) by giving the details of computation. Now set

\[(4.4) \quad \Lambda_{j,Q} = \left\{ \psi \mid \psi = \sum_{k=1}^{j} h_k(x)\sigma_k, \ h_k \in \mathcal{O}_{\Sigma_1,Q} \right\}, \ j = 1, 2, \ldots, 6.\]

Then \(\Lambda_{1,Q} \subset \Lambda_{2,Q} \subset \cdots \subset \Lambda_{6,Q}\) holds. It is easy to verify by direct computation that each \(\Lambda_{j,Q}\) is stable under the action of \(P_i, \ i = 1, 2, \ldots, 6\), namely, \(P_i \Lambda_{j,Q} \subset \Lambda_{j,Q}\) holds for any \(i\) and \(j\). Our basic strategy is to find local cohomology solutions from each \(\Lambda_{j,Q}\), \(j = 1, 2, \ldots, 6\).

Local cohomology solutions \(\psi_1\) and \(\psi_2\) are easily decided as (i) and (ii) in the previous subsection. Let us compute \(\psi_i\ (3 \leq i \leq 6)\).

(a) Put \(\tau = h_3(x)\sigma_3\). We have

\[(4.5) \quad P_3 \tau = -(3xh_3' + 2h_3)\sigma_1, \quad P_4 \tau = 3x(3xh_3' + 2h_3)\sigma_3 + h_3\sigma_1, \quad P_5 \tau = 0.\]

By \(3xh_3' + 2h_3 = 0\), \(h_3\) is \(x^{-\frac{2}{3}}\). Then the weighted degree of \(\tau\) is \(-\frac{11}{12}\). We try to get \(\psi_3\) by considering the homogeneity. Noticing \(h_3\sigma_1\) in the right-hand side of equation for \(P_4 \tau\), we set \(\psi_3 = x^\frac{2}{3}\sigma_3 + cx^\lambda\sigma_1\). Here \(c\) and \(\lambda\) are decided as follows.

Since the weighted degree \(\frac{1}{4}\lambda - \frac{2}{4}\) of \(x^\lambda\sigma_1\) is equal to \(-\frac{11}{12}\), we get \(\lambda = -\frac{5}{3}\). Then \(P_4 \psi_3 = (1 - 12c)x^{-\frac{2}{3}}\sigma_1, \ P_i \psi_3 = 0 (i \neq 4)\). Hence \(c = \frac{1}{12}\) and \(s = -\frac{11}{12}\) and \(\psi_3\) in (4.2) is verified.

(b) Putting \(\tau = h_4(x)\sigma_4\), we have

\[(4.6) \quad P_3 \tau = -2h_4'\sigma_1, \quad P_4 \tau = 3x(3xh_4' + h_4)\sigma_4 - 2h_4\sigma_3 + 2h_4'\sigma_1, \quad P_5 \tau = -2(3xh_4' + h_4)\sigma_2.\]

By \(3xh_4' + h_4 = 0\), we have \(h_4 = x^{-\frac{1}{3}}\) and

\[(4.7) \quad P_3 \tau = x^{-\frac{4}{3}}\sigma_1, \quad P_4 \tau = -2x^{-\frac{1}{3}}\sigma_3 - x^{-\frac{4}{3}}\sigma_1, \quad P_5 \tau = 0.\]

The weighted degree of \(\tau\) is \(-\frac{13}{12}\). For \(\varrho_3 = x^{-\frac{4}{3}}\sigma_3\), a simple computation shows

\[(4.8) \quad P_3 \varrho_3 = 2x^{-\frac{4}{3}}\sigma_1, \quad P_4 \varrho_3 = -6x^{-\frac{4}{3}}\sigma_3 + x^{-\frac{4}{3}}\sigma_1, \quad P_5 \varrho_3 = 0.\]

Comparing (4.7) and (4.8), we set \(\psi_4 = \tau - \frac{1}{3}\varrho_3 + cx^{-\frac{7}{3}}\sigma_1\) which is equal to \(x^{-\frac{1}{3}}\sigma_4 - \frac{1}{3}x^{-\frac{4}{3}}\sigma_3 + cx^{-\frac{7}{3}}\sigma_1\). Then we have \(P_4 \psi_4 = -(1 + 18c)x^{-\frac{4}{3}}\sigma_1, \ P_i \psi_4 = 0 (i \neq 4)\). Hence \(c = -\frac{1}{18}\) and \(s = -\frac{13}{12}\). Therefore \(\psi_4\) is checked.
(c) Put $\tau = h_5(x)\sigma_5$. Then $\tau$ satisfies
\begin{equation}
\begin{aligned}
P_3\tau &= -(3xh'_5 + 2h_5)\sigma_2, \\
P_4\tau &= 3x(3xh'_5 + 2h_5)\sigma_5 + 2h_5\sigma_2, \\
P_5\tau &= -2(3xh'_5 + 2h_5)\sigma_3 + 2h'_5\sigma_1.
\end{aligned}
\tag{4.9}
\end{equation}
From the above, $h_5 = x^{-\frac{8}{3}}$. The weighted degree of $\tau$ is $-\frac{7}{6}$. For $\varrho_2 = x^{-\frac{8}{3}}\sigma_2$, we have
\begin{equation}
P_3\varrho_2 = 0, \quad P_4\varrho_2 = -12x^{-\frac{8}{3}}\sigma_2, \quad P_5\varrho_2 = 8x^{-\frac{8}{3}}\sigma_1.
\end{equation}
The form of $\psi_5$ follows from (4.9) and (4.10).

(d) Putting $\tau = h_6(x)\sigma_6$, we get
\begin{equation}
\begin{aligned}
P_3\tau &= \frac{3}{2}x(3xh'_6 + 5h_6)\sigma_4, \\
P_4\tau &= -\frac{9}{2}x(3xh'_6 + 5h_6)\sigma_4 - \frac{9}{2}xh_6\sigma_4 - (3xh'_6 + 2h_6)\sigma_3, \\
P_5\tau &= 3x(3xh'_6 + 5h_6)\sigma_5 - (3xh'_6 + 2h_6)\sigma_2.
\end{aligned}
\tag{4.11}
\end{equation}
Let us decide $h_6$ so that $3xh'_6 + 5h_6 = 0$ holds, i.e., $h_6 = x^{-\frac{8}{3}}$. Then we have
\begin{equation}
P_3\varrho_4 = 0, \quad P_4\varrho_4 = -\frac{9}{2}x^{-\frac{8}{3}}\sigma_4 + 3x^{-\frac{8}{3}}\sigma_3, \quad P_5\varrho_4 = 3x^{-\frac{8}{3}}\sigma_2.
\end{equation}
The weighted degree of $\tau$ is $-\frac{17}{12}$. For $\varrho_4 = x^{-\frac{8}{3}}\sigma_4$, we see
\begin{equation}
P_3\varrho_4 = \frac{10}{3}x^{-\frac{8}{3}}\sigma_1, \quad P_4\varrho_4 = -12x^{-\frac{8}{3}}\sigma_4 - 2x^{-\frac{8}{3}}\sigma_3 - \frac{10}{3}x^{-\frac{8}{3}}\sigma_1, \quad P_5\varrho_4 = 8x^{-\frac{8}{3}}\sigma_2.
\end{equation}
Comparing (4.12) and (4.13), we put
\begin{equation}
\psi = \tau - \frac{3}{8}\varrho_4 = x^{-\frac{8}{3}}\sigma_6 - \frac{3}{8}x^{-\frac{8}{3}}\sigma_4.
\end{equation}
Then we have
\begin{equation}
P_3\psi = -\frac{5}{4}x^{-\frac{8}{3}}\sigma_1, \quad P_4\psi = \frac{15}{4}x^{-\frac{8}{3}}\sigma_3 + \frac{5}{4}x^{-\frac{8}{3}}\sigma_1, \quad P_5\psi = 0.
\end{equation}
Next, for $\varrho_3 = x^{-\frac{8}{3}}\sigma_3$, we get
\begin{equation}
P_3\varrho_3 = 6x^{-\frac{8}{3}}\sigma_1, \quad P_4\varrho_3 = -18x^{-\frac{8}{3}}\sigma_3 + x^{-\frac{8}{3}}\sigma_1, \quad P_5\varrho_3 = 0.
\end{equation}
Comparing (4.14) and (4.15), we set
\begin{equation}
\psi_6 = x^{-\frac{8}{3}}\sigma_6 - \frac{3}{8}x^{-\frac{8}{3}}\sigma_4 + \frac{5}{24}x^{-\frac{8}{3}}\sigma_3 + cx^{-\frac{11}{3}}\sigma_1.
\end{equation}
Then we have

\[(4.16) \quad P_4 \psi_6 = \left( \frac{35}{24} - 30c \right) x^{-\frac{8}{3}} \sigma_1, \quad P_i \psi_6 = 0 \ (i \neq 4). \]

This implies \( c = \frac{7}{144} \) and \( s = -\frac{17}{12} \), which completes the computation of \( \psi_6 \).

Therefore the result (4.2) is obtained.

**§ 4.2. Algebraic local cohomology solutions supported on \( \Sigma_0 \)**

Considering a form of algebraic local cohomology class supported on origin, we put

\[ \varphi = \sum a_{i,j,k} \left[ \frac{1}{x^i y^j z^k} \right]. \]

Let us compute \( \varphi \) annihilated by \( J_f \) and \( P_i \)'s. By \( J_f \varphi = 0 \), i.e., \( z^3 \varphi = (4y^3 + 2yz^2) \varphi = (3xz^2 + 2y^2z) \varphi = 0 \), the form of \( \varphi \) is specified as follows.

\[ \varphi = \sum_{i \geq 1} \sum_{j=1}^{3} a_{i,j,1} \left[ \frac{1}{x^i y^j z} \right] + \sum_{i \geq 1} \sum_{j=1}^{3} a_{i,j,2} \left[ \frac{1}{x^i y^j z^2} \right] \]

\[ + \sum_{i \geq 1} a_{i,1,3} \left[ \frac{1}{x^i y z^3} \right] + a_{1,2,3} \left[ \frac{1}{xy^2 z^3} \right] + a_{1,4,1} \left[ \frac{1}{xy^4 z} \right] \]

with the conditions

\[ (4.18) \quad 3a_{i+1,1,3} + 2a_{i,3,2} = 0 \ (i \geq 1) \quad \text{and} \quad a_{1,2,3} + 2a_{1,4,1} = 0. \]

Next, we seek \( \varphi \) annihilated by \( P_i \)'s. For \( \varphi \) of the form (4.17), we have

\[ P_5 \varphi = \sum_{i \geq 1} \sum_{j=1}^{2} 2(3i - 1)a_{i,j+1,1} \left[ \frac{1}{x^i y^j z} \right] - \sum_{i \geq 2} \sum_{j=1}^{2} 2(i - 1)a_{i-1,j+1,2} \left[ \frac{1}{x^i y^j z} \right] \]

\[ + \sum_{i \geq 1} 2(3i - 2)a_{i,j+1,2} \left[ \frac{1}{x^i y^j z^2} \right] + \sum_{i \geq 1} a_{i,1,3} \left[ \frac{1}{x^i y^2 z} \right] \]

\[ + 2(a_{1,2,3} + 2a_{1,4,1}) \left[ \frac{1}{xy^3 z} \right] - 2a_{1,2,3} \left[ \frac{1}{x^2 y^2 z^2} \right]. \]
By (4.18) and the right-hand side of (4.19), \( \varphi \) satisfying \( P_3 \varphi = 0 \) can be written in the form

\[
\varphi = \sum_{i \geq 1} a_{i,1,1} \left[ \frac{1}{x^i y z} \right] + \sum_{i \geq 1} a_{i,1,2} \left[ \frac{1}{x^i y z^2} \right] \\
+ a_{1,1,3} \left[ \frac{1}{xyz^3} \right] + a_{1,2,3} \left[ \frac{1}{xy^2 z^3} \right] + a_{1,3,1} \left[ \frac{1}{xy^3 z} \right] \\
+ a_{1,4,1} \left[ \frac{1}{xy^4 z} \right] + a_{2,2,2} \left[ \frac{1}{x^2 y^2 z^2} \right] + a_{3,2,1} \left[ \frac{1}{x^3 y^2 z} \right],
\]

(4.20)

where

\[
a_{1,1,3} + 4a_{1,3,1} = 0, \quad a_{1,2,3} + 2a_{1,4,1} = 0, \\
a_{1,2,3} - 4a_{2,2,2} = 0, \quad a_{2,2,2} - 4a_{3,2,1} = 0.
\]

(4.21)

Similarly, for \( \varphi \) of the form (4.20), we have

\[
P_3 \varphi = \sum_{i \geq 1} (3i - 2)a_{i,1,2} \left[ \frac{1}{x^i y z} \right] + 2a_{1,3,1} \left[ \frac{1}{x^2 y z} \right].
\]

(4.22)

Hence the following conditions are needed for \( P_3 \varphi = 0 \).

\[
a_i, 1, 2 = 0 \quad (i \neq 2) \quad \text{and} \quad a_{1,3,1} + 2a_{2,1,2} = 0.
\]

(4.23)

Finally, we calculate

\[
P_4 \varphi = -\sum_{i \geq 1} 3(3i + 2)a_{i+1,1,1} \left[ \frac{1}{x^i y z} \right] - (2a_{1,3,1} - a_{2,1,2}) \left[ \frac{1}{x^2 y z} \right]
\]

(4.24)

Therefore the coefficients must satisfy

\[
a_{i,1,1} = 0 \quad (i \neq 1, 3) \quad \text{and} \quad 2a_{1,3,1} - a_{2,1,2} + 24a_{3,1,1} = 0.
\]

(4.25)

To sum up, the form of \( \varphi \) is specified as

\[
\varphi = a_{1,1,1} \left[ \frac{1}{xyz} \right] + a_{1,1,3} \left[ \frac{1}{xyz^3} \right] + a_{1,2,3} \left[ \frac{1}{xy^2 z^3} \right] \\
+ a_{1,3,1} \left[ \frac{1}{xy^3 z} \right] + a_{1,4,1} \left[ \frac{1}{xy^4 z} \right] + a_{2,1,2} \left[ \frac{1}{x^2 y^2 z^2} \right] \\
+ a_{2,2,2} \left[ \frac{1}{x^2 y^2 z^2} \right] + a_{3,1,1} \left[ \frac{1}{x^3 y z} \right] + a_{3,2,1} \left[ \frac{1}{x^3 y^2 z} \right]
\]
with the conditions

\[ a_{1,2,3} = -2a_{1,4,1}, \quad a_{2,2,2} = -\frac{1}{2}a_{1,4,1}, \quad a_{3,2,1} = -\frac{1}{8}a_{1,4,1}, \]
\[ a_{1,3,1} = -\frac{1}{4}a_{1,1,3}, \quad a_{2,1,2} = \frac{1}{8}a_{1,1,3}, \quad a_{3,1,1} = \frac{5}{192}a_{1,1,3}. \]  

(4.26)

Then we have \( P_1 \varphi = P_2 \varphi = 0 \) and

\[ P_0 \varphi = -(4s+3)a_{1,1,1} \begin{bmatrix} 1 \\ xyz \end{bmatrix} \]
\[ -(4s+5) \left( a_{1,1,3} \begin{bmatrix} 1 \\ xyz^3 \end{bmatrix} + a_{1,3,1} \begin{bmatrix} 1 \\ xyz^3 \end{bmatrix} + a_{2,1,2} \begin{bmatrix} 1 \\ x^2yz^2 \end{bmatrix} + a_{3,1,1} \begin{bmatrix} 1 \\ x^3yz \end{bmatrix} \right) \]
\[ -(4s+6) \left( a_{1,2,3} \begin{bmatrix} 1 \\ x^2yz^3 \end{bmatrix} + a_{1,4,1} \begin{bmatrix} 1 \\ x^4z \end{bmatrix} + a_{2,2,2} \begin{bmatrix} 1 \\ x^2y^2z^2 \end{bmatrix} + a_{3,2,1} \begin{bmatrix} 1 \\ x^3y^2z \end{bmatrix} \right). \]  

(4.27)

By (4.27), we have linearly independent three algebraic local cohomology solutions supported on \( \Sigma_0 \):

\[ \varphi_1 = \begin{bmatrix} 1 \\ xyz \end{bmatrix}, \quad s = -\frac{3}{4}, \]
\[ \varphi_2 = \begin{bmatrix} 1 \\ xyz^3 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 \\ xy^3z \end{bmatrix} + \frac{1}{8} \begin{bmatrix} 1 \\ x^2yz^2 \end{bmatrix} + \frac{5}{192} \begin{bmatrix} 1 \\ x^3yz \end{bmatrix}, \quad s = -\frac{5}{4}, \]
\[ \varphi_3 = -2 \begin{bmatrix} 1 \\ xy^2z^3 \end{bmatrix} + \begin{bmatrix} 1 \\ xy^4z \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ x^2y^2z^2 \end{bmatrix} - \frac{1}{8} \begin{bmatrix} 1 \\ x^3y^2z \end{bmatrix}, \quad s = -\frac{3}{2}. \]

As a consequence, we have

\[ M_{-\frac{3}{4}} = D_X \varphi_1, \quad M_{-\frac{5}{4}} = D_X \varphi_2, \quad M_{-\frac{3}{2}} = D_X \varphi_3. \]

Note that all these three holonomic systems are simple as D-Module.

In 2010, K. Nishiyama and M. Noro ([8]) devised algorithms to compute local b-functions and stratifications associated with local b-functions. By using their algorithms implemented in Risa/Asir, we get \((4s+3)(4s+5)(2s+3)\) as a factor of the local b-function on the stratum \( \Sigma_0 \) in question of \( f \). Thus, our results of computation are also consistent with the local b-function on the stratum \( \Sigma_0 \) of \( f \).

In the rest of this section, we compute algebraic local cohomology solutions \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) by using alternative method, already mentioned in the preceding section, that utilizes the homogeneity.

Recall \( w_f = \frac{1}{4}(1, 1, 1) \).
**Case 1:** Let us consider the case of \( s = -\frac{3}{4} \). The weighted degree of \( \left\{ \begin{array}{l} 1 \\ x^i y^j z^k \end{array} \right\} \) is \( -\frac{3}{4} \) only when \((i, j, k) = (1, 1, 1)\). From this, \( \varphi_1 \) is verified.

**Case 2:** We consider the case of \( s = -\frac{5}{4} \). The combination of \((i, j, k)\) satisfying \(-\frac{1}{4}(i + j + k) = -\frac{5}{4}\) is given by

\[(i, j, k) = (1, 1, 3), (1, 2, 2), (1, 3, 1), (2, 1, 2), (2, 2, 1), (3, 1, 1).\]

Note that the algebraic local cohomology classes associated with above are annihilated by \( J_f \). Set

\[ \varphi = a \left\{ \begin{array}{l} 1 \\ x y z^3 \end{array} \right\} + b \left\{ \begin{array}{l} 1 \\ x y^2 z^2 \end{array} \right\} + c \left\{ \begin{array}{l} 1 \\ x y^3 z \end{array} \right\} + d \left\{ \begin{array}{l} 1 \\ x^2 y z^2 \end{array} \right\} + e \left\{ \begin{array}{l} 1 \\ x^2 y^2 z \end{array} \right\} + f \left\{ \begin{array}{l} 1 \\ x^3 y z \end{array} \right\} \]

with indeterminate coefficients \( a, b, c, d, e, f \). A simple computation gives

\[(4.28)\]

\[ P_3 \varphi = b \left\{ \begin{array}{l} 1 \\ x y^2 z \end{array} \right\} + 2(c + 2d) \left\{ \begin{array}{l} 1 \\ x^2 y z \end{array} \right\}, \]

\[ P_4 \varphi = (a - 2c - 12d) \left\{ \begin{array}{l} 1 \\ x y^2 z \end{array} \right\} + (2b - 15e) \left\{ \begin{array}{l} 1 \\ x y^3 z \end{array} \right\} - (2c - d + 24f) \left\{ \begin{array}{l} 1 \\ x^2 y^2 z \end{array} \right\}, \]

\[ P_5 \varphi = 2b \left\{ \begin{array}{l} 1 \\ x y^2 z \end{array} \right\} + (a + 4c) \left\{ \begin{array}{l} 1 \\ x y^3 z \end{array} \right\} - 2(b - 5e) \left\{ \begin{array}{l} 1 \\ x^2 y z \end{array} \right\}. \]

The relation of \( a, b, c, d, e, f \) are determined by the above.

**Case 3:** We finally consider the case of \( s = -\frac{3}{2} \). Since the combination of \((i, j, k)\) satisfying \(-\frac{1}{4}(i + j + k) = -\frac{3}{2}\) is given by

\[(1, 1, 4), (1, 2, 3), (1, 3, 2), (1, 4, 1), (2, 1, 3), (2, 2, 2), (2, 3, 1), (3, 1, 2), (3, 2, 1), (4, 1, 1).\]

Let \( H_\varphi \) be the vector space generated by the set of \( \left\{ \begin{array}{l} 1 \\ x^i y^j z^k \end{array} \right\} \) whose index \((i, j, k)\) is in the combination above. Then, if we set

\[ H_{\varphi, J_f} = \{ \varphi \in H_\varphi \mid J_f \varphi = 0 \}, \]
the following set is a basis of $H_{\Phi \beta f}$.

$$
\begin{align*}
\left[ \begin{array}{c}
1 \\
x^2y^2z^2
\end{array} \right], & \left[ \begin{array}{c}
1 \\
x^2y^3z
\end{array} \right], & \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right], & \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right], & \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right], & \left[ \begin{array}{c}
1 \\
x^4yz
\end{array} \right], \\
-2 \left[ \begin{array}{c}
1 \\
x^2y^2z^3
\end{array} \right] & + \left[ \begin{array}{c}
1 \\
x^4yz
\end{array} \right] & - \frac{2}{3} \left[ \begin{array}{c}
1 \\
x^3y^2z^3
\end{array} \right].
\end{align*}
$$

Therefore we set

$$
\varphi = a \left[ \begin{array}{c}
1 \\
x^2y^2z^2
\end{array} \right] + b \left( -2 \left[ \begin{array}{c}
1 \\
x^4z
\end{array} \right] + \left[ \begin{array}{c}
1 \\
x^4z
\end{array} \right] \right) + c \left( \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right] - \frac{2}{3} \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right] \right) \\
+ d \left[ \begin{array}{c}
1 \\
x^2y^2z^2
\end{array} \right] + e \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right] + f \left[ \begin{array}{c}
1 \\
x^3yz^2
\end{array} \right] + g \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right] + h \left[ \begin{array}{c}
1 \\
x^4yz
\end{array} \right].
$$

Here $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$ are undetermined coefficients. A direct computation gives

$$
P_3 \varphi = -a \left[ \begin{array}{c}
1 \\
x^2yz^3
\end{array} \right] + c \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right] + 2(b + 2d) \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right] + (4e + 7f) \left[ \begin{array}{c}
1 \\
x^3yz^2
\end{array} \right],
$$

$$
P_4 \varphi = (a + 2c) \left[ \begin{array}{c}
1 \\
x^2yz^3
\end{array} \right] - 6(b + 2d) \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right] + 3(c + e) \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right] \\
- \left( \frac{8}{3}c + 2e + 21f \right) \left[ \begin{array}{c}
1 \\
x^2y^2z^2
\end{array} \right] - 2(b - d + 12g) \left[ \begin{array}{c}
1 \\
x^2y^2z
\end{array} \right] \\
- (4e - f + 33h) \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right],
$$

$$
P_5 \varphi = (a + 2c) \left[ \begin{array}{c}
1 \\
x^2y^2z^2
\end{array} \right] + 4(b + 2d) \left[ \begin{array}{c}
1 \\
x^2y^2z^2
\end{array} \right] - \left( \frac{8}{3}c - 10e \right) \left[ \begin{array}{c}
1 \\
x^2y^2z
\end{array} \right] \\
- 4(d - 4g) \left[ \begin{array}{c}
1 \\
x^3y^2z
\end{array} \right].
$$

Therefore local cohomology solution $\varphi_3$ is determined by (4.31).

**Remark 4.2.** In the case of weighted homogeneous singularities, if we know, in advance, $\beta \in \mathbb{C}$ such that $\text{Supp}(M_\beta)$ is $\Sigma_0$, we can efficiently calculate $M_\beta$ by the above method.
References