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<th>Computing structures of holonomic D-modules associated with a simple line singularity (Several aspects of microlocal analysis)</th>
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<tr>
<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2016), B57: 125-140</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2016-05</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/241338">http://hdl.handle.net/2433/241338</a></td>
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<tr>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Computing structures of holonomic D-modules associated with a simple line singularity

By
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§ 1. Introduction

The purpose of this paper is to describe a method to analyze the structure of holonomic D-modules associated with a hypersurface with non-isolated simple line singularities. We show in particular, by using two examples, the proposed method effectively determines the monodromy structure, on the singular locus of a given hypersurface, of the local system induced by the relevant holonomic D-module.

Let $D_X$ be the sheaf of rings of holomorphic partial differential operators on a complex manifold $X$ and $D_X[s]$ the sheaf of rings $D_X[s] = D_X \otimes \mathbb{C} [s]$, where $s$ is an indeterminate. Let $f$ be a holomorphic function on $X$ and let $J_f$ be the Jacobian ideal generated by the partial derivatives of $f$.

In 1970's, the following three ideal were introduced in the theory of b-functions.

$\text{Ann}_{D_X[s]} f^s$, $\text{Ann}_{D_X[s]} f^s + D_X[s] f$, $\text{Ann}_{D_X[s]} f^s + D_X[s] J_f + D_X[s] f$,

where $\text{Ann}_{D_X[s]} f^s$ is the annihilator of $f^s$ in $D_X[s]$. The annihilator $\text{Ann}_{D_X[s]} f^s$ and the associated $D_X[s]$-module $D_X[s]/\text{Ann}_{D_X[s]} f^s$ were introduced and investigated by M. Kashiwara to prove the existence of b-functions and the rationality of their roots. M. Kashiwara also showed in the same paper [3] that the b-function $b_f(s)$ of $f$ can be defined as the minimal polynomial of the action of $s$ on the $D_X[s]$-module $D_X[s]/(\text{Ann}_{D_X[s]} f^s + D_X[s] f)$, defined by the second ideal presented above. The last ideal and the associated $D_X[s]$-module $D_X[s]/(\text{Ann}_{D_X[s]} f^s + D_X[s] J_f + D_X[s] f)$ were effectively utilized in [16] by T. Yano to compute b-functions for many cases. In 1997,
T. Oaku considered b-functions in the context of computational algebraic analysis and introduced an algorithm for computing annihilator $\text{Ann}_{D_X[s]} f^s$. Furthermore, using the ideal $\text{Ann}_{D_X[s]} f^s + D_X[s] f$, he succeeded to derive an algorithm for computing b-functions.

The authors of the present paper and T. Oaku [12] have recently examined, for $\beta \in \mathbb{C}$, holonomic $D_X$-modules

$$D_X[s]/(\text{Ann}_{D_X[s]} f^s + D_X[s] J_f + D_X[s] f + D_X[s](s - \beta))$$

associated with a hypersurface with simple line singularities. They have investigated in particular the monodromy structure of the local systems, on the singular locus $\Sigma$ of the hypersurface, induced by the relevant holonomic $D_X$-modules associated with 14 type simple line singularities given in [1].

In this paper, we focus our attention to two cases, the transverse $A_2$ singularity and the transverse $E_6$ singularity and illustrate the method to analyze the structure of holonomic $D_X$-modules in question. A key ingredient of the proposed method is the concept of local cohomology supported on the singular locus.

§ 2. Preparation

In this section we recall some basic results on holonomic D-modules and simple line singularities relevant to our study.

§ 2.1. Holonomic D-modules

Let $X$ be a complex manifold of dimension $n$ and $D_X$ the sheaf of rings on $X$ of holomorphic partial differential operators.

**Definition 2.1** ([2]). Let $M$ be a holonomic $D_X$-module on $X$. A stratification $X = \bigsqcup_{\alpha} S_{\alpha}$ of $X$ is said to be regular with respect to $M$ if it satisfies

(i) the Whitney conditions (a), (b)

(ii) the singular support of $M$ is contained in $\bigcup T_{S_{\alpha}}^* X$, the union of the conormal bundle of strata.

Let $Y$ be a complex submanifold of $X$ of codimension $d$. Let $B_{Y|X}$ denote the left $D_X$-module of algebraic local cohomology $\mathcal{H}^d_{[Y]}(\mathcal{O}_X)$.

**Theorem 2.2** ([2]). Let $M$ be a holonomic $D_X$-module whose support is contained in a submanifold $Y$ and whose singular support is contained in $T_Y^* X$. Then $M$ is locally isomorphic to the direct sum of finite copies of $B_{Y|X}$. 
Let $f$ be a holomorphic function on $X$ and let $J_f$ be the Jacobian ideal of $f$. Let $b_f(s)$ be the b-function of $f$ and let $\tilde{b}_f(s)$ denote the reduced b-function of $f$ defined to be $\tilde{b}_f(s) = b_f(s)/(s + 1)$.

**Theorem 2.3 ([16]).** Let $S$ be the hypersurface $\{x \in X \mid f(x) = 0\}$ and let $\Sigma$ denote the singular locus of the hypersurface $S$. Then, for a root $\beta$ of the reduced b-function of $f$, the $D_X$-module defined by

$$D_X[s]/(\Ann_{D_X[s]} f^s + D_X[s]J_f + D_x[s]f + D_X(s - \beta))$$

is a holonomic $D_X$-module whose support is contained in the singular set $\Sigma$.

### §2.2. Simple line singularities

Now let $f$ be a defining function of a hypersurface $S$ with simple line singularities introduced by T. de Jong ([1]). Then the singular locus $\Sigma$ of $S$ is a complex line. According to [1], the singular locus $\Sigma$ is stratified by two strata $\Sigma_1 = \Sigma - \{O\}$ and $\Sigma_0 = \{O\}$. Classical results, recalled in the preceding subsection, on holonomic $D_X$-modules and on the theory of b-functions yield that the support of the holonomic $D_X$-module

$$M_\beta = D_X[s]/(\Ann_{D_X[s]} f^s + D_X[s]J_f + D_x[s]f + D_X(s - \beta)),$$

for a root $\beta$ of the reduced b-function of $f$, is contained in $\Sigma$. If the stratification $\Sigma = \Sigma_1 \sqcup \Sigma_0$ is regular with respect to $M_\beta$, then, for a point $Q \in \Sigma_1$, the holonomic $D_X$-module $M_\beta$ is locally isomorphic to the direct sum of the finite copies of $B_{\Sigma_1[X]}$.

In order to understand the monodromy structure of the local system on $\Sigma_1$ induced by the holonomic $D_X$-module $M_\beta$, it is natural to consider the *multivaluedness* of algebraic local cohomology solutions of the system $M_\beta$ on the stratum $\Sigma_1$.

### §3. $A_{2-1}$ type

Let us consider $A_{2-1}$ type simple line singularity defined by $f(x, y, z) = xy^3 + z^2$. Let $S = \{(x, y, z) \in X \mid f(x, y, z) = 0\}$, where $X$ is a neighborhood of the origin $O$ in $\mathbb{C}^3$. Note that $f$ is a weighted homogeneous polynomial. Define the weight vector $w_f$ of $f$ by $\frac{1}{3}(1, 1, 2)$ so that the weighted degree of $f$ is equal to $1$. Since the Jacobian ideal $J_f$ generated by the partial derivatives $\partial_x f = \partial f/\partial x,$ $\partial_y f = \partial f/\partial y,$ $\partial_z f = \partial f/\partial z$ is $(y^3, 3xy^2, 2z)$, the singular locus of $S$ is $\Sigma = \{(x, y, z) \mid y = z = 0\}$, a complex line. The singular locus $\Sigma$ is stratified by two strata $\Sigma = \Sigma_0 \sqcup \Sigma_1$, where $\Sigma_1 = \Sigma \setminus \{O\}$ and $\Sigma_0 = \{O\}$. Note that since $\Sigma_1 \cong \mathbb{C} - \{O\}$, the fundamental group of the stratum $\Sigma_1$ is non-trivial.
By executing an algorithm derived in [11] by T. Oaku, we get the following partial differential operators as a set of generators of the annihilator $\text{Ann}_{D_X[s]} f^s$:

\[
\begin{cases}
P_1 = 2z\partial_y - 3xy^2\partial_z, & P_2 = 2z\partial_x - y^3\partial_z, \\
P_3 = -3x\partial_x + y\partial_y, & P_4 = 2y\partial_y + 3z\partial_z - 6s
\end{cases}
\]

with $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$ and $\partial_z = \partial/\partial z$. Let $E$ denote the Euler operator

\[E = \frac{1}{4}x\partial_x + \frac{1}{4}y\partial_y + \frac{1}{2}z\partial_z.\]

Then $2P_4 - P_3 = 12(E - s)$ holds. Since $P_i \in D_X[s]J_f \cap \text{Ann}_{D_X[s]} f^s$ ($i = 1, 2$), we have

\[(3.1) \quad I = \text{Ann}_{D_X[s]} f^s + D_X[s]J_f + D_X[s]f = D_X[s] (P_3, E - s, \partial_x f, \partial_y f, \partial_z f).\]

As previously described in §2, the holonomic $D$-module $M_\beta$ defines a local system on $\Sigma_1$ for a root $\beta$ of the reduced $b$-function $\tilde{b}_f(s)$. In §3.1, the monodromy structure for the local system is decided by the algebraic local cohomology solutions annihilated by $J_f$, $P_3$ and $E - s$. In §3.2, we calculate algebraic local cohomology solutions supported on $\Sigma_0$.

### § 3.1. Algebraic local cohomology solutions supported on $\Sigma_1$

Let $H^2_{[\Sigma_1]}(\mathcal{O}_X)$ be the sheaf of algebraic local cohomology supported on $\Sigma_1$, where $\mathcal{O}_X$ is the sheaf on $X$ of holomorphic functions. Set

\[H_{\Sigma_1} = \left\{ \psi \in H^2_{[\Sigma_1]}(\mathcal{O}_X) \mid J_f \psi = 0 \right\}.\]

Then any germ at a point $Q \in \Sigma_1$ of the sheaf $H_{\Sigma_1}$ can be represented as a linear combination

\[h_1(x) \begin{bmatrix} 1 \\ yz \end{bmatrix} + h_2(x) \begin{bmatrix} 1 \\ y^2z \end{bmatrix},\]

where $[ ]$ denotes the Grothendieck symbol and $h_1(x), h_2(x)$ are germs at $Q$ of holomorphic functions on $\Sigma_1$. Taking the representation of a local section of $H_{\Sigma_1}$ into account, we explicitly compute algebraic local cohomology classes $\psi$ that satisfy $J_f \psi = P_3 \psi = (E - s) \psi = 0$ as follows.

(i) Put $\psi_1 = h_1(x) \begin{bmatrix} 1 \\ yz \end{bmatrix}$. Then, $P_3$ acts on $\psi_1$ as $P_3 \psi_1 = -(3xh_1' + h_1) \begin{bmatrix} 1 \\ yz \end{bmatrix}$ with $h_1' = \frac{dh_1}{dx}$. By $3xh_1' + h_1 = 0$, $h_1(x)$ is decided as $\text{const } x^{-\frac{1}{3}}$. It follows from $(E - s) \psi_1 = -\left(\frac{1}{12} + \frac{1}{4} + \frac{1}{2} + s\right) \psi_1$ that $s = -\frac{5}{6}$. Thus, we have

\[\psi_1 = \text{const } x^{-\frac{1}{3}} \begin{bmatrix} 1 \\ yz \end{bmatrix}, \quad s = -\frac{5}{6}.\]
(ii) Put $\psi_2 = h_2(x) \left[ \frac{1}{y^2 z} \right]$. In the same manner as (i), we have

$$\psi_2 = \text{const} \ x^{-\frac{2}{3}} \left[ \frac{1}{y^2 z} \right], \quad s = -\frac{7}{6}. $$

Note that the monodromy structure on $\Sigma_1$ is shown by the multi-valuedness of $\psi_1$ and $\psi_2$ with respect to the variable $x$ ([15]).

**Remark 3.1.** Let $r_x(y, z) = f(x, y, z) = xy^3 + z^2$ for $x \neq 0$. Here $y$ and $z$ are variables and $x(\neq 0)$ corresponding to a point $(x, 0, 0) \in \Sigma_1$ is regarded as parameter. Then, $r_x$ is a weighted homogeneous polynomial in $y, z$ with respect to the weight vector $w_{r_x} = \left( \frac{1}{6}, 2, 3 \right)$. The weighted degree of the constructed solution with respect to $w_f$ and also $w_{r_x}$ is equal to the value of $s$ respectively, namely, the exponent $\lambda = -\frac{1}{3}$ of $\psi_1$ satisfies

$$\frac{1}{4} \times \lambda + \frac{1}{4} \times (-1) + \frac{2}{4} \times (-1) = \frac{2}{6} \times (-1) + \frac{3}{6} \times (-1) = -\frac{5}{6},$$

and the exponent $\lambda = -\frac{2}{3}$ of $\psi_2$ satisfies

$$\frac{1}{4} \times \lambda + \frac{1}{4} \times (-2) + \frac{2}{4} \times (-1) = \frac{2}{6} \times (-2) + \frac{3}{6} \times (-1) = -\frac{7}{6}.$$

**Remark 3.2.** Since the factor $s + 1$ of the b-function of $r_x$ given by

$$b_{r_x}(s) = (s + 1)(6s + 5)(6s + 7)$$

comes from the non-singular part, the reduced local b-function on the stratum $\Sigma_1$ of $f$ is equal to $(6s + 5)(6s + 7)$. Therefore, what we have computed implies in particular that the weighted degree of solutions are compatible with the roots of local b-function on the stratum $\Sigma_1$ of the defining function $f$.

§ 3.2. Algebraic local cohomology solutions supported on $\Sigma_0$

Any algebraic local cohomology class in $\mathcal{H}^3_{[O]}(O_X)$ supported at the origin can be written in the form

$$\sum a_{i,j,k} \left[ \frac{1}{x^i y^j z^k} \right].$$

Let us construct a solution $\varphi$ in $\mathcal{H}^3_{[O]}(O_X)$ satisfying $J_f \varphi = P_3 \varphi = (E - s) \varphi = 0$. Recall $J_f = (\partial_x f, \partial_y f, \partial_z f) = (y^3, 3xy^2, 2z)$. Any solution $\varphi$ in $\mathcal{H}^3_{[O]}(O_X)$ for $z \varphi = 0$ is expressed in the form

$$\varphi = \sum a_{i,j,1} \left[ \frac{1}{x^i y^j z} \right].$$
By putting (3.2) into $y^3\varphi = 0$ and $xy^2\varphi = 0$, we see

$$
\left\{ \varphi \in \mathcal{H}^3_{(O)}(O_X) \mid J_f \varphi = 0 \right\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ x^i \\ yz \end{bmatrix} (i \geq 1), \begin{bmatrix} 1 \\ x^i \\ y^2z \end{bmatrix} (i \geq 1), \begin{bmatrix} 1 \\ x^i \\ y^3z \end{bmatrix} \right\}.
$$

Since $P_3 \begin{bmatrix} 1 \\ x^i \\ y^jz \end{bmatrix} = (3i-j) \begin{bmatrix} 1 \\ x^i \\ y^jz \end{bmatrix}$, the relation $j = 3i$ holds for $\begin{bmatrix} 1 \\ x^i \\ y^jz \end{bmatrix}$. Therefore $\varphi = \text{const} \begin{bmatrix} 1 \\ xy^3z \end{bmatrix}$. Finally, $s$ is decided by $(E-s)\varphi = -(\frac{1}{4} + \frac{3}{4} + \frac{2}{4} + s)\varphi$.

Summing up, the algebraic local cohomology solution supported on $\Sigma_0$ is the following.

$$
\varphi = \text{const} \begin{bmatrix} 1 \\ xy^3z \end{bmatrix}, \quad s = -\frac{3}{2}.
$$

Notice that the weighted degree of $\varphi$ is equal to the value of $s$.

As a consequence, we have

$$
M_{-\frac{5}{6}} = D_X \left( x^{-\frac{1}{3}} \begin{bmatrix} 1 \\ yz \end{bmatrix} \right), \quad M_{-\frac{7}{6}} = D_X \left( x^{-\frac{2}{3}} \begin{bmatrix} 1 \\ y^2z \end{bmatrix} \right)
$$

and

$$
M_{-\frac{3}{2}} = D_X \left( \begin{bmatrix} 1 \\ xy^3z \end{bmatrix} \right).
$$

Moreover, we see

$$
\text{Ch}(M_{-\frac{5}{6}}) = \text{Ch}(M_{-\frac{7}{6}}) = T_{\Sigma_1}^* X \cup T_{\Sigma_0}^* X \quad \text{and} \quad \text{Ch}(M_{-\frac{3}{2}}) = T_{\Sigma_0}^* X,
$$

where $\text{Ch}(M_\beta)$ denotes the characteristic variety of $M_\beta$.

**Remark 3.3.** Since the global b-function $b_f(s)$ of $f$ is $b_f(s) = (s+1)(2s+3)(6s+5)(6s+7)$ and that of $r_x$ is $b_{r_x}(s) = (s+1)(6s+5)(6s+7)$, $s = -\frac{3}{2}$ is a root of the local b-function on the stratum $\Sigma_0$ of $f$. Therefore, the result which says that holonomic system $M_{-\frac{3}{2}}$ corresponding to the root $-\frac{3}{2}$ is supported on the stratum $\Sigma_0$ is consistent with this fact.

Note that the b-function $b_f(s)$ presented above is computed by using an algorithm implemented by M. Noro ([10]).

In the rest of this section, we propose here an alternative method to compute algebraic local cohomology solutions, that utilizes the homogeneity of solutions. We
start from the fact that $2s + 3$ is a factor of the local b-function on the stratum $\Sigma_0$ of $f$. The combination of $(i, j, k)$ satisfying $-\frac{1}{4}(i + j + 2k) = -\frac{3}{2}$ is given by

$$(i, j, k) = (1, 1, 2), (1, 3, 1), (2, 2, 1), (3, 1, 1).$$

Since the index $(i, j, k)$ of $\varphi$ satisfying $J_f \varphi = P_i \varphi = 0$ ($1 \leq i \leq 4$) is only $(1, 3, 1)$, we immediately get the desired solution $\varphi$.

§4. $E_{6-3}$ type

In this section, we consider the case of $E_{6-3}$ type simple line singularity. Let $f(x, y, z) = y^4 + xz^3 + y^2z^2$ and let $S = \{(x, y, z) \in X \mid f(x, y, z) = 0\}$, where $X$ is a neighborhood of the origin $O$ in $\mathbb{C}^3$. Define $w_f = \frac{1}{4}(1, 1, 1)$, then the weighted degree of $f$ is equal to 1. As the Jacobian ideal $J_f = (\partial_x f, \partial_y f, \partial_z f)$ of $f$ is $(z^3, 4y^3 + 2yz^2, 3xz^2 + 2y^2z)$, the singular locus of $S$ is given by $\Sigma = \{(x, y, z) \mid y = z = 0\} \subset S$ and $\Sigma$ is stratified by two strata $\Sigma = \Sigma_0 \sqcup \Sigma_1$, where $\Sigma_1 = \Sigma \setminus \{O\}$ and $\Sigma_0 = \{O\}$.

According to the Oaku’s algorithm [11] implemented in a computer algebra system Risa/Asir ([9]), a set of generators of the annihilator $\text{Ann}_{D_X[s]} f^s$ can be computed as follows:

$$
\begin{align*}
P_1 &= 2y(2y^2 + z^2)\partial_x - z^3\partial_y, \\
P_2 &= -(3xz + 2y^2)\partial_y + 2y(2y^2 + z^2)\partial_z, \\
P_3 &= -(3xz + 2y^2)\partial_x + z^2\partial_z, \\
P_4 &= (9x^2 + 2y^2)\partial_x - yz\partial_y - (3xz - 2y^2)\partial_z, \\
P_5 &= -2y(3x - z)\partial_x - z^2\partial_y + 2yz\partial_z, \\
P_6 &= x\partial_x + y\partial_y + z\partial_z - 4s.
\end{align*}
$$

Note that the operators $P_i$’s are of weighted homogeneous and $P_6 = 4(E - s)$, where $E$ is the Euler operator. It follows from $P_i \in D_X[s] J_f \cap \text{Ann}_{D_X[s]} f^s$ ($i = 1, 2$) that

$$
\text{Ann}_{D_X[s]} f^s + D_X[s] f + D_X[s] J_f = D_X[s] (P_3, P_4, P_5, P_6, \partial_x f, \partial_y f, \partial_z f).
$$

§4.1. Algebraic local cohomology solutions supported on $\Sigma_1$

Set $H_{\Sigma_1} = \left\{ \psi \in \mathcal{H}_{[\Sigma_1]}^0(\mathcal{O}_X) \mid J_f \psi = 0 \right\}$. Then it is easy to see, by using a method described in [6] if necessary, that for a point $Q \in \Sigma_1$, any germ at $Q$ of the sheaf $H_{\Sigma_1}$ is represented as a linear combination of the form $\sum_{i=1}^{6} h_i(x) \sigma_i$, where $h_i(x)$’s are germs at $Q$ of holomorphic functions on $\Sigma_1$ and algebraic local cohomology classes $\sigma_i$’s are
defined by
\[
\sigma_1 = \left\{ \begin{array}{l}
1 \\
yz
\end{array} \right\}, \quad \sigma_2 = \left\{ \begin{array}{l}
y^2z \\
yz
\end{array} \right\}, \quad \sigma_3 = \left\{ \begin{array}{l}
y^2z^2 \\
yz^2
\end{array} \right\}, \quad \sigma_4 = \left\{ \begin{array}{l}
y^3z \\
y^2z^2
\end{array} \right\},
\]
(4.1)
\[
\sigma_5 = \left\{ \begin{array}{l}
y^2z^2 \\
y^2z
\end{array} \right\}, \quad \sigma_6 = \left\{ \begin{array}{l}
y^3z^3 \\
y^3z
\end{array} \right\} - \frac{3}{2}x \left\{ \begin{array}{l}
y^3z^2 \\
y^3z
\end{array} \right\}.
\]
The following are bases of algebraic local cohomology solutions in question supported on \( \Sigma_1 \).
\[
\psi_1 = x^{-\frac{1}{3}} \left\{ \begin{array}{l}
1 \\
yz
\end{array} \right\}, \quad s = -\frac{7}{12},
\]
\[
\psi_2 = x^{-\frac{1}{3}} \left\{ \begin{array}{l}
y^2z \\
yz
\end{array} \right\}, \quad s = -\frac{5}{6},
\]
\[
\psi_3 = x^{-\frac{2}{3}} \left\{ \begin{array}{l}
y^2z^2 \\
yz^2
\end{array} \right\} + \frac{1}{12}x^{-\frac{5}{3}} \left\{ \begin{array}{l}
y^2z \\
yz
\end{array} \right\}, \quad s = -\frac{11}{12},
\]
(4.2)
\[
\psi_4 = x^{-\frac{1}{3}} \left\{ \begin{array}{l}
y^3z \\
y^2z^2
\end{array} \right\} - \frac{1}{3}x^{-\frac{5}{3}} \left\{ \begin{array}{l}
y^3z^2 \\
yz^2
\end{array} \right\} - \frac{1}{18}x^{-\frac{5}{3}} \left\{ \begin{array}{l}
y^3z \\
y^2z
\end{array} \right\}, \quad s = -\frac{13}{12},
\]
\[
\psi_5 = x^{-\frac{2}{3}} \left\{ \begin{array}{l}
y^2z^2 \\
y^2z
\end{array} \right\} + \frac{1}{6}x^{-\frac{5}{3}} \left\{ \begin{array}{l}
y^2z \\
yz^2
\end{array} \right\}, \quad s = -\frac{7}{6},
\]
\[
\psi_6 = x^{-\frac{5}{3}} \left( \left\{ \begin{array}{l}
y^3z^3 \\
y^3z^2
\end{array} \right\} - \frac{3}{2}x \left\{ \begin{array}{l}
y^3z^2 \\
y^3z
\end{array} \right\} - \frac{3}{8}x^{-\frac{5}{3}} \left\{ \begin{array}{l}
y^3z \\
y^3z
\end{array} \right\} ight)
+ \frac{5}{24}x^{-\frac{8}{3}} \left\{ \begin{array}{l}
y^2z^2 \\
y^2z
\end{array} \right\} + \frac{7}{144}x^{-\frac{4}{3}} \left\{ \begin{array}{l}
y^3z \\
y^2z
\end{array} \right\}, \quad s = -\frac{17}{12}.
\]
Notably, the monodromy structure on the stratum \( \Sigma_1 \) encoded originally in the ideal \( \text{Ann}_{D_X[s]} f^s \) is revealed by computing local cohomology solutions of the relevant holonomic systems.

**Remark 4.1.** Let \( r_x(y, z) = y^4 + xz^3 + y^2z^2 \) denote the function of two variable \( y, z \), where \( x(\neq 0) \) is regarded as the parameter that corresponds to a point \((x, y, z) = (x, 0, 0) \in \Sigma_1 \). Then, \( r_x(y, z) = 0 \) has an isolated singular point at \((y, z) = (0, 0) \) for each \( x \) with \( x \neq 0 \). The polynomial \( r_x \) is not of weighted homogeneous whereas one can easily see, by using a method described in [7, 14] for instance, that \( r_x \) is a quasi-homogeneous function. Hence, the \( b \)-function of \( r_x \) is given by
\[
b_{r_x}(s) = (s+1)(6s+5)(6s+7)(12s+7)(12s+11)(12s+13)(12s+17).
\]
(4.3)
Therefore, the reduced local \( b \)-function on the stratum \( \Sigma_1 \) of \( f \) is equal to
\[
(6s+5)(6s+7)(12s+7)(12s+11)(12s+13)(12s+17).
\]
Thus the results (4.2) presented above is also consistent with the local b-function.

We show the method to obtain the result (4.2) by giving the details of computation. Now set

\[(4.4) \quad \Lambda_{j,Q} = \left\{ \psi \mid \psi = \sum_{k=1}^{j} h_k(x)\sigma_k, \ h_k \in \mathcal{O}_{\Sigma_1,Q} \right\}, \quad j = 1, 2, \ldots, 6.\]

Then \(\Lambda_1,Q \subset \Lambda_2,Q \subset \cdots \subset \Lambda_6,Q\) holds. It is easy to verify by direct computation that each \(\Lambda_{j,Q}\) is stable under the action of \(P_i, \ i = 1, 2, \ldots, 6\), namely, \(P_i\Lambda_{j,Q} \subset \Lambda_{j,Q}\) holds for any \(i\) and \(j\). Our basic strategy is to find local cohomology solutions from each \(\Lambda_{j,Q}\), \(j = 1, 2, \ldots, 6\).

Local cohomology solutions \(\psi_1\) and \(\psi_2\) are easily decided as (i) and (ii) in the previous subsection. Let us compute \(\psi_i\) (3 \(\leq i \leq 6\)).

(a) Put \(\tau = h_3(x)\sigma_3\). We have

\[(4.5) \quad P_3\tau = -(3xh'_3 + 2h_3)\sigma_1, \quad P_4\tau = 3x(3xh'_3 + 2h_3)\sigma_3 + h_3\sigma_1, \quad P_5\tau = 0.\]

By \(3xh'_3 + 2h_3 = 0\), \(h_3\) is \(x^{-\frac{2}{3}}\). Then the weighted degree of \(\tau\) is \(-\frac{11}{12}\). We try to get \(\psi_3\) by considering the homogeneity. Noticing \(h_3\sigma_1\) in the right-hand side of equation for \(P_4\tau\), we set \(\psi_3 = x^\frac{2}{3}\sigma_3 + cx^\lambda\sigma_1\). Here \(c\) and \(\lambda\) are decided as follows.

Since the weighted degree \(\frac{1}{4}\lambda - \frac{2}{4}\) of \(x^\lambda\sigma_1\) is equal to \(-\frac{11}{12}\), we get \(\lambda = -\frac{5}{3}\). Then \(P_4\psi_3 = (1 - 12c)x^{-\frac{8}{3}}\sigma_1, \quad P_i\psi_3 = 0 (i \neq 4)\). Hence \(c = \frac{1}{12}\) and \(s = -\frac{11}{12}\) and \(\psi_3\) in (4.2) is verified.

(b) Putting \(\tau = h_4(x)\sigma_4\), we have

\[(4.6) \quad P_3\tau = 2h'_4\sigma_1, \quad P_4\tau = 3x(3xh'_4 + h_4)\sigma_4 - 2h_4\sigma_3 + 2h'_4\sigma_1, \quad P_5\tau = -2(3xh'_4 + h_4)\sigma_2.\]

By \(3xh'_4 + h_4 = 0\), we have \(h_4 = x^{-\frac{1}{3}}\) and

\[(4.7) \quad P_3\tau = \frac{2}{3}x^{-\frac{3}{3}}\sigma_1, \quad P_4\tau = -2x^{-\frac{1}{3}}\sigma_3 - \frac{2}{3}x^{-\frac{4}{3}}\sigma_1, \quad P_5\tau = 0.\]

The weighted degree of \(\tau\) is \(-\frac{13}{12}\). For \(\varrho_3 = x^{-\frac{4}{3}}\sigma_3\), a simple computation shows

\[(4.8) \quad P_3\varrho_3 = 2x^{-\frac{4}{3}}\sigma_1, \quad P_4\varrho_3 = -6x^{-\frac{4}{3}}\sigma_3 + x^{-\frac{5}{3}}\sigma_1, \quad P_5\varrho_3 = 0.\]

Comparing (4.7) and (4.8), we set \(\tau = -\frac{1}{3}\varrho_3 + cx^{-\frac{7}{3}}\sigma_1\) which is equal to \(x^{-\frac{1}{3}}\sigma_4 - \frac{1}{3}x^{-\frac{4}{3}}\sigma_3 + cx^{-\frac{7}{3}}\sigma_1\). Then we have \(P_4\psi_4 = -(1 + 18c)x^{-\frac{4}{3}}\sigma_1, \quad P_i\psi_4 = 0 (i \neq 4)\). Hence \(c = -\frac{1}{18}\) and \(s = -\frac{13}{12}\). Therefore \(\psi_4\) is checked.
(c) Put $\tau = h_5(x)\sigma_5$. Then $\tau$ satisfies

$$
\begin{align*}
P_3 \tau &= -(3xh_5' + 2h_5)\sigma_2, \\
P_4 \tau &= 3x(3xh_5' + 2h_5)\sigma_5 + 2h_5\sigma_2, \\
P_5 \tau &= -2(3xh_5' + 2h_5)\sigma_3 + 2h_5'\sigma_1.
\end{align*}
$$

From the above, $h_5 = x^{-\frac{8}{3}}$. The weighted degree of $\tau$ is $-\frac{7}{6}$. For $\varrho_2 = x^{-\frac{8}{3}}\sigma_2$, we have

$$
P_3 \varrho_2 = 0, \quad P_4 \varrho_2 = -12x^{-\frac{8}{3}}\sigma_2, \quad P_5 \varrho_2 = 8x^{-\frac{8}{3}}\sigma_1.
$$

The form of $\psi_5$ follows from (4.9) and (4.10).

(d) Putting $\tau = h_6(x)\sigma_6$, we get

$$
\begin{align*}
P_3 \tau &= \frac{3}{2}x(3xh_6' + 5h_6)\sigma_4, \\
P_4 \tau &= -\frac{9}{2}x(3xh_6' + 5h_6)\sigma_6 - \frac{9}{2}xh_6\sigma_4 - (3xh_6' + 2h_6)\sigma_3, \\
P_5 \tau &= 3x(3xh_6' + 5h_6)\sigma_5 - (3xh_6' + 2h_6)\sigma_2.
\end{align*}
$$

Let us decide $h_6$ so that $3xh_6' + 5h_6 = 0$ holds, i.e., $h_6 = x^{-\frac{5}{3}}$. Then we have

$$
P_3 \varrho_4 = 0, \quad P_4 \varrho_4 = -\frac{9}{2}x^{-\frac{8}{3}}\sigma_4 + 3x^{-\frac{5}{3}}\sigma_3, \quad P_5 \varrho_4 = 3x^{-\frac{5}{3}}\sigma_2.
$$

The weighted degree of $\tau$ is $-\frac{17}{12}$. For $\varrho_4 = x^{-\frac{8}{3}}\sigma_4$, we see

$$
P_3 \varrho_4 = \frac{10}{3}x^{-\frac{8}{3}}\sigma_1, \quad P_4 \varrho_4 = -12x^{-\frac{8}{3}}\sigma_4 - 2x^{-\frac{8}{3}}\sigma_3 - \frac{10}{3}x^{-\frac{8}{3}}\sigma_1, \quad P_5 \varrho_4 = 8x^{-\frac{8}{3}}\sigma_2.
$$

Comparing (4.12) and (4.13), we put

$$
\psi = \tau - \frac{3}{8}\varrho_4 = x^{-\frac{5}{3}}\sigma_6 - \frac{3}{8}x^{-\frac{8}{3}}\sigma_4.
$$

Then we have

$$
P_3 \psi = -\frac{5}{4}x^{-\frac{8}{3}}\sigma_1, \quad P_4 \psi = \frac{15}{4}x^{-\frac{8}{3}}\sigma_3 - \frac{5}{4}x^{-\frac{8}{3}}\sigma_1, \quad P_5 \psi = 0.
$$

Next, for $\varrho_3 = x^{-\frac{8}{3}}\sigma_3$, we get

$$
P_3 \varrho_3 = 6x^{-\frac{8}{3}}\sigma_1, \quad P_4 \varrho_3 = -18x^{-\frac{8}{3}}\sigma_3 + x^{-\frac{8}{3}}\sigma_1, \quad P_5 \varrho_3 = 0.
$$

Comparing (4.14) and (4.15), we set

$$
\psi_6 = x^{-\frac{8}{3}}\sigma_6 - \frac{3}{8}x^{-\frac{8}{3}}\sigma_4 + \frac{5}{24}x^{-\frac{8}{3}}\sigma_3 + cx^{-\frac{11}{3}}\sigma_1.
$$
Then we have

\begin{equation}
P_4 \psi_6 = \left( \frac{35}{24} - 30c \right) x^{-\frac{8}{3}} \sigma_1, \quad P_i \psi_6 = 0 \ (i \neq 4).
\end{equation}

This implies $c = \frac{7}{144}$ and $s = -\frac{17}{12}$, which completes the computation of $\psi_6$.

Therefore the result (4.2) is obtained.

§ 4.2. Algebraic local cohomology solutions supported on $\Sigma_0$

Considering a form of algebraic local cohomology class supported on origin, we put

$$
\varphi = \sum a_{i,j,k} \left( x^i y^j z^k \right)
$$

Let us compute $\varphi$ annihilated by $J_f$ and $P_i$'s. By $J_f \varphi = 0$, i.e., $z^3 \varphi = (4y^3 + 2yz^2) \varphi = (3xz^2 + 2y^2z) \varphi = 0$, the form of $\varphi$ is specified as follows.

\begin{equation}
\varphi = \sum_{i \geq 1} \sum_{j=1}^{3} a_{i,j,1} \left[ \begin{array}{c}
1 \\
x^i y^j z \\
\end{array} \right] + \sum_{i \geq 1} \sum_{j=1}^{3} a_{i,j,2} \left[ \begin{array}{c}
1 \\
x^i y^j z^2 \\
\end{array} \right]
\end{equation}

\begin{equation}
+ \sum_{i \geq 1} a_{i,1,3} \left[ \begin{array}{c}
1 \\
x^i y z^3 \\
\end{array} \right] + a_{1,2,3} \left[ \begin{array}{c}
1 \\
x y^2 z^3 \\
\end{array} \right] + a_{1,4,1} \left[ \begin{array}{c}
1 \\
x y^4 z \\
\end{array} \right]
\end{equation}

with the conditions

\begin{equation}
3a_{i+1,1,3} + 2a_{i,3,2} = 0 \ (i \geq 1) \quad \text{and} \quad a_{1,2,3} + 2a_{1,4,1} = 0.
\end{equation}

Next, we seek $\varphi$ annihilated by $P_i$'s. For $\varphi$ of the form (4.17), we have

\begin{equation}
P_5 \varphi = \sum_{i \geq 1} \sum_{j=1}^{2} 2(3i - 1)a_{i,j+1,1} \left[ \begin{array}{c}
1 \\
x^i y^j z \\
\end{array} \right] - \sum_{i \geq 2} \sum_{j=1}^{2} 2(i - 1)a_{i-1,j+1,2} \left[ \begin{array}{c}
1 \\
x^i y^j z \\
\end{array} \right]
\end{equation}

\begin{equation}
+ \sum_{i \geq 1} \sum_{j=1}^{2} 2(3i - 2)a_{i,j+1,2} \left[ \begin{array}{c}
1 \\
x^i y^j z^2 \\
\end{array} \right] + \sum_{i \geq 1} a_{i,1,3} \left[ \begin{array}{c}
1 \\
x^i y^2 z \\
\end{array} \right]
\end{equation}

\begin{equation}
+ 2(a_{1,2,3} + 2a_{1,4,1}) \left[ \begin{array}{c}
1 \\
x y^3 z \\
\end{array} \right] - 2a_{1,2,3} \left[ \begin{array}{c}
1 \\
x^2 y^2 z \\
\end{array} \right].
\end{equation}
By (4.18) and the right-hand side of (4.19), $\varphi$ satisfying $P_5 \varphi = 0$ can be written in the form

$$
\varphi = \sum_{i \geq 1} a_{i,1,1} \left[ \frac{1}{x^i y z} \right] + \sum_{i \geq 1} a_{i,1,2} \left[ \frac{1}{x^i y z^2} \right] + \sum_{i \geq 1} a_{i,1,3} \left[ \frac{1}{x y z^3} \right] + a_{1,2,3} \left[ \frac{1}{x y z^3} \right] + a_{1,3,1} \left[ \frac{1}{x y z^3} \right] + a_{1,4,1} \left[ \frac{1}{x y z^3} \right] + a_{2,1,2} \left[ \frac{1}{x^2 y z^2} \right] + a_{2,2,2} \left[ \frac{1}{x^2 y z^2} \right] + a_{2,3,1} \left[ \frac{1}{x^2 y z^2} \right] + a_{2,4,1} \left[ \frac{1}{x^2 y z^2} \right] + a_{3,1,1} \left[ \frac{1}{x^3 y z} \right] + a_{3,2,1} \left[ \frac{1}{x^3 y z^2} \right] + a_{3,3,1} \left[ \frac{1}{x^3 y z^2} \right] + a_{3,4,1} \left[ \frac{1}{x^3 y z^2} \right],
$$

where

$$a_{1,1,3} + 4a_{1,3,1} = 0, \quad a_{1,2,3} + 2a_{1,4,1} = 0,
$$

$$a_{1,2,3} - 4a_{2,2,2} = 0, \quad a_{2,2,2} - 4a_{3,2,1} = 0.$$

Similarly, for $\varphi$ of the form (4.20), we have

$$P_3 \varphi = \sum_{i \geq 1} (3i - 2) a_{i,1,2} \left[ \frac{1}{x^i y z} \right] + 2a_{1,3,1} \left[ \frac{1}{x^2 y z} \right].
$$

Hence the following conditions are needed for $P_3 \varphi = 0$.

$$a_{i,1,2} = 0 \quad (i \neq 2) \quad \text{and} \quad a_{1,3,1} + 2a_{2,1,2} = 0.
$$

Finally, we calculate

$$P_4 \varphi = -\sum_{i \geq 1} 3(3i + 2) a_{i+1,1,1} \left[ \frac{1}{x^i y z} \right] - (2a_{1,3,1} - a_{2,1,2}) \left[ \frac{1}{x^2 y z} \right].
$$

Therefore the coefficients must satisfy

$$a_{i,1,1} = 0 \quad (i \neq 1,3) \quad \text{and} \quad 2a_{1,3,1} - a_{2,1,2} + 24a_{3,1,1} = 0.
$$

To sum up, the form of $\varphi$ is specified as

$$\varphi = a_{1,1,1} \left[ \frac{1}{x y z} \right] + a_{1,1,3} \left[ \frac{1}{x y z^3} \right] + a_{1,2,3} \left[ \frac{1}{x y z^3} \right] + a_{1,3,1} \left[ \frac{1}{x y z^3} \right] + a_{1,4,1} \left[ \frac{1}{x y z^3} \right] + a_{2,1,2} \left[ \frac{1}{x^2 y z^2} \right] + a_{2,2,2} \left[ \frac{1}{x^2 y z^2} \right] + a_{2,3,1} \left[ \frac{1}{x^2 y z^2} \right] + a_{2,4,1} \left[ \frac{1}{x^2 y z^2} \right] + a_{3,1,1} \left[ \frac{1}{x^3 y z} \right] + a_{3,2,1} \left[ \frac{1}{x^3 y z^2} \right] + a_{3,3,1} \left[ \frac{1}{x^3 y z^2} \right] + a_{3,4,1} \left[ \frac{1}{x^3 y z^2} \right].$$
with the conditions
\begin{equation}
(4.26)
\begin{aligned}
a_{1,2,3} &= -2a_{1,4,1}, & a_{2,2,2} &= -\frac{1}{2}a_{1,4,1}, & a_{3,2,1} &= -\frac{1}{8}a_{1,4,1}, \\
a_{1,3,1} &= -\frac{1}{4}a_{1,1,3}, & a_{2,1,2} &= \frac{1}{8}a_{1,1,3}, & a_{3,1,1} &= \frac{5}{192}a_{1,1,3}.
\end{aligned}
\end{equation}

Then we have $P_1\varphi = P_2\varphi = 0$ and

\begin{equation}
(4.27)
P_6\varphi = -(4s+3)a_{1,1,1} \left[ \begin{array}{c} 1 \\ x \\ y \\ z \end{array} \right]
- (4s+5) \left( a_{1,1,3} \left[ \begin{array}{c} 1 \\ xyz^3 \end{array} \right] + a_{1,3,1} \left[ \begin{array}{c} 1 \\ xy^3z \end{array} \right] + a_{2,1,2} \left[ \begin{array}{c} 1 \\ x^2yz^2 \end{array} \right] + a_{3,1,1} \left[ \begin{array}{c} 1 \\ x^3yz \end{array} \right] \right)
- (4s+6) \left( a_{1,2,3} \left[ \begin{array}{c} 1 \\ xy^2z^3 \end{array} \right] + a_{1,4,1} \left[ \begin{array}{c} 1 \\ xy^4z \end{array} \right] + a_{2,2,2} \left[ \begin{array}{c} 1 \\ x^2y^2z^2 \end{array} \right] + a_{3,2,1} \left[ \begin{array}{c} 1 \\ x^3y^2z \end{array} \right] \right).
\end{equation}

By (4.27), we have linearly independent three algebraic local cohomology solutions supported on $\Sigma_0$:

\begin{align*}
\varphi_1 &= \left[ \begin{array}{c} 1 \\ xyz \end{array} \right], \quad s = -\frac{3}{4}, \\
\varphi_2 &= \left[ \begin{array}{c} 1 \\ xy^3z \end{array} \right] - \frac{1}{4} \left[ \begin{array}{c} 1 \\ xy^3z \end{array} \right] + \frac{1}{8} \left[ \begin{array}{c} 1 \\ x^2yz^2 \end{array} \right] + \frac{5}{192} \left[ \begin{array}{c} 1 \\ x^3yz \end{array} \right], \quad s = -\frac{5}{4}, \\
\varphi_3 &= -2 \left[ \begin{array}{c} 1 \\ xy^2z^3 \end{array} \right] + \left[ \begin{array}{c} 1 \\ xy^4z \end{array} \right] - \frac{1}{2} \left[ \begin{array}{c} 1 \\ x^2y^2z^2 \end{array} \right] - \frac{1}{8} \left[ \begin{array}{c} 1 \\ x^3y^2z \end{array} \right], \quad s = -\frac{3}{2}.
\end{align*}

As a consequence, we have

\begin{align*}
M_{-\frac{3}{4}} &= DX\varphi_1, \quad M_{-\frac{5}{4}} = DX\varphi_2, \quad M_{-\frac{3}{2}} = DX\varphi_3.
\end{align*}

Note that all these three holonomic systems are simple as D-Module.

In 2010, K. Nishiyama and M. Noro ([8]) devised algorithms to compute local b-functions and stratifications associated with local b-functions. By using their algorithms implemented in Risa/Asir, we get $(4s+3)(4s+5)(2s+3)$ as a factor of the local b-function on the stratum $\Sigma_0$ in question of $f$. Thus, our results of computation are also consistent with the local b-function on the stratum $\Sigma_0$ of $f$.

In the rest of this section, we compute algebraic local cohomology solutions $\varphi_1$, $\varphi_2$ and $\varphi_3$ by using alternative method, already mentioned in the preceding section, that utilizes the homogeneity.

Recall $w_f = \frac{1}{4}(1, 1, 1)$. 
Case 1: Let us consider the case of \( s = -\frac{3}{4} \). The weighted degree of \( \left[ \begin{array}{c} 1 \\ x^i y^j z^k \end{array} \right] \) is \( -\frac{3}{4} \) only when \((i, j, k) = (1, 1, 1)\). From this, \( \varphi_1 \) is verified.

Case 2: We consider the case of \( s = -\frac{5}{4} \). The combination of \((i, j, k)\) satisfying \(-\frac{1}{4}(i + j + k) = -\frac{5}{4}\) is given by

\[
(i, j, k) = (1, 1, 3), (1, 2, 2), (1, 3, 1), (2, 1, 2), (2, 2, 1), (3, 1, 1).
\]

Note that the algebraic local cohomology classes associated with above are annihilated by \( J_f \). Set

\[
\varphi = a \left[ \begin{array}{c} 1 \\ x y^2 z^3 \end{array} \right] + b \left[ \begin{array}{c} 1 \\ x y^2 z^2 \end{array} \right] + c \left[ \begin{array}{c} 1 \\ x y^3 z \end{array} \right] + d \left[ \begin{array}{c} 1 \\ x^2 y z^2 \end{array} \right] + e \left[ \begin{array}{c} 1 \\ x^2 y^2 z \end{array} \right] + f \left[ \begin{array}{c} 1 \\ x^3 y z \end{array} \right]
\]

with indeterminate coefficients \( a, b, c, d, e, f \). A simple computation gives (4.28)

\[
P_3\varphi = b \left[ \begin{array}{c} 1 \\ x y^2 z \end{array} \right] + 2(c + 2d) \left[ \begin{array}{c} 1 \\ x^2 y z \end{array} \right],
\]

\[
P_4\varphi = (a - 2c - 12d) \left[ \begin{array}{c} 1 \\ x y z^2 \end{array} \right] + (2b - 15e) \left[ \begin{array}{c} 1 \\ x y^2 z \end{array} \right] - (2c - d + 24f) \left[ \begin{array}{c} 1 \\ x^2 y z \end{array} \right],
\]

\[
P_5\varphi = 2b \left[ \begin{array}{c} 1 \\ x y z^2 \end{array} \right] + (a + 4c) \left[ \begin{array}{c} 1 \\ x y^2 z \end{array} \right] - 2(b - 5e) \left[ \begin{array}{c} 1 \\ x^2 y z \end{array} \right].
\]

The relation of \( a, b, c, d, e, f \) are determined by the above.

Case 3: We finally consider the case of \( s = -\frac{3}{2} \). Since the combination of \((i, j, k)\) satisfying \(-\frac{1}{4}(i + j + k) = -\frac{3}{2}\) is given by

\[
(1, 1, 4), (1, 2, 3), (1, 3, 2), (1, 4, 1), (2, 1, 3),
\]

\[
(2, 2, 2), (2, 3, 1), (3, 1, 2), (3, 2, 1), (4, 1, 1).
\]

Let \( H_\Phi \) be the vector space generated by the set of \( \left[ \begin{array}{c} 1 \\ x^i y^j z^k \end{array} \right] \) whose index \((i, j, k)\) is in the combination above. Then, if we set

\[
H_{\Phi, J_f} = \{ \varphi \in H_\Phi \mid J_f \varphi = 0 \},
\]
the following set is a basis of $H_{\Phi,j_f}$.

$$\begin{pmatrix} 1 \\ xy^2z^2 \\ x^2y^2z^2 \\ x^2y^3z \\ x^2yz^2 \\ x^3y^2z \\ x^3y^2z^2 \\ x^4yz \end{pmatrix}, \quad \left\{ \begin{array}{l} 1 \\ xyz^4 \\ x^4yz^3 \\ x^4y^2z^3 \\ x^3y^2z^3 \\ x^2y^3z^3 \\ x^2y^2z^3 \\ x^4yz \end{array} \right\}$$

(4.29)

Therefore we set

$$\varphi = a \begin{pmatrix} 1 \\ xy^2z^2 \end{pmatrix} + b \begin{pmatrix} -2 \\ xy^2z^2 \end{pmatrix} + c \begin{pmatrix} 1 \\ x^2y^2z^2 \end{pmatrix} + d \begin{pmatrix} 1 \\ x^2y^2z^2 \end{pmatrix} + e \begin{pmatrix} 1 \\ x^2y^3z \end{pmatrix} + f \begin{pmatrix} 1 \\ x^3y^2z \end{pmatrix} + g \begin{pmatrix} 1 \\ x^3y^2z \end{pmatrix} + h \begin{pmatrix} 1 \\ x^4yz \end{pmatrix}$$

(4.30)

Here $a, b, c, d, e, f, g, h$ are undetermined coefficients. A direct computation gives

$$P_3\varphi = -a \begin{pmatrix} 1 \\ xy^2z^2 \end{pmatrix} + c \begin{pmatrix} 1 \\ x^2y^2z^2 \end{pmatrix} + 2(b + 2d) \begin{pmatrix} 1 \\ x^2y^2z^2 \end{pmatrix} + (4e + 7f) \begin{pmatrix} 1 \\ x^3yz \end{pmatrix}$$

$$P_4\varphi = (a + 2c) \begin{pmatrix} 1 \\ xy^2z^2 \end{pmatrix} - 6(b + 2d) \begin{pmatrix} 1 \\ x^2y^2z^2 \end{pmatrix} + 3(c + e) \begin{pmatrix} 1 \\ x^3yz \end{pmatrix}$$

(4.31)

$$P_5\varphi = (a + 2c) \begin{pmatrix} 1 \\ x^2y^2z^2 \end{pmatrix} + 4(b + 2d) \begin{pmatrix} 1 \\ x^2y^2z^2 \end{pmatrix} - \frac{8}{3} c - 10e \begin{pmatrix} 1 \\ x^2y^2z^2 \end{pmatrix}$$

Therefore local cohomology solution $\varphi_3$ is determined by (4.31).

**Remark 4.2.** In the case of weighted homogeneous singularities, if we know, in advance, $\beta \in \mathbb{C}$ such that $\text{Supp}(M_\beta)$ is $\Sigma_0$, we can efficiently calculate $M_\beta$ by the above method.
References