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On the asymptotic expansion of oscillatory integrals with smooth phases in two dimensions

By

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§ 1. Introduction

The main purpose of this article is to announce a part of results in our forthcoming paper [24]. In [24], the local zeta function will be mainly treated, while this article focuses on the study of oscillatory integrals. To be more specific, we investigate the behavior of *oscillatory integrals* in two dimensions:

$$I(t;\varphi) = \int_{\mathbb{R}^2} e^{itf(x_1,x_2)}\varphi(x_1,x_2)dx_1dx_2 \quad \text{as } t \to +\infty, \tag{1.1}$$

where f and φ are real-valued (C^∞) smooth functions defined on an open neighborhood U of the origin in $\mathbb{R}^2$ and the support of φ is contained in U. Here, f and φ are called the *phase* and the *amplitude* respectively. In this article, we always assume that

$$f(0,0) = 0 \quad \text{and} \quad \nabla f(0,0) = (0,0).$$

The above first condition is only for the convenience. The second condition follows from the principle of stationary phase, i.e., the main contribution in the behavior of oscillatory integrals is given by local properties of the phase f around its critical point. The investigations of the behavior of $I(t;\varphi)$ as $t \to +\infty$ are very important subjects occurring in harmonic analysis, partial differential equations, several complex variables,

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probability theory, number theory, etc. We refer to the beginning of chapter 6 of [1] and the end of chapter 8 of [31] for an overview of many of these applications.

In the real analytic phase case, precise results about asymptotic expansion of oscillatory integrals have been obtained. By using a famous Hironaka's resolution of singularities [13], it is known (see [19], [25]) that if f is real analytic and the support of φ is contained in a sufficiently small open neighborhood of the origin, then the integral $I(t;\varphi)$ has an asymptotic expansion of the form

$$I(t;\varphi) \sim \sum_{\alpha} (C_\alpha(\varphi) t^\alpha \log t + C'_\alpha(\varphi) t^\alpha) \quad \text{as } t \to +\infty, \tag{1.2}$$

where α runs through a finite number of arithmetic progressions, not depending on the amplitude, which consist of negative rational numbers. (Recently, Greenblatt [11] gives a new proof of the above result by using the elementary resolution of singularities constructed in his paper [8].) Moreover, in an important work of Varchenko [32] (see also [1]), the exponents of the terms, especially the leading terms, in the expansions (1.2) are precisely expressed by using some geometrical data of Newton polyhedra of the phases. Since the geometrical data of Newton polyhedra largely depends on chosen coordinates, it is an important problem to look for *good* coordinates in the asymptotic analysis of oscillatory integrals. Indeed, Varchenko showed that so called *adapted coordinates* (see Section 2.4) always exist in the two-dimensional case and he uses these coordinates to express precise behavior of oscillatory integrals. More recently, the above investigation of Varchenko has been developed in [28], [9], [14], [15], [16], [12], [5] from another point of view, which is inspired by the work of Phong and Stein on oscillatory integral operators in their seminal paper [27]. In particular, Greenblatt [9] introduces *superadapted coordinates*, which is more refined version of the above adapted coordinates (see Section 2.4) and shows that these coordinates are very useful in the two-dimensional case.

More generally, let us consider the case when the phase is a (C^∞) smooth function. Since resolution of singularities by using rational transform may not exist in the smooth case, it is difficult to investigate oscillatory integrals in the sense of *asymptotic expansion* like (1.2). Indeed, there are only a few results in this sense, which need strong assumptions (see [30], [20]). On the other hand, there are interesting results about the behavior of oscillatory integrals in the form of *estimates* or *limits* in [9], [14], [16], which have been already referred as above. The existences of adapted coordinates or superadapted coordinates are shown in the smooth case ([9], [15]) and the behavior of oscillatory integrals is investigated by using these coordinates in the above papers (see Remark 3 in Section 3). They treat with the case when the behaviors are similar to those in the real analytic case. Here, it must be noticed that there exists an example of non-real analytic phase, in which the behavior of oscillatory integral is quite different from that in the real analytic case, which is shown by Iosevich and Sawyer [18] (see

Remark 5 in Section 3).

As is observed from the above example given by Iosevich and Sawyer, it seems difficult to understand the asymptotic behavior of oscillatory integrals in the general smooth case. In this article, we consider the case when oscillatory integrals have a relatively good asymptotic behavior. To be more precise, we give a sufficient condition on f for which oscillatory integrals can be asymptotically expanded similarly to the form (1.2), but these expansions may have only finitely terms. This condition is naturally characterized by using the Newton polyhedron of f with respect to superadapted coordinates.

In our analysis of oscillatory integrals, it is crucial to understand geometrical properties of the singular variety

$$V_f = \{(x_1, x_2) \in U : f(x_1, x_2) = 0\},$$

where f, U are as in (1.1). Indeed, the most important part of Varchenko's analysis in [32] (see also [1], [20]) is to quantitatively express f in a normal crossing form by using a toric resolution of singularities of f. Furthermore, these properties have been more generally understood in the real analytic case ([27]). Unfortunately, it is impossible to get complete resolution of singularities of f by using rational transform in the general smooth case. But some weak condition implies that f can be expressed in an "almost" normal crossing form by using a toric blowing-up in [32]. Note that the result of Rychkov [29] plays an important role in this construction. We apply a Van der Corput-type lemma in this form and can get necessary analytical information. Note that this kind of lemma plays important roles in the analysis in [9], [14], [15], [16], [12].

It is known (c.f. [17], [1]) that the asymptotic analysis of oscillatory integral (1.1) is closely related to an investigation of the poles of the *local zeta function*

$$Z(s; \varphi) = \int_{\mathbb{R}^2} |f(x_1, x_2)|^s \varphi(x_1, x_2) dx_1 dx_2, \tag{1.3}$$

where f, φ are the same as in (1.1). Our substantial analysis is to investigate the meromorphy and the properties of poles of the functions $Z_\pm(s; \varphi)$, which are similar to the above local zeta function, under associated assumptions.

As was mentioned in the beginning, this article announces a part of results about oscillatory integrals in [24]. Since actual proof of main results is too long, we only give its sketch in this article. On the other hand, this article contains an original result in Section 5, which can be easily obtained from an application of the main theorem.

Notation and symbols.

- We denote by $\mathbb{Z}_+, \mathbb{R}_+$ the subsets consisting of all nonnegative numbers in $\mathbb{Z}$, $\mathbb{R}$, respectively. For $s \in \mathbb{C}$, $\mathrm{Re}(s)$ expresses the real part of s.

- For $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, define $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2}$.
- For $A, B \subset \mathbb{R}^2$, we set $A + B = \{a + b \in \mathbb{R}^2 : a \in A \text{ and } b \in B\}$.

§ 2. Preliminaries

§ 2.1. Polyhedra

Let us explain fundamental notions in the theory of convex polyhedra in two dimensions, which are necessary for our investigation. Refer to [33] for general theory of convex polyhedra.

For $(a, l) = ((a_1, a_2), l) \in \mathbb{R}^2 \times \mathbb{R}$, let $H(a, l)$ and $H^+(a, l)$ be a straight line and a closed halfspace in $\mathbb{R}^2$ defined by

$$H(a, l) := \{(x_1, x_2) \in \mathbb{R}^2 : a_1 x_1 + a_2 x_2 = l\},$$
$$H^+(a, l) := \{(x_1, x_2) \in \mathbb{R}^2 : a_1 x_1 + a_2 x_2 \geq l\},$$

respectively. A (*convex rational*) *polyhedron* is an intersection of closed halfspaces: a set $P \subset \mathbb{R}^2$ presented in the form $P = \bigcap_{j=1}^N H^+(a^j, l_j)$ for some $a^1, \ldots, a^N \in \mathbb{Z}^2$ and $l_1, \ldots, l_N \in \mathbb{Z}$.

Let P be a polyhedron in $\mathbb{R}^2$. A pair $(a, l) \in \mathbb{Z}^2 \times \mathbb{Z}$ is said to be *valid* for P if P is contained in $H^+(a, l)$. A *face* of P is any set of the form $F = P \cap H(a, l)$, where (a, l) is valid for P. Since $(0, 0)$ is always valid, we consider P itself as a trivial face of P; the other faces are called *proper faces*. The *dimension* of a face F is the dimension of its affine hull of F (i.e., the intersection of all affine flats that contain F). The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively. In particular, the noncompact edge of P which is contained in the line $\{(c, \alpha) \in \mathbb{R}^2 : \alpha \in \mathbb{R}\}$ (resp. $\{(\alpha, c) \in \mathbb{R}^2 : \alpha \in \mathbb{R}\}$) with some $c \in \mathbb{R}$ is called the *vertical edge* (resp. *the horizontal edge*).

§ 2.2. Newton polyhedra

Let f be a real-valued smooth function defined on a neighborhood of the origin in $\mathbb{R}^2$, which has the Taylor series at the origin:

$$f(x_1, x_2) \sim \sum_{\alpha \in \mathbb{Z}_+^2} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2}. \tag{2.1}$$

The *Newton polyhedron* $\Gamma_+(f)$ of f is defined by the convex hull of the set $\bigcup\{\alpha + \mathbb{R}_+^2; c_\alpha \neq 0\}$. Of course, the Newton polyhedron is a polyhedron. We say that f is *flat* if $\Gamma_+(f) = \emptyset$ (i.e., all derivatives of f vanish at the origin). The *center* of the boundary of the Newton polyhedron $\Gamma_+(f)$ is the point at which the bisectrix $\alpha_1 = \alpha_2$ intersects the boundary of $\Gamma_+(f)$. The *Newton distance* of f is given by the coordinate d of the center (d,d), which is denoted by $d(f)$. Of course, this distance depends on the coordinates. In order to make clear the chosen coordinate x, we sometimes write this distance as $d_x(f)$. The *principal face* γ_* of the Newton polyhedron of f is the smallest face of $\Gamma_+(f)$ containing the center $(d(f),d(f))$. The *multiplicity* of the Newton distance is given by the codimension of γ_*, which is denoted by $m(f)$. That is to say,

$$\begin{cases} m(f)=1 & \text{if } \gamma_* \text{ is an edge of } \Gamma_+(f), \\ m(f)=2 & \text{if } \gamma_* \text{ is a vertex of } \Gamma_+(f). \end{cases} \tag{2.2}$$

§ 2.3. The γ-part

Let f be a nonflat real-valued smooth function defined on an open neighborhood V of the origin in $\mathbb{R}^2$ with the Taylor series (2.1).

Definition 2.1. Let γ be a face of $\Gamma_+(f)$. We say that f *admits the γ-part* on an open neighborhood $U \subset V$ of the origin if for any x in U the limit:

$$\lim_{t\to 0} \frac{f(t^{a_1}x_1, t^{a_2}x_2)}{t^l} \tag{2.3}$$

exists for *all* valid pairs $(a,l)=((a_1,a_2),l)\in \mathbb{Z}_+^2\times\mathbb{Z}_+$ defining γ. When f admits the γ-part, it is known in [20], Proposition 5.2 (iii), that the above limits take the same value for any valid pair $(a,l)\in\mathbb{Z}_+^2\times\mathbb{Z}_+$ defining γ, which is denoted by $f_\gamma(x)$. Let us consider f_γ as a function on U, which is called the γ-part of f on U.

We summarize important properties of the γ-part. See [20] for the details.

1. The γ-part f_γ is a smooth function defined on U.
2. If f admits the γ-part f_γ on U, then f_γ has the quasihomogeneous property:
$$f_\gamma(t^{a_1}x_1, t^{a_2}x_2) = t^l f_\gamma(x_1,x_2) \quad \text{for} \quad t\in(0,1) \text{ and } (x_1,x_2)\in U,$$
where $(a,l)=((a_1,a_2),l)\in\mathbb{Z}_+^2\times\mathbb{Z}_+$ is a valid pair defining γ.
3. For a compact face γ of $\Gamma_+(f)$, f always admits the γ-part near the origin. Then f_γ is the same as the γ-part of f defined in [32], [1], i.e., $f_\gamma(x_1,x_2)=\sum_{\alpha\in\gamma\cap\mathbb{Z}_+^2} c_\alpha x_1^{\alpha_1}x_2^{\alpha_2}$.
4. If f is real analytic, then f always admits the γ-part on U for any face γ of $\Gamma_+(f)$. Moreover, f_γ is real analytic and is equal to a convergent power series $\sum_{\alpha\in\gamma\cap\mathbb{Z}_+^2} c_\alpha x_1^{\alpha_1}x_2^{\alpha_2}$ on some neighborhood of the origin.
5. Let f be a smooth function and γ a noncompact edge of $\Gamma_+(f)$. Then, f does not admit the γ-part in general. If f admits the γ-part, then the Taylor series of f_γ at the origin is $\sum_{\alpha\in\gamma\cap\mathbb{Z}_+^2} c_\alpha x_1^{\alpha_1}x_2^{\alpha_2}$.

6. When a noncompact edge γ of $\Gamma_+(f)$ is contained in some coordinate axis, f always admits the γ-part on U. Indeed, for every valid pair (a, l) defining γ, we have $l = 0$ and so the limit (2.3) exists.

7. When f is smooth and γ is a noncompact edge, there are many examples in which f does not admit the γ-part. For example, consider the case when $f(x_1, x_2) = x_2^2 + e^{-1/|x_1|^p}$ with $p > 0$ and the face γ defined by $\{(\alpha_1, \alpha_2) : \alpha_1 \geq 0, \alpha_2 = 2, \}$. See also Remark 5 in Section 3 below.

§ 2.4. Adapted coordinates and superadapted coordinates

Let f be a nonflat real-valued smooth function defined near the origin in $\mathbb{R}^2$ with $f(0, 0) = 0$ and $\nabla f(0, 0) = (0, 0)$. The *height* of real analytic (resp. smooth) function f is defined by

$$h(f) := \sup_x d_x(f),$$

where the supremum is taken over all local analytic (resp. smooth) coordinate systems x at the origin and $d_x(f)$ is the Newton distance of f in the coordinate x. We easily see that $h(f) \geq 1$ since $f(0, 0) = 0$ and $\nabla f(0, 0) = (0, 0)$.

Definition 2.2. A coordinate x is *adapted* to f (or f is in an *adapted* coordinate x) if $h(f) = d_x(f)$.

When f is real analytic, the existence of adapted coordinates is shown by Varchenko [32] by means of two-dimensional resolution of singularities and by Phong-Stein-Sturm [28] by means of the Puiseux series expansion of roots of f. Moreover, Ikromov and Müller [15] apply Varchenko's algorithm for the construction of the coordinates to the method of Phong-Stein [27] and give stronger results for the existence and the criterion for the adaptedness. Indeed, they shows the existence in the case when f is smooth. We remark that in dimension higher than two, adapted coordinates may not exist, as Varchenko shows in [32].

We gives some remarks on adapted coordinates. See [15] for the details.

1. When γ_* is a vertex or a noncompact edge of $\Gamma_+(f)$ in a coordinate, this coordinate is adapted to f.

2. A coordinate is adapted to f if and only if for any compact edge γ of $\Gamma_+(f)$ containing the center of the boundary of $\Gamma_+(f)$, any real zero of the functions $f_\gamma(\pm 1, \cdot)$ or $f_\gamma(\cdot, \pm 1)$ has order less than or equal to $d(f)$.

3. When f is in adapted coordinates, if a compact face γ of $\Gamma_+(f)$ does not contain the center of the boundary of $\Gamma_+(f)$, then any real zero of $f_\gamma(\cdot, \pm 1)$ and $f_\gamma(\pm 1, \cdot)$ has order less than $d(f)$.

4. The multiplicity $m(f)$ of $d(f)$ depends on taking adapted coordinates.

Greenblatt [9] introduces the following special adapted coordinates, called *superadapted coordinates.* Though his coordinates are slightly different from the adapted coordinates, they are much more useful for our analysis.

Definition 2.3. A coordinate x is *superadapted* to f (or f is said to be in a *superadapted* coordinate x) if for any compact edge γ of $\Gamma_+(f)$ containing the center of the boundary of $\Gamma_+(f)$, any real zero of the functions $f_\gamma(\pm 1, \cdot)$ or $f_\gamma(\cdot, \pm 1)$ has order less than $d_x(f)$.

For any smooth function f, the existence of superadapted coordinates is shown by Greenblatt [9].

We gives some remarks on superadapted coordinates. See [9] for the details.

1. Any superadapted coordinate system is adapted.
2. If the principal face of the Newton polyhedron $\Gamma_+(f)$ is a noncompact edge, then the function f is in superadapted coordinates.
3. For any superadapted coordinates, the multiplicity $m(f)$ of $d(f)$ is uniquely determined. (i.e., The multiplicity $m(f)$ does not depend on taking superadapted coordinates.)

§ 3. Main results

In this section, we always assume that f and φ satisfy the following: Let U be an open neighborhood of the origin in $\mathbb{R}^2$.

(A) f is a nonflat real-valued (C^∞) smooth function defined on U satisfying that $f(0,0) = 0$ and $\nabla f(0,0) = (0,0)$;

(B) φ is a real-valued (C^∞) smooth function whose support is contained in U.

As was mentioned in the Introduction, when the phase is real analytic, there exists an asymptotic expansion of the form (1.2). On the other hand, in the smooth case, $I(t;\varphi)$ does not always admit the asymptotic expansion (1.2) (see Remark 5, below). But, as the following theorem shows, under some assumption, a similar expansion exists but this expansion may not have infinitely many terms. In the theorem, the first coefficient of this expansion is explicitly computed.

Theorem 3.1. *Suppose that f is in a superadapted coordinate x. We denote $h := h(f)$ and $m := m(f)$ for simplicity. We assume that $h > 1$ and that if a noncompact* *edge γ of $\Gamma_+(f)$ contains the center of the boundary of $\Gamma_+(f)$, then $f(x_1,x_2)$ admits the γ-part on U. If the support of φ is contained in a sufficiently small neighborhood of the origin, the following holds. There exist a positive number δ, a subset S_δ in $\mathbb{Q}$, depending only on f, such that*

$$\left| I(t;\varphi) - \sum_{\alpha \in S_\delta} (C_\alpha(\varphi) t^\alpha \log t + C'_\alpha(\varphi) t^\alpha) \right| < C t^{-1/h-\delta-\epsilon}, \tag{3.1}$$

where $C_\alpha(\varphi)$ are constants and C is a positive constant and ϵ is any small positive number. Moreover, the set $S_\delta \subset \mathbb{Q}$ is the restriction of S to the region $[-1/h-\delta, -1/h]$, where S consists of a finite number of arithmetic progressions produced by some algorithm, which can be given by using the theory of toric varieties based on Newton polyhedra concerning f.

Concerning the first term in the expansion (3.1), we have the limit:

$$\lim_{t\to\infty} t^{1/h} (\log t)^{-m+1} \cdot I(t;\varphi) = C(\varphi), \tag{3.2}$$

where $C(\varphi)$ is explicitly given as follows: For a negative number A, let $A^{-1/h} := |A|^{-1/h} e^{-\pi i/h}$ and $\Gamma(\cdot)$ is the gamma function.

(a) Suppose that the principal face γ_* of $\Gamma_+(f)$ is a compact edge defined by a valid pair $(a,l) = ((a_1,a_2),l) \in \mathbb{Z}_+^2 \times \mathbb{Z}_+$. Then

$$C(\varphi) = \frac{\Gamma(1/h) e^{\pi i/2h}}{h(a_2/a_1+1)} \varphi(0,0) \int_{-\infty}^{\infty} \left(f_{\gamma_*}(1,u)^{-1/h} + f_{\gamma_*}(-1,u)^{-1/h} \right) du; \tag{3.3}$$

(b) Suppose that the principal face γ_* is the vertical edge. Then

$$C(\varphi) = \frac{\Gamma(1/h) e^{\pi i/2h}}{h} \int_{-\infty}^{\infty} \left(f_{\gamma_*}(1,u)^{-1/h} + f_{\gamma_*}(-1,u)^{-1/h} \right) \varphi(0,u) du; \tag{3.4}$$

(c) Suppose that the principal face γ_* is the horizontal edge. Then

$$C(\varphi) = \frac{\Gamma(1/h) e^{\pi i/2h}}{h} \int_{-\infty}^{\infty} \left(f_{\gamma_*}(u,1)^{-1/h} + f_{\gamma_*}(u,-1)^{-1/h} \right) \varphi(u,0) du; \tag{3.5}$$

(d) Suppose that the principal face γ_* of $\Gamma_+(f)$ is a vertex. Let $((a_1,a_2),l_1)$ and $((b_1,b_2),l_2)$ be valid pairs in $\mathbb{Z}_+^2 \times \mathbb{Z}_+$ for $\Gamma_+(f)$ which define the two edges containing γ_* and satisfy that $0 \le a_2/a_1 \le b_2/b_1 \le \infty$. If h is odd, then

$$C(\varphi) = \frac{4\Gamma(1/h)\cos(\pi/2h)}{h^2} \varphi(0,0) |f_{\gamma_*}(1,1)|^{-1/h} \left(\frac{1}{a_2/a_1+1} - \frac{1}{b_2/b_1+1} \right). \tag{3.6}$$

If h is even, then

$$C(\varphi) = \frac{4\Gamma(1/h) e^{\pi i/2h}}{h^2} \varphi(0,0) |f_{\gamma_*}(1,1)|^{-1/h} \left(\frac{1}{a_2/a_1+1} - \frac{1}{b_2/b_1+1} \right). \tag{3.7}$$

In particular, $C(\varphi)$ does not vanish if $\varphi(0,0)$ is positive (resp. negative) and φ is nonnegative (resp. nonpositive) on U.

Remarks

1. We see that the integrals in (3.3), (3.4), (3.5) are convergent since any real zero of $f_{\gamma_*}(\pm 1, \cdot)$ or $f_{\gamma_*}(\cdot, \pm 1)$ has order less than the height $h(f)$. Indeed, in superadapted coordinates, any real zero of $f_\gamma(\pm 1, \cdot)$ or $f_\gamma(\cdot, \pm 1)$ for each compact face γ has order less than $h(f)$. When the principal face is noncompact, careful computations gives the same assertion under the assumption of the theorem.

2. It is shown in [9] that in a superadapted coordinate to f satisfying $\nabla f(0,0) = (0,0)$, the critical point of f at the origin is nondegenerate, i.e., $\nabla^2 f(0,0)$ is invertible, if and only if $h(f) = 1$. In general nondegenerate case, the asymptotic expansions of oscillatory integrals are completely computed by using the Morse lemma and Fresnel integrals (See Section 2.3, Chapter VIII in [31]).

3. Until now, there have been many studies concerning the limit (3.2) in [32],[6],[9],[16], etc. In the general dimensional real analytic phase case, Varchenko [32] gives a sufficient conditions for determining the leading terms of the asymptotic expansions of oscillatory integrals under some nondegeneracy conditions. Moreover, the first coefficient $C(\varphi)$ is computed in many cases in [30], [6], [26], [20].

   In the two-dimensional case, there have been interesting investigation without the above nondegeneracy condition. In order to show the strength of our result, let us recall important known results. In [9], Greenblatt introduces superadapted coordinates and obtain similar results about the real analytic phase case to that in [32], [6] by using the properties of superadapted coordinates. In the same paper, furthermore, he also has the following weaker results with nonflat smooth phases: Suppose that $h(f) > 1$ and that $\varphi(0,0) > 0$ and φ is nonnegative on U. If the principal face is compact, then

$$\limsup_{t\to\infty} \left| t^{1/h(f)} (\log t)^{-m(f)+1} \cdot I(t;\varphi) \right| > 0;$$

   If the principal face is noncompact, then for any $\delta > 0$,

$$\limsup_{t\to\infty} \left| t^{1/h(f)+\delta} \cdot I(t;\varphi) \right| = \infty.$$

   In the smooth phase case, Ikromov and Müller [16] prove that if the principal face is compact and $h(f) > 1$, then

$$\lim_{t\to\infty} t^{1/h(f)} (\log t)^{-m(f)+1} \cdot I(t;\varphi) = C\varphi(0,0),$$

where C is a nonzero constant and depends only on the phase f (but they do not give an explicit value).

4. In some special smooth phase cases, asymptotic expansions of $I(t;\varphi)$ have been computed in the form (1.2). In [30], Schulz obtains the asymptotic expansion of $I(t;\varphi)$ when the phase is convex and satisfies the finite line type condition. The authors [20] introduce the class of smooth functions admitting the γ-parts for any face γ of the Newton polyhedron of f and naturally generalize the general dimensional results of Verchenko [32] in the case when f belongs to this class under some nondegeneracy condition. We remark that the assumption of Theorem 3.1 is much weaker than that of the corresponding two-dimensional results in [20].

5. Without the assumption in the theorem, the limit (3.2) may not hold. In fact, Iosevich and Sawyer [18] give an estimate from the above in the case when the phase is $f(x_1,x_2) = x_2^2 + e^{-1/|x_1|^p}$ with $p > 0$. More precisely, the following limit is shown in [23]:

$$\lim_{t\to\infty} t^{1/2}(\log t)^{1/p}\cdot I(t;\varphi) = 2\sqrt{\pi}e^{i\pi/4}\cdot\varphi(0,0). \tag{3.8}$$

The above limit implies that $I(t;\varphi)$ does not have the asymptotic expansion of the form (1.2). In this example, the coordinate x is superadapted to f since the principal face is a noncompact edge, and we see that $h(f) = 2$. (See also Remark 7 in Section 2.3.) When the case of smooth functions is treated, it must be noticed that the geometrical information of the Newton polyhedra does not always give sufficient analytical information. To be more specific, though flat functions do not appear in the information of the Newton polyhedron, they may affect the behavior of the oscillatory integrals. In particular, it must be careful to deal with the case when the complement of their Newton polyhedra in $\mathbb{R}_+^n$ is noncompact.

§ 4. Outline of a proof of Theorem 3.1

Let us overview our proof of Theorem 3.1. A detailed proof will appear in the paper [24].

§ 4.1. Local zeta-type functions $Z_\pm(s;\varphi)$

In order to prove the theorem, we investigate local zeta-type functions of the form

$$\begin{aligned} Z_+(s;\varphi) &= \int_{\mathbb{R}^2} f(x_1,x_2)_+^s\varphi(x_1,x_2)dx_1dx_2, \\ Z_-(s;\varphi) &= \int_{\mathbb{R}^2} f(x_1,x_2)_-^s\varphi(x_1,x_2)dx_1dx_2, \end{aligned} \tag{4.1}$$

where $s \in \mathbb{C}$, f and φ satisfy the conditions (A),(B) in Section 3 and

$$f(x_1, x_2)_+ = \max\{f(x_1, x_2), 0\}, \quad f(x_1, x_2)_- = \max\{-f(x_1, x_2), 0\}.$$

Since the integrals in (4.1) converge locally uniformly on the region Re(s) > 0, which implies that $Z_\pm(\cdot;\varphi)$ can be regarded as holomorphic functions there. Moreover, in the case when f is real analytic, it is shown in [3], [2] (see also Section 4.3) that if the support of φ is sufficiently small, then $Z_\pm(s;\varphi)$ can be analytically continued as meromorphic functions to the whole complex plane.

§ 4.2. Relationships between $I(t;\varphi)$ and $Z_\pm(s;\varphi)$

Let us overview the relationships between $I(t;\varphi)$ and $Z_\pm(s;\varphi)$ (see [17], [1] for the details).

Suppose that the support of φ is sufficiently small. Define the Gelfand-Leray function: $K : \mathbb{R} \to \mathbb{R}$ as

$$K(c) = \int_{W_c} \varphi(x_1, x_2)\omega,$$

where $W_c = \{x \in \mathbb{R}^2 : f(x_1, x_2) = c\}$ and ω is the line element on W_c which is determined by $df \wedge \omega = dx_1 \wedge dx_2$. Here, $I(t;\varphi)$ and $Z_\pm(s;\varphi)$ can be expressed by using $K(c)$: Changing the integral variables in (1.1) and (4.1), we have

$$I(t;\varphi) = \int_{-\infty}^{\infty} e^{itc} K(c)dc = \int_0^\infty e^{itc} K(c)dc + \int_0^\infty e^{-itc} K(-c)dc, \tag{4.2}$$

$$Z_\pm(s;\varphi) = \int_0^\infty c^s K(\pm c)dc, \tag{4.3}$$

respectively. Applying the inverse formula of the Mellin transform to (4.3), we have

$$K(\pm c) = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} Z_\pm(s;\varphi) c^{-s-1} ds, \tag{4.4}$$

where $r > 0$ and the integral contour follows the line $\mathrm{Re}(s) = r$ upwards. Let us consider the case that $Z_\pm(s;\varphi)$ are meromorphic functions on $\mathbb{C}$ and their poles exist on the negative part of the real axis. By deforming the integral contour as r tends to $-\infty$ in (4.4), the residue formula gives the asymptotic expansions of $K(c)$ as $c \to \pm 0$. (Of course, it must be checked that this deformation can be done.) Substituting these expansions of $K(c)$ into (4.2), we can get an asymptotic expansion of $I(t;\varphi)$ as $t \to +\infty$.

§ 4.3. Meromorphic continuation of $Z_\pm(s;\varphi)$

The main theorem in [7] due to Greenblatt implies that $Z_\pm(s;\varphi)$ can be analytically continued as holomorphic functions to the region $\mathrm{Re}(s) > -1/h(f)$, where $h(f)$ is the height of f. In the same paper, Greenblatt gives an example in which $Z_{\pm}(s;\varphi)$ cannot be meromorphically continued to a wider region $\mathrm{Re}(s) > -1/h(f) - \epsilon$ for any positive number ϵ. But we show that $Z_{\pm}(s;\varphi)$ can be meromorphically continued in a wider region under the assumption in Theorem 3.1 as follows.

Theorem 4.1. *Let f,h,m be the same as in Theorem 3.1. If the support of φ is contained in a sufficiently small neighborhood of the origin, then the following hold:*

(i) There exists a positive number δ independent of φ such that the functions $Z_{\pm}(s;\varphi)$ can be analytically continued as meromorphic functions to the region $\mathrm{Re}(s) >$ *$-1/h - \delta$.*

(ii) The poles of the functions $Z_{\pm}(s;\varphi)$ in the region $\mathrm{Re}(s) > -1/h - \delta$ *belong to finitely many arithmetic progressions which are precisely obtained by using the theory of toric varieties based on the geometry of the Newton polyhedron concerning f.*

(iii) When $Z_{\pm}(s;\varphi)$ have poles at $s = -1/h$, their orders are at most m. Define the coefficients of the poles of $Z_{\pm}(s;\varphi)$ at $s = -1/h$:

$$C_{\pm}(\varphi) := \lim_{s \to -1/h} (s + 1/h)^m Z_{\pm}(s;\varphi).$$

If $\varphi(0,0)$ is positive (resp. negative) and φ is nonnegative (resp. nonpositive) on U, then $C_{\pm}(\varphi)$ are nonnegative (resp. nonpositive) and, moreover, $C_{+}(\varphi) + C_{-}(\varphi)$ is positive (resp. negative). (The explicit formulae for $C_{\pm}(\varphi)$ can be given, but we omit them here (see [24]).

Through the relationship between $I(t;\varphi)$ and $Z_{\pm}(s;\varphi)$ in the previous subsection, we can obtain Theorem 3.1 by using the above theorem.

§ 4.4. Geometrical properties of the singular variety V_f

When f is a monomial, we can completely see the meromorphic continuation of $Z_{\pm}(s;\varphi)$ on $\mathbb{C}$ through an elementary method (see [1] etc.). By observing this typical case, it is an essential issue to look for an appropriate map such that f can be locally expressed in a normal crossing form. In the other words, this issue is to construct an appropriate resolution of singularity of the variety

$$V_f = \{(x_1, x_2) \in U : f(x_1, x_2) = 0\}.$$

In the real analytic case, Hironaka's theorem [13] implies an abstract existence of this resolution. To be more specific, Varchenko [32] applies the theory of toric varieties based on the geometry of the Newton polyhedron of f and obtains quantitative resolution of singularities of V_f when f is real analytic and satisfies some nondegeneracy condition.

As a result, he gives a precise result about the meromorphic continuation of $Z_{\pm}(s;\varphi)$. More recently, Phong and Stein [27] give another approach to understand the geometry of the variety V_f. (They use Puiseux series expansion to express V_f.)

In the smooth case, there does not always exist resolution of singularity of V_f by using rational transform. But Rychkov [29] improves the result of Phong and Stein in the smooth case. Applying this Rychkov's result to Varchenko's analysis using toric resolution, we show that "almost" resolution of singularities of V_f can be obtained under the assumption in Theorem 3.1: if a noncompact edge γ of $\Gamma_+(f)$ contains the center of the boundary of $\Gamma_+(f)$, then $f(x_1,x_2)$ admits the γ-part.

§ 4.5. Outline of a proof of Theorem 4.1

By using the above "almost" resolution of singularity of V_f, we split up the integrals $Z_{\pm}(s;\varphi)$ to many integrals with suitable cut-off functions. Roughly speaking, there are three kinds of integrals, which can be treated as follows.

1. In the integrals of the first type, f can be expressed in a normal crossing form. These integrals can be meromorphically continued on the whole complex plane $\mathbb{C}$ and the properties of poles can be precisely seen.

2. In the integrals of the second type, f still has some kind of singularities. These integrals can be represented in some *weighted* form. Applying the analysis in [4], [22], we can see the situation of these meromorphic continuations.

3. In the integrals of the third type, flat functions exist in f. Applying a Van der Corput-type lemma, we show these integrals are negligible for our analysis. This kind of lemma plays useful roles in the analysis in [9], [14], [15], [16], [12],

After careful analysis of each integrals, we can obtain the properties of $Z_{\pm}(s;\varphi)$ in the theorem. In this analysis, properties of superadapted coordinates are deeply used.

§ 5. Applications to the higher-dimensional case

Let $U \subset \mathbb{R}^n$ be an open neighborhood of the origin and f, φ smooth functions defined on U and the support of φ is contained in U. For a smooth function f, the Newton polyhedron $\Gamma_+(f)$, the γ-part f_γ with a compact face γ, the Newton distance $d(f)$ and its multiplicity $m(f)$ can be naturally generalized in the higher-dimensional case (see [20] for their exact definitions). For $x=(x_1,\dots,x_n)\in\mathbb{R}^n$, we similarly define the oscillatory integral:

$$I(t;\varphi)=\int_{\mathbb{R}^n} e^{itf(x)}\varphi(x)dx \qquad t>0,$$

In the general dimensional case, Varchenko [32] shows the following.

Theorem 5.1 ([32]). *Suppose that*

1. *f is real analytic on U;*
2. *$d(f) > 1$;*
3. *f is nondegenerate with respect to $\Gamma_+(f)$, i.e., for every compact face γ of $\Gamma_+(f)$, the polynomial $f_\gamma(x)$ satisfies*

$$\nabla f_\gamma = \left(\frac{\partial f_\gamma}{\partial x_1}, \dots, \frac{\partial f_\gamma}{\partial x_n}\right) \neq (0, \dots, 0) \quad \text{on the set } (\mathbb{R} \setminus \{0\})^n. \tag{5.1}$$

If the support of φ is sufficiently small, then

$$\lim_{t \to \infty} t^{1/d(f)} (\log t)^{-m(f)+1} \cdot I(t; \varphi) = C(\varphi), \tag{5.2}$$

where $C(\varphi)$ is a nonzero constant when $\varphi(0) > 0$ and $\varphi \geq 0$ on U.

We remark that the equation (5.2) holds when the phase belongs to a certain class of smooth functions satisfying the above nondegeneracy condition (5.1) (see [20]).

As an easy application of Theorem 3.1, we can see that the equation (5.2) holds in the three-dimensional case when the phase does not always belong to the above class of smooth functions. (That is to say, this phase does not always satisfy the nondegeneracy condition (5.1).) Hereafter, let f, φ, h, m be the same as in Theorem 3.1 and let g, ψ be smooth functions defined on $\mathbb{R}$ satisfying that $g(0) = g'(0) = 0$ and the support of g is contained in a small open interval of the origin. Let us define

$$F(x) = F(x_1, x_2, x_3) = f(x_1, x_2) + g(x_3),$$
$$\Phi(x) = \Phi(x_1, x_2, x_3) = \varphi(x_1, x_2)\psi(x_3).$$

We consider the integral of the form:

$$I_F(t; \Phi) = \int_{\mathbb{R}^3} e^{itF(x)} \Phi(x) dx \qquad t > 0.$$

As far as we know, the following theorem cannot be covered by the earlier investigations of oscillatory integrals.

Theorem 5.2. *Suppose that $g(0) = g'(0) = \cdots = g^{(k-1)}(0) = 0$ and $g^{(k)}(0) \neq 0$. If the support of Φ is sufficiently small, then*

$$\lim_{t \to \infty} t^{1/d(F)} (\log t)^{-m(F)+1} \cdot I_F(t; \Phi) = a(g) \cdot C(\varphi) \cdot \psi(0),$$

where $C(\varphi)$ is as in Theorem 3.1 and $a(g)$ is nonzero constant defined by

$$a(g) = \begin{cases} 2\Gamma(1/k+1) \cdot \left(\frac{k!}{g^{(k)}(0)}\right)^{1/k} \cdot e^{\frac{\pi}{2k} i} & (k \text{ is even}); \\ 2\Gamma(1/k+1) \cdot \left(\frac{k!}{g^{(k)}(0)}\right)^{1/k} \cdot \cos \frac{\pi}{2k} & (k \text{ is odd}). \end{cases}$$

Remarks.

1. It is easy to see that $1/d(F) = 1/h + 1/k$ and $m(F) = m(f)$.
2. In [10], Greenblatt deals with more general cases when the phase is real analytic but his results concern with the estimate.

On the other hand, the following theorem gives a three-dimensional counterexample to the equation (5.2) in the general smooth phase case.

Theorem 5.3. *Suppose that $g(x_3) = e^{-1/|x_3|^p}$ where p is a positive real number. If the support of Φ is sufficiently small, then*

$$\lim_{t\to\infty} t^{1/h}(\log t)^{-m+1+1/p} \cdot I_F(t;\Phi) = 2C(\varphi)\cdot\psi(0),$$

where $C(\varphi)$ is as in Theorem 3.1.

Proofs of Theorems 5.2 and 5.3. From Fubini's theorem, we have

$$I_F(t;\Phi) = \int_{\mathbb{R}^2} e^{itf(x_1,x_2)}\varphi(x_1,x_2)dx_1dx_2 \cdot \int_{\mathbb{R}} e^{itg(x_3)}\psi(x_3)dx_3. \tag{5.3}$$

When g satisfies the condition in Theorem 5.2, the computation in [31], Chapter VIII, implies that

$$\lim_{t\to\infty} t^{1/k}\cdot\int_{\mathbb{R}} e^{itg(x)}\psi(x)dx = a(g)\cdot\psi(0),$$

where $a(g)$ is as in Theorem 5.2. When $g(x_3) = e^{-1/|x_3|^p}$, Lemma 2.1 in [23] implies that

$$\lim_{t\to\infty}(\log t)^{1/p}\cdot\int_{\mathbb{R}} e^{ite^{-1/|x|^p}}\psi(x)dx = 2\psi(0).$$

Applying the above limits and Theorem 3.1 to the equation (5.3), we can easily obtain the assertions in Theorems 5.2 and 5.3. $\square$

Remark. The assertions in the theorems can be easily generalized in the higher-dimensional case ($n \geq 4$). For example, consider $F(x) = f(x_1,x_2) + g(x_3,\ldots,x_n)$ where g is real analytic and satisfies the nondegeneracy condition (5.1) in the case of Theorem 5.2 and consider $F(x) = f(x_1,x_2) + e^{-1/|x_3|^{p_3}} + \cdots + e^{-1/|x_n|^{p_n}}$ where p_j are positive real numbers for $j = 3,\ldots,n$ in the case of Theorem 5.3.

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