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A review of the results on second analytic singularities in diffraction problems

By

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Abstract

We survey the microlocal singularity theory of diffraction problems in the analytic category, in particular, J. Sjöstrand's theory [8, 9, 10] on propagation of micro-analyticity along the boundary, and G. Lebeau's celebrated theory on the construction of the parametrix in [3]. The title of [3] looks like a paper on Gevrey singularities, but his parametrix is constructed in the second analytic category, that is, modulo distributions with one holomorphic parameter. So, we can conclude that the boundary values of the diffraction wave have second analyticity in the shadow. We also give such a parametrix for a typical example of diffraction problems with some well-known properties of Airy integrals.

§ 1. What is a diffraction problem?

Let $\Omega \subset \mathbb{R}^2$ be a domain with real analytic boundary $\partial\Omega$. Consider a wave equation in $\Omega \times \mathbb{R}_t$ with the Dirichlet condition:

$$\begin{cases} (c^{-2}\partial_t^2 - \triangle_x)f(x,t) = 0, & (t,x) \in \mathbb{R}_t \times \Omega, \\ \quad f(x,t) = 0, & (t,x) \in \mathbb{R}_t \times \partial\Omega. \end{cases}$$

Then, the diffractive singularities of the solution f appear along the convex boundary points of the obstacle $\mathbb{R}^2 \setminus \Omega$. We call such a convex boundary point "a glancing point". In a small neighborhood U of a glancing point $\mathring{x} = (\mathring{x}_1, \mathring{x}_2) \in \partial\Omega$, we may write

$$\Omega \cap U = \{(x_1, x_2) \mid \varepsilon > x_2 - \varphi(x_1) > 0,\ x_1 \in U'\},$$

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where U' is a small neighborhood of $\overset{\circ}{x}_1$, and $\varphi \in C^\omega(U')$ satisfies $\overset{\circ}{x}_2 = \varphi(\overset{\circ}{x}_1)$, $\varphi'(\overset{\circ}{x}_1) = 0$, and $-\varphi''(x_1) \geq 0 \quad \text{in}\, U'$. Here, we consider the following coordinate transformation to flatten the boundary equation $x_2 - \varphi(x_1) = 0$ to $u = 0$:

$$u = x_2 - \varphi(x_1), \quad v = x_1, \quad w = ct.$$

Therefore, the wave operator is transformed into the following:

$$P := \partial_w^2 - \frac{1}{1+\varphi'(v)^2}\partial_v^2 - (1+\varphi'(v)^2)\left(\partial_u - \frac{\varphi'(v)}{1+\varphi'(v)^2}\partial_v\right)^2 + (\text{lower order terms}).$$

Further, we perform the following coordinate transformation $(u, v, w) \Rightarrow (u^*, v^*, w^*)$ to eliminate the crossing terms of second order differential operators:

$$u^* = u, \quad v^* = V(u, v), \quad w^* = w,$$

where V is an analytic function defined by the following differential equations :

$$\frac{\partial V}{\partial u} = \frac{\varphi'(v)}{1+\varphi'(v)^2}\frac{\partial V}{\partial v}, \quad V|_{u=0} = v.$$

Hence the operator $(-1/(1+\varphi'(v)^2))P$ is expressed as

$$\widetilde{P} := \partial_{u^*}^2 + \frac{V_v(u,v)^2}{1+\varphi'(v)^2}\partial_{v^*}^2 - \frac{1}{1+\varphi'(v)^2}\partial_{w^*}^2 + (\text{lower order terms}).$$

Putting $R(u^*, v^*, w^*, \partial_{v^*}, \partial_{w^*}) := \widetilde{P} - \partial_{u^*}^2$, we can prove that

$$\big(\partial_{u^*}\sigma(R)\big)(u^*, v^*, w^*; i\xi^*, i\eta^*) \geq 0$$

in a neighborhood of $\{\sigma(R) = 0\}$ (for any real $\xi^*, \eta^* \Leftrightarrow \partial_{u^*}/i, \partial_{v^*}/i$). Indeed, since

$$\frac{\partial v}{\partial u^*} = -\frac{V_u}{V_v} = -\frac{\varphi'(v)}{1+\varphi'(v)^2}, \quad V_{uv} = \partial_v\left(\varphi'(v)V_v/(1+\varphi'(v)^2)\right),$$

we have $\quad -\partial_{u^*}\sigma(R) =$

$$\left(\partial_u - \frac{\varphi'(v)}{1+\varphi'(v)^2}\partial_v\right)\left(\frac{V_v(u,v)^2\xi^{*2} - \eta^{*2}}{1+\varphi'(v)^2}\right) = \frac{2\varphi''(v)\big(V_v(u,v)^2\xi^{*2} - \varphi'(v)^2\eta^{*2}\big)}{(1+\varphi'(v)^2)^3}.$$

Here, the second factor of the numerator coincides with

$$V_v(u,v)^2(1-\varphi'(v)^2)\xi^{*2}$$

on $\{\sigma(R) = 0\}$, and so this is positive because $|\varphi'(v)| < 1$. In order to get a more explicit form of $\widetilde{P}$, we expand $V(u, v)$ into a power series in u at $u = 0$:

$$(v^* =)V(u,v) = v + \frac{\varphi'(v)}{1+\varphi'(v)^2}u + O(u^2).$$

Hence we obtain

$$V_v = 1 + \frac{(1-\varphi'(v)^2)\varphi''(v)}{(1+\varphi'(v)^2)^2}u + O(u^2), \quad v = v^* - \frac{\varphi'(v^*)}{1+\varphi'(v^*)^2}u^* + O((u^*)^2).$$

Consequently we have

$$\begin{aligned}\widetilde{P} =& \partial_{u^*}^2 + \frac{(1+\varphi'(v^*)^2)^2 + \{2+O(1)\varphi'(v^*)\}\varphi''(v^*)u^* + O((u^*)^2)}{(1+\varphi'(v^*)^2)^4}\partial_{v^*}^2 \\ &- \frac{1+O(1)\varphi'(v^*)\varphi''(v^*)u^* + O((u^*)^2)}{(1+\varphi'(v^*)^2)^2}\partial_{w^*}^2 + \text{(lower order terms)}.\end{aligned}$$

Therefore our diffraction problem reduces to the analysis of the following problem in $\mathbb{R}_t \times \mathbb{R}_x^n$:

$$\begin{cases}\big(\partial_t^2 - tA(t,x,\partial_x) + B(x,\partial_x) + \text{(lower)}\big)u(t,x) = 0 \ \ (t>0), \\ u(t,x)|_{t=+0} = 0.\end{cases}$$

Here A, B satisfy the conditions below:

- A, B are C^ω differential operators at $(0,0)$ of order 2.
- $\sigma(A)(t,x,i\xi), \sigma(B)(x,i\xi)/i$ are real-valued for real t,x,ξ.
- $\partial_t\big(t\,\sigma(A)(t,x,i\xi)\big) \le 0$, $d\sigma(B)(x,i\xi) \ne 0$.

Further, we note that the singularities should be considered microlocally because the rays correspond to the bicharacteristic curves on $T^*(\mathbb{R}_t \times \mathbb{R}_x^n)$ for

$$P = \partial_t^2 - tA(t,x,\partial_x) + B(x,\partial_x) + \text{(lower)}.$$

In a diffractive case, since the bicharacteristic curve γ is tangent to the boundary $t=0$, we take the microlocal point on the boundary as

$$\mathring{p} = (0,0;\mathring{\tau}dt + \mathring{\xi}dx) \in T^*(\mathbb{R}_t \times \mathbb{R}_x^n) = \{(t,x;\tau dt + \xi dx)\}.$$

Therefore, $\mathring{p}$ should satisfy

$$\sigma(P)(\mathring{p}) = 0, \quad \{\sigma(P), t\}(\mathring{p}) = 0, \quad d\sigma(P)\wedge dt \wedge \omega \ne 0 \text{ at } \mathring{p},$$

where $\omega = \tau dt + \xi_1 dx_1 + \cdots + \xi_n dx_n$, and

$$\{f,g\} := \frac{1}{i}\Big[\partial_\tau f \cdot \partial_t g - \partial_t f \cdot \partial_\tau g + \sum_{j=1}^n \big(\partial_{\xi_j} f \cdot \partial_{x_j} g - \partial_{x_j} f \cdot \partial_{\xi_j} g\big)\Big].$$

Hence we have the conditions:

$$\mathring{\tau} = 0, \quad \sigma(B)(0, i\mathring{\xi}) = 0, \quad d\sigma(B) \wedge \big(\sum \xi_j dx_j\big) \ne 0.$$

(The last condition is necessary for a non-vanishing flow on the cotangent spherical bundle of the boundary). Then, after a suitable contact transformation on the boundary, we may assume

$$\sigma(B) = i\xi_1 b(x, i\xi)$$

with some non-vanishing first-order symbol $b(x, i\xi)$. Thus, we obtain the following standard form of P and $\mathring{p}$:

A Standard Form of P and $\mathring{p}$ for Diffraction Models:

$$\begin{cases} P = \partial_t^2 - t\,A(t,x,\partial_x) + b(x,\partial_x)\partial_{x_1} + (\text{lower order terms}), \\ \mathring{p} = (0, 0; dx_n), \end{cases} \tag{1.1}$$

where $n \geq 2$, $b(x, i\xi)/i \neq 0$ is a real, first-order symbol, and $A(t, x, i\xi)$ is a real, second-order symbol satisfying

$$\partial_t(t\,A(t, x, i\xi)) \leq 0 \text{ in a neighborhood of } \{t = \xi_1 = 0\}. \tag{1.2}$$

Indeed, the convexity condition of the obstacle is expressed as

$$0 \geq -\{\sigma(P), \{\sigma(P), t\}\} = 2\partial_t\big(t\,A(t, x, i\xi)\big), \tag{1.3}$$

which was obtained for our wave operator as before.

Example 1.1. The boundary: $\{t = 0\}$, and the domain: $\{t > 0\}$.

- M. Sato's model of diffraction (the strictly convex case):

$$P = \partial_t^2 - t\partial_{x_2}^2 + \partial_{x_1}\partial_{x_2} \quad \text{at } (0, 0, 0; dx_2).$$

- Non-strictly convex models : For $\ell = 1, 2, 3, \ldots$

$$P_\ell = \partial_t^2 - t x_1^{2\ell}\partial_{x_2}^2 + \partial_{x_1}\partial_{x_2} \quad \text{at } (0, 0, 0; dx_2).$$

Remark. 1) The original Sato's model operator is $P = \partial_t^2 - (t - x_1)\partial_{x_2}^2$, which is transformed into the above one by using a quantized contact transformation with respect to only $(x_1, x_2; \xi_1, \xi_2)$. Though the original Sato's operator is not so good for illustrating the propagation of the singularity because the direction of the propagation is ∂_{ξ_1}, this form gave a key idea in KK [2]; any solution $u(t, x)$ to

$$\big(\partial_t^2 - (t - x_1)A(t, x, \partial_x) + \text{lower order}\big)u = 0 \text{ in } \{t > 0, |t| + |x| < \delta\}$$

extends to a solution in

$$\bigcup_{0 \leq \theta \leq 1} \{(t, x); t - \theta x_1 > 0, |t| + |x| < \delta'\}$$

for a small $\delta' > 0$, where the symbol $A(t, x, i\xi) < 0$ in a neighborhood of $t = x_1 = 0$.
2) R. B. Melrose [5] proved that any standard model operator is equivalent to Sato's operator under some C^∞ class contact transformation preserving the boundary $t = 0$. However, T. Oshima [6] proved that such an equivalence does not hold in C^ω category by giving a counter example.

At the last of this section we recall the celebrated results due to J. Sjöstrand [8, 9, 10, 11] on propagation of analytic singularities for generic diffractive operators. Let us consider the following Dirichlet problem:

$$\begin{cases} P(t, x, \partial_t, \partial_x)u(t, x) = 0, & \text{in } \{0 < t < \delta, |x| < \delta\}, \\ u(+0, x) = 0, & \text{in } \{|x| < \delta\}. \end{cases} \tag{1.4}$$

for $P = \partial_t^2 - t\, A(t, x, \partial_x) + b(x, \partial_x)\partial_{x_1} +$ (lower order terms), where $n \geq 2$, $ib(x, i\xi) \neq 0$ is a real first-order symbol, and $A(t, x, i\xi)$ is a real second-order symbol satisfying $A(t, x, i\xi) < 0$ in a neighborhood of $\{t = 0, x = 0, \xi_1 = 0, \xi' \neq 0\}$. Let $\mathcal{G}, \mathcal{H}, \mathcal{E}$ be the sets of all diffractive points, all hyperbolic points, and all elliptic points, respectively;

$$\begin{cases} \mathcal{G} & = \{(t, x; \tau dt + \xi dx) \in T^*(\mathbb{R}_t \times \mathbb{R}^n_x); t = 0, \tau = 0, \xi_1 = 0, A(t, x, i\xi) < 0\}, \\ \mathcal{H} & = \{(t, x; \tau dt + \xi dx) \in T^*(\mathbb{R}_t \times \mathbb{R}^n_x); t = 0, i\xi_1 b(x, i\xi) > 0, A(t, x, i\xi) < 0\}, \\ \mathcal{E} & = \{(t, x; \tau dt + \xi dx) \in T^*(\mathbb{R}_t \times \mathbb{R}^n_x); t = 0, i\xi_1 b(x, i\xi) < 0, A(t, x, i\xi) < 0\}. \end{cases} \tag{1.5}$$

Then, we have a vector field ∂_{x_1} on $\mathcal{G}$ because ∂_{x_1} is written as a linear combination of Hamilton vector fields iH_t and $iH_{\sigma(P)}$, where $H_{f(t,x,\tau,\xi)}$ is the Hamilton vector field defined by $H_f(g) := \{f, g\}$. Indeed, since $H_t = i\partial_\tau, H_{\sigma(P)} = b(x, 0, i\xi')\partial_{x_1} - iA(0, x, 0, i\xi')\partial_\tau$ on $\mathcal{G}$, we have

$$\partial_{x_1} = (ib(x, 0, i\xi'))^{-1} \left(iH_{\sigma(P)} + A(0, x, 0, i\xi')iH_t\right).$$

Let p be a point of $\mathcal{G} \cap \{t = 0, x = 0\}$. Then we have two integral curves γ_0, γ passing through p for $\partial_{x_1}, iH_{\sigma(P)}$, respectively as in Figure 1. Divide $\gamma_0 \setminus \{p\}, \gamma \setminus \{p\}$ into connected components $\gamma_{01} \sqcup \gamma_{03}$, $\gamma_2 \sqcup \gamma_4$ respectively such that $q_1 \in \gamma_{01}$, $q_3 \in \gamma_{03}$, $q_2, \in \gamma_2$, $q_4 \in \gamma_4$ as in Figure 1. Further, we identify $\mathcal{G}$ with

$$\mathcal{G}_0 = \{(x; \xi dx) \in T^*\mathbb{R}^n_x; \xi_1 = 0, A(0, x, 0, i\xi') < 0\} \subset T^*(\{t = 0\} \times \mathbb{R}^n_x).$$

Theorem 1.2. *Let $u(t, x)$ be a (hyperfunction) solution of (1.4). Take a diffractive point $p := (0, \mathring{x}; 0, 0, \mathring{\xi}') \in \mathcal{G}$ with $\mathring{\xi}' \neq 0$ as in Figure 2. Then, we have the following conclusions:*

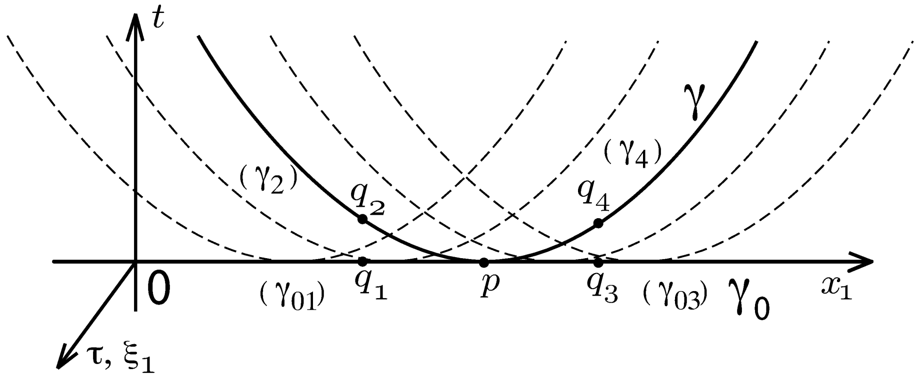

Figure 1. diffractive rays

(1) *If* $\gamma_{01} \cap \mathrm{SS}(\partial_t u(+0,x)) = \emptyset$ *and* $\gamma_2 \cap \mathrm{SS}(u) = \emptyset$ (*or* $\gamma_{03} \cap \mathrm{SS}(\partial_t u(+0,x)) = \emptyset$ *and* $\gamma_4 \cap \mathrm{SS}(u) = \emptyset$), *then* $p \notin \mathrm{SS}(\partial_t u(+0,x))$.

(2) *If* $(\gamma_2 \cup \gamma_4) \cap \mathrm{SS}(u) = \emptyset$, *then* $p \notin \mathrm{SS}(\partial_t u(+0,x))$.

(3) *If* $\gamma_{03} \cap \mathrm{SS}(\partial_t u(+0,x)) = \emptyset$ *and* $\gamma_2 \cap \mathrm{SS}(u) = \emptyset$ (*or* $\gamma_{01} \cap \mathrm{SS}(\partial_t u(+0,x)) = \emptyset$ *and* $\gamma_4 \cap \mathrm{SS}(u) = \emptyset$), *then* $p \notin \mathrm{SS}(\partial_t u(+0,x))$.

(4) *The condition* $(\gamma_{01} \cup \gamma_{03}) \cap \mathrm{SS}(\partial_t u(+0,x)) = \emptyset$ *does not necessarily imply* $p \notin \mathrm{SS}(\partial_t u(+0,x))$.

Remark.

i) As stated in the introduction of J. Sjöstrand [10], (1) is the conclusion of [8], (2) is the main result of [9], and (3), (4) are the main results of [10] (Theorem 0.3, Theorem 4.5, respectively). The result (1) is extended to more general boundary value problems in J. Sjöstrand [11].

ii) KK [2] proved that (2) holds without any boundary condition, which is generalized by Théorème 2 of G. Lebeau [4] to any degenerate cases (non-strictly convex cases). The proofs of these 3 results are completely different from each other. Further, M. Rouleux [7] proved that (2) holds under Gevrey wave front $\mathrm{SS}^s(*)$ for any $s \geq 1$, and that the conclusion of KK [2] is false for any $s > 1$.

iii) We introduce an interesting counter example for (4) due to the idea of J. Sjöstrand [10] in Example 3.6 (a solution with non-gliding analytic singularities).

§ 2. Lebeau's diffraction theory for the strictly convex case

In [3], G. Lebeau succeeded in the construction of the parametrix with a given Dirichlet data in C^ω category. His method is roughly as follows:

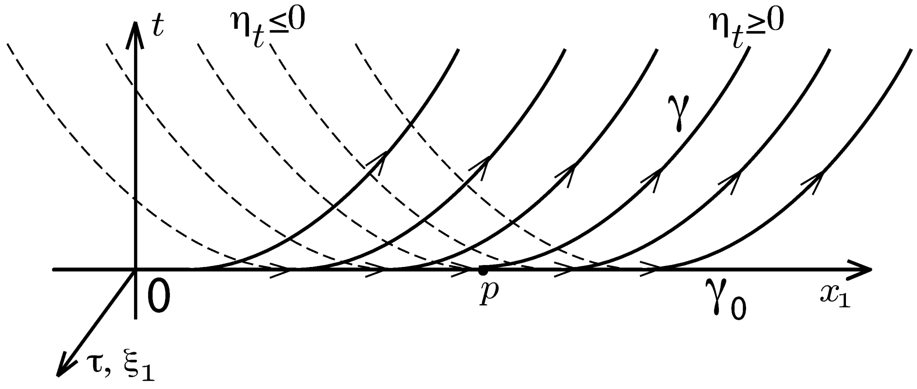

Figure 2. half rays in the shadow

- A construction by some oscillatory integral using analytic phases and analytic amplitudes, but he permitted $\sqrt{\xi_1}$ - singularities. Hence, it leads to the second microlocal analysis along $\{\xi_1 = 0\}$.
- At the last step, such integrals reduce essentially to Airy integrals. So the necessary quantities are calculable: For example, the invertibility of some important operator is deduced from the information of zeros of the Airy function.

By using this parametrix, he proved the following: *In the shadow, namely, when the analytic singularities are only on the half bicharacteristic curves of the same type issuing from diffraction region $\mathcal{G}$ as in Figure 2, the Neumann data of the Dirichlet problem is second analytic along the boundary bicharacteristic flow.* For a standard model operator

$$P = \partial_t^2 - tA(t, x, \partial_x) + b(x, \partial_x)\partial_{x_1} + (\text{lower}),$$

this means that the Neumann data $\partial_t u(+0, x)$ of the solution $u(t, x)$ in $t > 0$ with Dirichlet condition is partially holomorphic in the variable x_1 in the shadow; more precisely, there is a hyperfunction $h(z_1, x')$ defined in $\{(z_1, x') \in \mathbb{C} \times \mathbb{R}^{n-1}; |\operatorname{Im} z_1| < \delta, |\operatorname{Re} z_1 - \mathring{x}_1| + |x' - \mathring{x}'| < \delta\}$ such that $[h(x_1, x')] = \partial_t u(+0, x)$ as a microfunction. It is well-known that the microanalyticity of a hyperfunction with holomorphic parameter z_1 propagates along ∂_{z_1}. Hence, the second analyticity of $\partial_t u(+0, x)$ implies (1) and (3) of Theorem 1.2. The main result of Lebeau is the following:

Theorem 2.1. (Theorem 0 in [3]). *Let $u(t, x)$ be a distribution in $\{t > 0, |x - \mathring{x}| < \epsilon\}$ which is prolongeable to $\{t \le 0\}$ as a distribution. Assume that*

$$\begin{cases} (P(t, x, \partial_t, \partial_x)u(t, x) = 0 \quad (t > 0), \\ u(+0, x) = 0, \\ \mathrm{SS}^{G_3}(u) \cap \gamma_2 = \emptyset \ (\mathit{or}\ \mathrm{SS}^{G_3}(u) \cap \gamma_4 = \emptyset). \end{cases}$$

Then we have $(\overset{\circ}{x}, \overset{\circ}{\xi}) \notin \mathrm{SS}^{G_3}(\partial_t u(+0, x))$*, where* γ_2, γ_4 *are half bicharacteristic curves passing through* $(0, \overset{\circ}{x}; 0, \overset{\circ}{\xi})$ *for* $\sigma(P)$ *defined in Theorem* 1.2*, and* $\mathrm{SS}^{G_3}(f(x))$ *is the Gevrey wave front set of index* 3 *for* f*.*

Though the statement of this theorem is concerning neither analytic wave front set, nor second analyticity, he constructed the parametrix in analytic category, and so the result includes second analyticity of the parametrix. More precisely, he constructed the parametrix in some Sjöstrand space (the space of holomorphic functions with a real large parameter λ):

$$\begin{aligned}\mathcal{H}_\varphi(X) :=& \{F(z, \lambda); F(z, \lambda) \in \mathscr{O}(X) \text{ for any } \lambda > 0, \text{ and} \\ & \sup_{K,\lambda} |F(z, \lambda)| e^{-\lambda(\varphi(z)+\varepsilon)} < \infty \text{ for any compact } K \subset X, \text{ and any } \varepsilon > 0\},\end{aligned}$$

where an open subset $X \subset \mathbb{C}^n$, and $\varphi(z)$ is a real valued C^ω-function such that $\partial\varphi(z)/\partial z \neq 0$, and $(\partial^2\varphi(z)/\partial z_j \partial \overline{z}_k)_{jk}$ is positive definite for any $z \in X$. For example, setting $\varphi(z) = (\operatorname{Im} z)^2/2$, we can transform a distribution or a hyperfunction $f(x)$ with compact support into $F(z, \lambda) \in \mathcal{H}_\varphi(\mathbb{C}^n)$ by an integral transformation

$$F(z, \lambda) := \int_{\mathbb{R}^n} e^{-\lambda(z-x)^2/2} f(x) dx.$$

Indeed, $(\overset{\circ}{x}; \overset{\circ}{\xi}) \notin \mathrm{SS}(f)$ with $|\overset{\circ}{\xi}| = 1$ is equivalent to $(\overset{\circ}{x} - i\overset{\circ}{\xi}) \notin \mathrm{SS}^X_\varphi(F)$; namely, a decay estimate $|F(z, \lambda)| < \delta^{-1} \exp\big(\lambda(\varphi(z) - \delta)\big)$ in $\{|z - (\overset{\circ}{x} - i\overset{\circ}{\xi})| < \delta, \lambda > 1\}$ for some small $\delta > 0$. Hence the microlocal analysis on $T^*\mathbb{R}^n$ for $f(x)$ is transformed into the microlocal analysis for $F(z, \lambda)$ on $\Lambda_\varphi := \{(z; -2i\partial\varphi/\partial z) \in T^*X\}$. Further, for microlocal boundary value problems one needs an extension $\mathcal{H}^{+,\rho}_\varphi(X)$ of $\mathcal{H}_\varphi(X)$; an $F(t, z, \lambda) \in \mathcal{H}^{+,\rho}_\varphi(X)$ induces an element of $\mathcal{H}_\varphi(X)$ for any fixed $t \in [0, \rho] (\subset \mathbb{R})$ (we omit the details), and the singular spectrum of $F(t, z, \lambda)$ is defined as $\mathrm{SS}^{X,\rho}_{\partial,\varphi}(F) \subset T^*[0, \rho] \times \Lambda_\varphi$. Let $\widetilde{P}$ be the transformed operator of P:

$$(\widetilde{P}F)(t, z, \lambda) := \frac{1}{\lambda^2}\partial_t^2 F + \left(\frac{2\pi}{\lambda}\right)^n \int_{\Sigma_0} e^{i\lambda(z-\widetilde{z})\eta} R(t, z, \eta, \lambda) F(t, \widetilde{z}, \lambda) d\widetilde{z} d\eta,$$

where the symbol $R(t, z, \eta, \lambda)$ corresponds to $tA(t, x, \partial_x) - b(x, \partial_x)\partial_{x_1}$ (because his operator is of form $P = \partial_t^2 + R(t, x, -i\partial_x)$), and

$$\Sigma_0 := \big\{(\widetilde{z}, \eta) = \big(\overset{\circ}{z} + w, -2i\partial_z\varphi(\overset{\circ}{z}) - iC_0\overline{w}\big);\ w \in \mathbb{C}^n, |w - \overset{\circ}{z}| \leq r_0\big\}$$

with $C_0 \gg 1, 0 < r_0 \ll C_0^{-1}$. Let $\overset{\circ}{z} \in \mathbb{C}^n$ be a point such that $(0, \overset{\circ}{z}; 0, -2i\partial_z\varphi(\overset{\circ}{z}))$ is a diffractive point of $\widetilde{P}$ on $t = 0$, and let $G(z, \lambda) \in \mathcal{H}_\varphi(X)$ be any section satisfying:

$$\begin{cases} \mathrm{SS}^X_\varphi(G) \Subset \{(z; -2i\partial_z\varphi(z));\ |z - \overset{\circ}{z}| < \rho_0\}, \quad \{|z - \overset{\circ}{z}| < \rho_0\} \subset W \Subset X, \\ |G(z, \lambda)| \leq \lambda^M e^{\lambda\varphi(z)} \quad (\forall z \in X, \forall\lambda > 1) \end{cases}$$

with a constant $M > 0$. Then, Lebeau constructed a solution $F(t, z, \lambda) \in \mathcal{H}_{\varphi}^{+,\rho}(X)$ satisfying the following:

(C-1) $\mathrm{SS}_{\partial,\varphi}^{W,\rho}(\widetilde{P}F) = \emptyset$ ($\Leftrightarrow Pf = 0$ in $\{t > 0\}$ microlocally).

(C-2) $F(0, z, \lambda) = G(z, \lambda)$ ($\Leftrightarrow f(+0, x) = g(x)$).

(C-3) $\mathrm{SS}_{\partial,\varphi}^{W,\rho}(F) \subset \mathcal{F}^{+}\Big(\mathrm{SS}_{\varphi}^{X}(G) \cup \bigcup_{s\geq 0} \exp\big(sH_{-\sigma(R)(0,z,\eta)}(\mathrm{SS}_{\varphi}^{X}(G) \cap \mathcal{G})\big)\Big)$ ($\Leftrightarrow$ roughly, $\mathrm{SS}(f)$ is only on the half bicharacteristic curves ($\subset \{\eta_t \geq 0\}$) issuing from diffraction region $\mathcal{G}$ or from $\mathcal{H}_{+}$ as in Figure 2).

Here, $\mathcal{G}$ corresponds to the set in (1.5), and for a subset L of Λ_{φ} we define a subset $\mathcal{F}^{+}(L) \subset \Lambda_{\varphi} \cup (T^{*}\mathbb{R}_{+} \times \Lambda_{\varphi})$ by

$$\mathcal{F}^{+}(L) := L \cup \bigcup_{s>0} \exp\big(sH_{\sigma(\widetilde{P})(t,z,\eta_t,\eta)}(L \cap (\mathcal{G} \cup \mathcal{H}_{+}))\big),$$

where $\exp(sH_{Q(t,z,\eta_t,\eta)}(*))$ is the map induced by the Hamilton vector field H_Q with time parameter s (the definition of H_Q is due to Lebeau's one). Further,

$$\mathcal{H}_{+} := \{(t, z; \eta_t, \eta) \in \mathcal{H}; \sigma(\widetilde{P}) = 0, \eta_t \geq 0\}.$$

We remark here that the condition (C-3) in his paper uses "$sH_{\sigma(R)(0,z,\eta)}$ and $\eta_t \leq 0$" instead of "$sH_{-\sigma(R)(0,z,\eta)}$ and $\eta_t \geq 0$", respectively. They are equivalent to each other under the change of parameter $s \to -s$.

Theorem 2.2. (Theorem 1 in [3]). *Let $G(z, \lambda), F(t, z, \lambda)$ be as above. Let $z_j \in W$ $(j = 0, 1)$ be two points such that $p_j = (z_j; -2i\partial_z\varphi(z_j))$ $(j = 0, 1)$ are on the same boundary bicharacteristic curve $\gamma(\subset \mathcal{G})$ for $\widetilde{P}$. Assume $p_1 = \exp(s_0 H_{-\sigma(R)(0,z,\eta)}(p_0))$ for some small $s_0 > 0$, and $\exp(sH_{-\sigma(R)(0,z,\eta)}(p_0)) \notin \mathrm{SS}_{\varphi}(G)$ for any $s > 0$. Then, on some small neighborhood ω of z_1, we have the following estimate: $\forall\varepsilon > 0, \exists C_\varepsilon > 0$,*

$$|\partial_t F(+0, z, \lambda)| \leq C_\varepsilon \exp\Big(\lambda\varphi_g(z) + \lambda^{1/3}\tilde{\psi}(z) + \varepsilon\lambda^{1/3}\Big) \quad (\forall z \in \omega,\ \forall\lambda \geq 1). \tag{2.1}$$

Here, $\varphi_g(z)$ is the modification of $\varphi(z)$ concerning the partial complexification $\widetilde{\mathcal{G}}$ of $\mathcal{G}$ induced by the foliation $\exp(sH_{-\sigma(R)(0,z,\eta)}(*))$; namely, $\varphi_g(z) \leq \varphi(z)$ $(\forall z)$, $\varphi_g(z) = \varphi(z)$ for $\forall(z; -2i\partial_z\varphi(z)) \in \mathcal{G}$, and φ_g is harmonic concerning the partial complex variable. Further $\tilde{\psi}(z)$ is a weight function on Λ_{φ_g}.

Remark. The inequality (2.1) implies the second analyticity of the Neumann data $\partial_t F(+0, z, \lambda)$ along the foliation of $\widetilde{\mathcal{G}}$. Therefore, in our situation, we have the following conclusion: Let $I \subset \mathcal{G}$ be an interval on an integral curve γ for ∂_{x_1}. Assume that $I \cap \mathrm{SS}(f(+0, x)) = \emptyset$. Then, $\partial_t f(+0, x)$ is second-analytic in the variable x_1 in a neighborhood of I (see Example 3.4).

Hereafter we explain Lebeau's theory in our point of view under our notation in Section 1. The strictly convex case corresponds to the case $A(t,x,i\xi) < 0$ for the standard model operator

$$P = \partial_t^2 - tA(t,x,\partial_x) + b(x,\partial_x)\partial_{x_1} + (\text{lower}),$$

(see (1.3)). So the equation $\sigma(P) \equiv (i\tau)^2 - t\cdot\sigma(A)(t,x,i\xi) + \sigma(b)(x,i\xi)\, i\xi_1 = 0$ is rewritten as

$$t + q(x, \tau/\xi_n, \xi'/\xi_n) = 0$$

with some analytic function $q(x,\widetilde{\tau},\widetilde{\xi'})$, where τ corresponds to ∂_t, $\xi' = (\xi_1, ..., \xi_{n-1})$, $\widetilde{\tau} = \tau/\xi_n$, and $\widetilde{\xi'} = \xi'/\xi_n$. Indeed, since

$$t = \frac{\sigma(b)(x,\xi)\,\xi_1 + \tau^2}{\sigma(A)(0,x,\xi)}(1 + O(t)),$$

we have a Taylor expansion in $\widetilde{\tau}$:

$$q(x,\widetilde{\tau},\widetilde{\xi'}) = \alpha(x,\widetilde{\xi'})\,\widetilde{\xi_1} + \beta(x,\widetilde{\xi'})\widetilde{\tau}^2 + O(\widetilde{\tau}^4)$$

with some analytic functions $\alpha(x,\widetilde{\xi'}), \beta(x,\widetilde{\xi'})$ satisfying $\alpha \neq 0, \beta \neq 0$. We construct a solution to $P(t,x,\partial_t,\partial_x)u(t,x) = 0$ of the form

$$u(t,x) = \int e^{i\lambda\left(t\tau + \varphi(\tau,x,y,\eta')\right)} \left(\sum_{j=0}^{\infty} \psi_j(\tau,x,y,\eta')\lambda^{-j}\right) d\tau d\eta' d\lambda.$$

Then the phase $\varphi(\tau,x,y,\eta')$ should satisfy

$$\sigma(P)\,(-\varphi_\tau, x, i\lambda\tau, i\lambda\varphi_x) = 0.$$

That is,

$$\varphi_\tau = q\big(x, \tau/\varphi_{x_n}, \varphi_{x'}/\varphi_{x_n}\big).$$

Therefore, we solve the initial value problem:

$$\begin{cases} \dfrac{\partial\varphi}{\partial\tau} = q\left(x, \tau/\varphi_{x_n}, \varphi_{x'}/\varphi_{x_n}\right), \\ \varphi|_{\tau=0} = (x'-y')\cdot\eta' + x_n - y_n. \end{cases}$$

Then, we have the Taylor expansion in τ of $\varphi(\tau,x,y,\eta')$:

$$\begin{aligned} \varphi =& (x'-y')\cdot\eta' + x_n - y_n + \alpha(x,\eta')\eta_1\tau + \gamma(x,\eta')\eta_1\tau^2 \\ &+ \frac{1}{3}\big(\beta(x,\eta') + \delta(x,\eta')\eta_1\big)\tau^3 + O(\tau^4) \end{aligned}$$

with some analytic functions γ, δ of x, η'. Since α, β never vanish on $\eta_1 = 0$, the critical points $\tau_\pm$ for $\varphi(\tau, x, y, \eta')$ in τ near $\tau = 0$ (that is, the solutions τ of the equation below:)

$$0 = \varphi_\tau = \alpha(x,\eta')\eta_1 + 2\gamma(x,\eta')\eta_1\tau + \big(\beta(x,\eta') + \delta(x,\eta')\eta_1\big)\tau^2 + O(\tau^3),$$

are written as

$$\tau_\pm(x,\eta') = \frac{\pm\sqrt{(-\alpha\beta)\eta_1}}{\beta} + O(\eta_1) \ \ (\text{as } \eta_1 \to 0)$$

when $\alpha\beta\eta_1 < 0$. Indeed, $\tau_\pm(x,\eta')$ are analytic functions of $x, \sqrt{\eta_1}, \eta_2, ..., \eta_{n-1}$. Then Lebeau proved the following: There is an analytic function

$$w(\tau, x, \eta') = \sum_{j=0}^{\infty} w_j(x,\eta')\tau^j$$

satisfying $w_1 \neq 0$, and

$$\varphi(\tau, x, y, \eta') - ((x' - y')\cdot\eta' + x_n - y_n) = \eta_1\rho(x,\eta')w - \frac{1}{3}w^3 - B(x,\eta')$$

with analytic functions $\rho(x,\eta'), B(x,\eta')$ in x, η'. Such functions ρ and B are determined by the difference of the critical values of both sides. Indeed, the critical value

$$\varphi(\tau_\pm, x, \eta') - ((x' - y')\cdot\eta' + x_n - y_n)$$

of $\varphi(\tau, x, \eta') - ((x' - y')\cdot\eta' + x_n - y_n)$ in τ has the form:

$$(\eta_1)^{3/2}\{f(x,\eta') + \sqrt{\eta_1}g(x,\eta')\}$$

with some analytic functions f, g of x, η', where $f \neq 0$. Further the critical value of $\eta_1\rho(x,\eta')w - \frac{1}{3}w^3$ in w is

$$(2/3)(\eta_1\rho(x,\eta'))^{3/2}.$$

So,

$$\rho(x,\eta') := (3f(x,\eta')/2)^{2/3}, \quad B(x,\eta') := -\eta_1^2 g(x,\eta').$$

By changing the integration variable τ into w, we obtain a new expression of the parametrix

$$\int e^{i\lambda\left(t\cdot\tau(w,x,\eta') + (x'-y')\eta' + x_n - y_n + \eta_1\rho(x,\eta')w - \frac{1}{3}w^3 - B(x,\eta')\right)}$$
$$\times \left(\sum_{j=0}^{\infty}\phi_j(w,x,y,\eta')\lambda^{-j}\right) dw d\eta' d\lambda.$$

Therefore the boundary values of this function on $t = 0$ is written by using Airy function (of course, asymptotically).

§ 3. Examples

Before giving examples of solutions for diffraction problems, we recall the definition and some important properties of Airy function.

Definition 3.1. Airy function $\mathrm{Ai}(z)$ is an entire function given by

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty e^{-i\delta}}^{\infty e^{i\delta}} e^{izt+i(t^3/3)}dt$$

with any small $\delta > 0$. Easily to see, we have the following equation:

$$\mathrm{Ai}''(z) - z\mathrm{Ai}(z) = 0.$$

Proposition 3.2. $\overline{\mathrm{Ai}(z)} = \mathrm{Ai}(\overline{z})$ *for every* $z \in \mathbb{C}$*, and for any small* $\delta > 0$ *we have a uniform estimate on* $|\arg z| \le \pi - \delta$ *as* $|z| \to +\infty$

$$\mathrm{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} z^{-1/4} e^{-\frac{2}{3}z^{3/2}}.$$

Further, $\mathrm{Ai}(0) = \frac{\sqrt{3}}{2\pi}\int_0^\infty e^{-t^3/3}dt > 0$.

Proof. The first one is clear by the expression. Concerning the second one we use the steepest descent method. Namely, we find the critical point in t of the phase:

$$\partial_t(zt + t^3/3) = 0; \text{ that is, } t = \pm i\sqrt{z}.$$

So we change the variable as $t = z^{1/2}(z^{-3/4}y + i)$ where $|\arg z| < \pi$. Then, putting $\theta := \arg z \in [-\pi + \delta, \pi - \delta]$, for any small $\epsilon > 0$ ($\epsilon \ll \delta$) we have

$$\begin{aligned}
\mathrm{Ai}(z) &= \frac{1}{2\pi}\int_{-\infty e^{i(\theta-\epsilon)/4}}^{\infty e^{i(\theta+\epsilon)/4}} e^{iz^{3/2}(z^{-3/4}y+i)+iz^{3/2}(z^{-3/4}y+i)^3/3} z^{-1/4}dy \\
&= \frac{e^{-\frac{2}{3}z^{3/2}}}{2\pi z^{1/4}}\int_{-\infty e^{i(\theta-\epsilon)/4}}^{\infty e^{i(\theta+\epsilon)/4}} e^{-y^2+iz^{-3/4}y^3/3}dy.
\end{aligned}$$

Since $\mathrm{Re}\, y^2 = |y|^2\cos\{(\theta+\epsilon)/2\} > 0$ for $y \in (0,\infty)e^{i(\theta+\delta)/4}$, we get

$$2\pi z^{1/4}e^{\frac{2}{3}z^{3/2}}\mathrm{Ai}(z) = \int_0^{+\infty}\left(e^{-s^2e^{i\theta/2}+i|z|^{-3/4}s^3/3}e^{i\theta/4} + e^{-s^2e^{-i\theta/2}-i|z|^{-3/4}s^3/3}e^{-i\theta/4}\right)ds.$$

Further, since the integrands are estimated by an integrable function $e^{-s^2\sin(\delta/2)}$, we get a uniform convergence on $|\arg z| \le \pi - \delta$:

$$\lim_{|z|\to\infty} 2\pi z^{1/4}e^{\frac{2}{3}z^{3/2}}\mathrm{Ai}(z) = 2\,\mathrm{Re}\left(\int_0^{+\infty} e^{-s^2e^{i\theta/2}}e^{i\theta/4}ds\right) = \sqrt{\pi}.$$

The last equality follows directly from the change of the integration path. □

Proposition 3.3. $\{z \in \mathbb{C}; \mathrm{Ai}(z) = 0\} \subset \{x \in \mathbb{R}; x < 0\}$.

Proof. Let $z_0 \in \mathbb{C}$ be a complex number satisfying $\mathrm{Ai}(z_0) = 0$. First we show that $z_0 \in \mathbb{R}$. Consider $u(t) := \mathrm{Ai}(z_0 + t)$. Then, $u''(t) = (z_0 + t)\, u(t)$. Hence, for real t, we have

$$\overline{u(t)}u''(t) - \overline{u''(t)}u(t) = (z_0 - \overline{z_0})|u(t)|^2.$$

Therefore

$$\begin{aligned}(z_0 - \overline{z_0}) \int_0^\infty |u(t)|^2 dt &= \int_0^\infty \left(\overline{u(t)}u''(t) - \overline{u''(t)}u(t)\right) dt \\ = \int_0^\infty \frac{d}{dt}\left(\overline{u(t)}u'(t) - \overline{u'(t)}u(t)\right) dt &= -\left(\overline{u(0)}u'(0) - \overline{u'(0)}u(0)\right) = 0\end{aligned}$$

because $u(0) = \mathrm{Ai}(z_0) = 0$. Hence $z_0 - \overline{z_0} = 0$. If $z_0 = x_0$ is a non-negative real number, $x_0 > 0$ by the preceding Proposition. On the other hand, let x_1 be the minimum positive zero of $\mathrm{Ai}(x)$, then $\mathrm{Ai}'(x_1) < 0$. So, by the equation $\mathrm{Ai}''(x) = x\mathrm{Ai}(x)$, we can easily conclude that $\mathrm{Ai}(x) \le (x - x_1)\mathrm{Ai}'(x_1) \to -\infty$ as $x \to +\infty$. Contradiction! So there is no zero in $[0, +\infty)$. Thus the zeros of $\mathrm{Ai}(z) = 0$ is included in the negative real line. $\square$

Example 3.4. We consider the following Dirichlet problem for Sato's operator:

$$\begin{cases}(\partial_t^2 - t\partial_{x_2}^2 + \partial_{x_1}\partial_{x_2})u(t,x) = 0 \quad (t > 0), \\ u(+0, x) = \delta(x), \\ \mathrm{SS}(u) \cap \{(t, x_1, x_2; \tau, \eta_1, \eta_2);\ t > 0, \eta_2 > 0\} \subset \{\tau \ge 0\}\end{cases}$$

These conditions correspond to Lebeau's (C-1), (C-2), (C-3) (see Theorem 2.2). Then by Fourier analysis in x we obtain

$$u(t,x) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \frac{\mathrm{Ai}(e^{-\pi i/3}|\eta_2|^{2/3}(t - \eta_1/\eta_2))}{\mathrm{Ai}(e^{-\pi i/3}|\eta_2|^{2/3}(-\eta_1/\eta_2))} e^{i(\eta_1 x_1 + \eta_2 x_2)} d\eta_1 d\eta_2.$$

It is easy to see that the first and second equations are formally satisfied. Further, we can get the last estimation of singular spectrum of $u(t,x)$ by considering a partial holomorphic extension $u(t+is, x)$ to $\{t + is \in \mathbb{C}; 0 < s < t\}$ of $u(t,x)$ as a distribution:

$$u(t+is, x) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \frac{H(|\eta_2|^{2/3}(t + is - \eta_1/\eta_2))}{H(|\eta_2|^{2/3}(-\eta_1/\eta_2))} e^{i(\eta_1 x_1 + \eta_2 x_2)} d\eta_1 d\eta_2,$$

where $H(z) := \mathrm{Ai}(e^{-\pi i/3}z)$. Indeed, setting $\mu := \eta_1/\eta_2 \in \mathbb{R}$, $\rho := |\eta_2|^{2/3}$, we have

$$\begin{aligned}|H(\rho(t+is-\mu))/H(-\rho\mu)| &\le C(1 + |\rho\mu|)|H(\rho(t+is-\mu))| \exp(-2(\rho\mu)_+^{3/2}/3) \\ &\le C'(1 + |\rho\mu|) \exp\left(-\frac{2}{3}\rho^{3/2}((\mu)_+^{3/2} - \mathrm{Re}(\mu - t - is)^{3/2}))\right),\end{aligned}$$

where some constants $C, C' > 0$, $(x)_+ = (x+|x|)/2$, and $-\pi < \arg(\mu - t - is) < 0$. Hence we have $|H(\rho(t+is-\mu))/H(-\rho\mu)| \le C'(1+|\rho\mu|)$, and so $u(t+is,x)$ is well-defined if

$$\mathrm{Re}(\mu - t - is)^{3/2} \le (\mu)_+^{3/2} \quad \text{for } \forall\mu\in\mathbb{R}, \forall t, \forall s\ (0<s<t). \tag{3.1}$$

Indeed, when $\mu - t \le 0$, (3.1) holds because $\mathrm{Re}(\mu-t-is)^{3/2} \le 0$. When $\mu - t > 0$, we have $|\mu - t - is| \le |\mu - t| + s \le \mu - t + t = \mu$. Hence we get (3.1) because $\mathrm{Re}(\mu - t - is)^{3/2} \le |\mu - t - is|^{3/2} \le \mu^{3/2} = (\mu)_+^{3/2}$.

Concerning the Neumann data of u, we have

$$\partial_t u(+0,x) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \frac{\mathrm{Ai}'(e^{-\pi i/3}|\eta_2|^{2/3}(-\eta_1/\eta_2))}{\mathrm{Ai}(e^{-\pi i/3}|\eta_2|^{2/3}(-\eta_1/\eta_2))} e^{-\pi i/3}|\eta_2|^{2/3} e^{i(\eta_1x_1+\eta_2x_2)} d\eta_1 d\eta_2.$$

Hence we obtain the results of Lebeau in Theorem 2.2 from Proposition 3.5; namely, an estimate of $\mathrm{SS}(\partial_t u(+0,x))$ and the second-analyticity of $\partial_t u(+0,x)$.

Proposition 3.5. *Let $f(x)$ be a function defined by*

$$\begin{aligned} f(x) &:= \int_{\mathbb{R}\times[0,\infty)} \frac{\mathrm{Ai}'(e^{-\pi i/3}|\eta_2|^{2/3}(-\eta_1/\eta_2))}{\mathrm{Ai}(e^{-\pi i/3}|\eta_2|^{2/3}(-\eta_1/\eta_2))} |\eta_2|^{2/3} e^{i(\eta_1x_1+\eta_2x_2)} d\eta_1 d\eta_2 \\ &= \int_0^\infty \lambda^{5/3} d\lambda \int_{-\infty}^\infty G(e^{-\pi i/3}\lambda^{2/3}(-\mu)) e^{i\lambda(\mu x_1 + x_2)} d\mu \end{aligned}$$

with $G(w) := \mathrm{Ai}'(w)/\mathrm{Ai}(w)$. Then, we have

$$\mathrm{SS}(f)\cap\{\eta_2>0\} \subset \{x=0\}\cup\{(x_1,x_2;\eta_1,\eta_2); x_2=0, x_1\ge 0, \eta_1=0\}. \tag{3.2}$$

Further, $f(x_1,x_2)|_{x_1>0}$ extends to $\{(z_1,x_2)\in\mathbb{C}\times\mathbb{R}; z_1\ne 0, |\arg z_1|<\delta\}$ as a distribution with holomorphic parameter z_1 for a sufficiently small $\delta>0$.

Proof. Note that $B(w) := \log(\mathrm{Ai}(w))$ is holomorphic in $\Omega_\delta := \{w\in\mathbb{C}; |w| < \delta|\} \cup \{|\arg w| \le \pi - \delta\}$ for some small $\delta > 0$ and satisfies an asymptotic expansion

$$B(w) \sim -\frac{2}{3}w^{3/2} - \frac{\log w}{4} - \log(2\sqrt{\pi}) \quad (|w|\to\infty,\ w\in\Omega).$$

Hence $G(w) = B'(w)$ is a holomorphic function satisfying $|G(w)| \le C(1+|w|)^{1/2}$ in $\{|\arg w| \le 5\pi/6\}$ for some $C>0$. Therefore $f(x)$ is a kernel function of an analytic pseudodifferential operator with constant coefficients in $\{(*,*;\eta_1,\eta_2); \eta_2>0, \eta_1\ne 0\}$. Thus we have

$$\mathrm{SS}(f)\cap\{\eta_2>0,\eta_1\ne 0\} \subset \{(x_1,x_2;\eta_1,\eta_2); x=0\}.$$

In order to prove

$$\mathrm{SS}(f)\cap\{\eta_2>0,\eta_1=0\} \subset \{(x_1,x_2;\eta_1,\eta_2); x_2=0, x_1\ge 0\}, \tag{3.3}$$

it is sufficient to prove (3.3) for $f^\epsilon(x)$ with

$$f^\epsilon(x) := \int_0^\infty \lambda^{5/3} d\lambda \int_{-\epsilon}^{\epsilon} G(e^{-\pi i/3}\lambda^{2/3}(-\mu)) e^{i\lambda(\mu x_1 + x_2)} d\mu$$

instead of f for some small $\epsilon > 0$. On the other hand, at any fixed point $(\overset{\circ}{x}_1, \overset{\circ}{x}_2)$ with $\overset{\circ}{x}_2 \neq 0$, taking $\epsilon > 0$ as $\epsilon|\overset{\circ}{x}_1| < |\overset{\circ}{x}_2|$, we get the analyticity of f^ϵ at $(\overset{\circ}{x}_1, \overset{\circ}{x}_2)$ by the change of the integration path $[0, +\infty) \to [0, +\infty)e^{\pm i\epsilon'}$ of λ for some small $\epsilon' > 0$ (choose the same signature with $\overset{\circ}{x}_2$). Thus we have

$$\text{(3.4)} \qquad \mathrm{SS}(f) \cap \{\eta_2 > 0, \eta_1 = 0\} \subset \{x_2 = 0\}.$$

Moreover, since $G(e^{-\pi i/3}w)$ is a holomorphic function in $\{w \in \mathbb{C}; \operatorname{Im} w \geq 0\}$ with a polynomial growth order as $|w| \to \infty$, we can change the integration path $\mathbb{R}$ of μ into $\mathbb{R} - iN$ for any large $N > 0$. Hence we have $f(x) = 0$ if $x_1 < 0$: In particular,

$$\mathrm{SS}(f) \subset \{x_1 \geq 0\}.$$

(Similar interesting arguments were given in [1]). This concludes the estimate (3.2). The partial holomorphic extension $f(z_1, x_2)$ of $f(x_1, x_2)|_{x_1 > 0}$ is also obtained similarly. Namely, change the integration path $\mathbb{R}$ of μ into $\mathbb{R}e^{-i\theta}$ where $\theta = \arg z_1$ for $z_1 \neq 0, |\arg z_1| \ll 1$. □

Example 3.6. (A solution with non-gliding analytic singularities). By the same argument with Theorem 4.5 of [10], we get a distribution solution $u(t, x_1, x_2)$ of

$$\text{(3.5)} \qquad \begin{cases} (\partial_t^2 - t\partial_{x_2}^2 + \partial_{x_1}\partial_{x_2})u(t,x) = 0 \quad (t > 0), \\ u(+0, x) = 0, \quad \partial_t u(+0, x) = 1/(x_2 + ix_1^4 + i0). \end{cases}$$

Since the analytic singularity of $\partial_t u(+0, x)$ is at just one point $\{x = 0\}$, it gives a counter example for (4) of Theorem 1.2. Indeed, let $G_j(w)$ $(j = 1, 2)$ be a system of independent solutions of $G''(w) + wG(w) = 0$. Then, a formal solution of (3.5) is given by the Fourier inverse transformation with respect to η_1, η_2 of

$$\widetilde{u}(t, \eta) := \frac{|\eta_2|^{-\frac{2}{3}}}{W} \left(\frac{G_2(|\eta_2|^{\frac{2}{3}}t + \varphi(\eta))}{G_2(\varphi(\eta))} - \frac{G_1(|\eta_2|^{\frac{2}{3}}t + \varphi(\eta))}{G_1(\varphi(\eta))} \right) G_1(\varphi(\eta))G_2(\varphi(\eta))\, \widetilde{u}_1(\eta),$$

where $W = G_1(w)G_2'(w) - G_2(w)G_1'(w) \equiv G_1(0)G_2'(0) - G_1'(0)G_2(0) \neq 0$, $\varphi(\eta) = |\eta_2|^{2/3}(-\eta_1/\eta_2)$, and $\widetilde{u_1}(\eta)$ is the Fourier transformation of $1/(x_2 + ix_1^4 + i0)$. Setting $G_1(w) = \mathrm{Ai}(e^{\pi i/3}w), G_2(w) = \mathrm{Ai}(e^{-\pi i/3}w)$, we know that $G_j(|\eta_2|^{\frac{2}{3}}t + \varphi(\eta))/G_j(\varphi(\eta))$ is a slowly increasing continuous function with respect to η in $\{t \geq 0, \eta_2 \geq 1\}$ for $j = 1, 2$.

Therefore we have only to show that $G_1(\varphi(\eta))G_2(\varphi(\eta))\,\widetilde{u}_1(\eta)$ is a slowly increasing function as $|\eta| \to \infty$ in $\eta_2 \geq 1$. Note that $|G_1(\varphi(\eta))G_2(\varphi(\eta))| \leq C\exp(4|\eta_2|(\eta_1/\eta_2)_+^{3/2}/3)$ for a constant $C > 0$ (here, $(t)_+ = \max\{t, 0\}$), and that

$$\widetilde{u}_1(\eta) = 2\pi i\eta_2^{-1/4}Y(\eta_2)\int_{-\infty}^{\infty}\exp\big(-i\eta_1\eta_2^{-1/4}y - y^4\big)dy.$$

On the other hand, by the proof in Lemma 4.4 in [10] we have an estimate

$$\left|\int_{-\infty}^{\infty}\exp\big(-i\eta_1\eta_2^{-1/4}y - y^4\big)dy\right| \leq C'\exp\left(-\frac{|\eta_1\eta_2^{-1/4}|^{4/3}}{C'}\right)$$

for a large constant C'. Therefore, since $4/3 < 3/2$, we get an estimate

$$|G_1(\varphi(\eta))G_2(\varphi(\eta))\,\widetilde{u}_1(\eta)| \leq 2\pi CC'\exp\left(-|\eta_2|\left(\frac{|\eta_1/\eta_2|^{4/3}}{C'} - \frac{4|\eta_1/\eta_2|^{3/2}}{3}\right)\right) \leq 2\pi CC'$$

in $\{\eta_2 \geq 1, |\eta_1/\eta_2| < (3/(4C'))^6\}$. Hence $u_1(t,x)$ gives a microlocal distribution solution in $\{(t, x_1, x_2; \tau, \eta_1, \eta_2); t > 0, |\eta_1| < (3/(4C'))^6\eta_2\}$ to (3.5).

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