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Boundary value problem for Hyperfunction solutions to Fuchsian systems

By

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Abstract

In the framework of algebraic analysis, a general boundary value morphism is defined for any hyperfunction solutions to the Fuchsian system of analytic linear partial differential equations in derived category, and the injectivity of this morphism in zero-th cohomologies (that is, the Holmgren type theorem) is proved. Moreover, under a kind of hyperbolicity condition, it is proved that this morphism is surjective (that is, the solvability). These results extend that of H. Tahara and Laurent-Monteiro Fernandes to general Fuchsian systems.

Introduction

In this article, we announce of results about boundary value problems for hyperfunction solutions along an initial boundary to the *Fuchsian system* of analytic linear differential equations in the framework of *Algebraic Analysis.*

Fuchsian partial differential operator was first defined by Baouendi-Goulaouic [1], This class includes non-characteristic type as a special case, and Cauchy-Kovalevskaja type theorem (that is, unique solvability for Cauchy problem) was proved in [1] under the conditions of characteristic exponents. Next, Tahara [22] defined a *Fuchsian Volevič system* as a generalization of Fuchsian partial differential operator, and proved a Cauchy-Kovalevskaja type theorem in the complex domain for holomorphic solutions under the conditions of characteristic exponents. Moreover Laurent-Monteiro Fernandes [10] defined a Fuchsian $\mathscr{D}_X$-Module, and they proved a Cauchy-Kovalevskaja type theorem in the complex domain in general settings; that is, without conditions of characteristic

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exponents. Here and in what follows, we shall write a *Ring* or a *Module* etc. with capital letters, instead of a *sheaf of rings* or a *sheaf of left modules* etc. We remark that the notion of Fuchsian $\mathscr{D}_X$-Modules includes Fuchsian Volevič systems.

For Cauchy problem in the framework of hyperfunctions on the real domain, Tahara [22] obtained a Cauchy-Kovalevskaja type theorem for hyperfunction solutions with a real analytic parameter to *hyperbolic Fuchsian Volevič system* under the conditions of characteristic exponents. As for the uniqueness of hyperfunction solutions, Oaku [16] and Oaku-Yamazaki [19] extended the uniqueness result to Fuchsian systems. Further Yamazaki [25] obtained the unique solvability theorem of Cauchy problem for general Fuchsian hyperbolic systems in the framework of hyperfunctions (that is, hyperfunctions with a real analytic parameter, or mild hyperfunctions) without the conditions of characteristic exponents.

Next, for a boundary value problems for hyperfunction solutions, Laurent-Monteiro Fernandes [11] give a general framework, and using results of [9], for any regular-specializable system (i.e. Fuchsian with constant characteristic exponents case), they defined an injective boundary value morphism (see also [14], [15]), and discussed solvability. For a microlocal counterpart, see Yamazaki [24].

In this paper, along the line of [11] and [24], we shall define an injective boundary value morphism for hyperfunction solutions to general Fuchsian system and state the unique solvability theorem for the boundary value problem in the category of hyperfunctions. For this purpose, by using precise analysis due to Tahara [22] and an idea of Oaku [18], we shall define a sort of nearby cycles for general Fuchsian Modules.

Details of this article will be appeared in a forthcoming paper.

§ 1. Preliminaries

In this section, we shall fix the notation and recall known results used in later sections. Our main reference is Kashiwara-Schapira [7].

We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the sets of all the integers, real numbers and complex numbers respectively. Moreover we set $\mathbb{N} := \{n \in \mathbb{Z}; n \geqslant 1\} \subset \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}_{>0} := \{r \in \mathbb{R}; r > 0\} \subset \mathbb{R}_{\geqslant 0} := \{r \in \mathbb{R}; r \geqslant 0\}$ and $\mathbb{C}^{\times} := \mathbb{C} \setminus \{0\}$.

In this paper, all the manifolds are assumed to be *paracompact*. Let Z be a manifold. For a subset $A \subset Z$, we denote by $\operatorname{Int} A$ and $\operatorname{Cl} A$ the interior and the closure of A respectively. Let $\mathcal{A}$ be a Ring on Z. We denote by $\mathcal{A}^{\mathrm{op}}$ the opposed Ring, and we regard right $\mathcal{A}$-Modules as (left) $\mathcal{A}^{\mathrm{op}}$-Modules. We denote by $\mathfrak{Mod}(\mathcal{A})$ the category of $\mathcal{A}$-Modules, and by $\mathfrak{Coh}(\mathcal{A})$ the full subcategory of $\mathfrak{Mod}(\mathcal{A})$ consisting of coherent $\mathcal{A}$-Modules. Further we denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ the bounded derived category of complexes of $\mathcal{A}$-Modules, and by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{A})$ the full subcategory of $\mathbf{D}^{\mathrm{b}}(\mathcal{A})$ consisting of objects with coherent cohomologies. We set $\mathbf{D}^{\mathrm{b}}(Z) := \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Z)$ etc. for short. Set $* \otimes * :=$

$* \underset{\mathbb{C}_Z}{\otimes} *$ etc. We denote by or_Z the orientation sheaf. Let $f\colon W \to Z$ be a continuous mapping between manifolds. Then the relative orientation sheaf is defined by $or_{W/Z} := or_W \otimes f^{-1} or_Z$. Further $\omega_{W/Z} = or_{W/Z}[\dim W - \dim Z]$ denotes the dualizing complex, and $\omega_{W/Z}^{\otimes -1} = or_{W/Z}[\dim Z - \dim W]$ its dual. If $\tau\colon E \to Z$ is a vector bundle over a manifold Z, we set $\dot{E} := E \setminus Z$ and $\dot{\tau}$ the restriction of τ to $\dot{E}$. Let $\pi\colon E^* \to Z$ the dual bundle. We set

$$P_E^+ := \{(v, \xi) \in E \underset{Z}{\times} E^*;\ \langle v, \xi \rangle > 0\},$$

and denote by $\hat{p}_1\colon P_E^+ \to E$ and $\hat{p}_2\colon P_E^+ \to E^*$ the canonical projections. Let $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(E) \subset \mathbf{D}^{\mathrm{b}}(E)$ be the subcategory of the bounded derived category of sheaves such that each cohomology is conic. If $\tau\colon E \to Z$ is a complex vector bundle, we denote by $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}^\times}(E)$ the subcategory of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(E)$ consisting of objects $\mathscr{F}$ such that for any $i \in \mathbb{Z}$, each $H^i(\mathscr{F})$ is locally constant on the orbits of the action $\mathbb{C}^\times$.

Let $\mathscr{F} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(E)$. We denote by $\mathscr{F}^\wedge$ the Fourier-Sato transform of $\mathscr{F}$.

1.1. Proposition ([23, Corollary A.2], cf. [20, Chapter I]). *Let $\mathscr{F} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(E)$. Then there exists the following distinguished triangle*:

$$\mathscr{F} \to \tau^! \boldsymbol{R}\tau_! \mathscr{F} \to \boldsymbol{R}\hat{p}_{1*}\hat{p}_2^! \mathscr{F}^\wedge \xrightarrow{+1}. \tag{1.1}$$

Let $\mathcal{F}$ be an object of $\mathbf{D}^{\mathrm{b}}(Z)$, and $T^*Z \to Z$ the cotangent bundle of Z. We denote by $\mathrm{SS}(\mathcal{F})$ the *microsupport* of $\mathcal{F}$ due to Kashiwara-Schapira (see [7]). $\mathrm{SS}(\mathcal{F})$ is a closed conic involutive subset of T^*Z and described as follows: Let $\mathring{p}$ be a point of T^*Z. Then $\mathring{p} \notin \mathrm{SS}(\mathcal{F})$ if the following condition holds: there exists a neighborhood U of $\mathring{p}$ in T^*Z such that for any $\mathring{z} \in Z$ and any real valued real analytic function ψ defined on a sufficiently small neighborhood of $\mathring{z}$ satisfying $(\mathring{z}; d\psi(\mathring{z})) \in U$, it follows that

$$\boldsymbol{R}\Gamma_{\{z;\, \psi(z) \geqslant \psi(\mathring{z})\}}(\mathcal{F})_{\mathring{z}} = 0.$$

Note that $\mathrm{SS}(\mathcal{F}) \cap T^*_Z Z = \operatorname{supp} \mathcal{F}$.

Next, let Z be a complex manifold with a local coordinate system $z = x + \sqrt{-1}\, y$, we use the following identifications as in [20, Chapter I]:

$$TZ \ni (z; \langle v, \partial_z \rangle) \leftrightarrow (x, y; \langle \operatorname{Re} v, \partial_x \rangle + \langle \operatorname{Im} v, \partial_y \rangle) \in TZ^{\mathbb{R}},$$
$$T^*Z \ni (z; \langle \zeta, dz \rangle) \leftrightarrow (x, y; \langle \operatorname{Re} \zeta, dx \rangle - \langle \operatorname{Im} \zeta, dy \rangle) \in T^*Z^{\mathbb{R}},$$

where $Z^{\mathbb{R}}$ denotes the underlying real manifold of Z. Thus, for the complex dual inner product $\langle *, * \rangle\colon TZ \underset{Z}{\times} T^*Z \to \mathbb{C}$, the corresponding real dual inner product is $\operatorname{Re}\langle *, * \rangle\colon TZ \underset{Z}{\times} T^*Z \to \mathbb{R}$.

§ 2. General Boundary Values

Let M be an $(n+1)$-dimensional real analytic manifold and N a one-codimensional closed real analytic submanifold of M. Let X and Y be complexifications of M and N respectively such that Y is a closed submanifold of X and that $Y \cap M = N$. Let $\tilde{z} = \tilde{x} + \sqrt{-1}\,\tilde{y}$ be a local coordinate system of X such that $\tilde{x}$ is a local coordinate system of M. We assume that there exists a $(2n+1)$-dimensional real analytic submanifold L of X containing both M and Y such that the triplet (N, M, L) is locally isomorphic to the triplet $(\{(x,0) \in \mathbb{R}^n \times \{0\}\}, \{(x,t) \in \mathbb{R}^{n+1}\}, \{(z,t) \in \mathbb{C}^n \times \mathbb{R}\})$ by a local coordinate system $\tilde{z} = (z, \tau)$ with $\tilde{x} = (x_1, \ldots, x_n, t) = (x, t)$, $z = x + \sqrt{-1}\,y$ and $\tau = t + \sqrt{-1}\,s$ around each point of N (i.e. L is a partial complexification). We say such a local coordinate system *admissible*, and under this local coordinate system, we have:

$$
\begin{array}{ccccc}
N = \mathbb{R}^n_x \times \{0\} & \xrightarrow{f_N} & M = \mathbb{R}^n_x \times \mathbb{R}_t & \xrightarrow{\iota} & \\
\downarrow \iota_N & & \downarrow \iota_M & & \\
Y = \mathbb{C}^n_z \times \{0\} & \xrightarrow{g} & L = \mathbb{C}^n_z \times \mathbb{R}_t & \xrightarrow{h} & X = \mathbb{C}^n_z \times \mathbb{C}_\tau .
\end{array}
\qquad (Y \xrightarrow{f} X) \tag{2.1}
$$

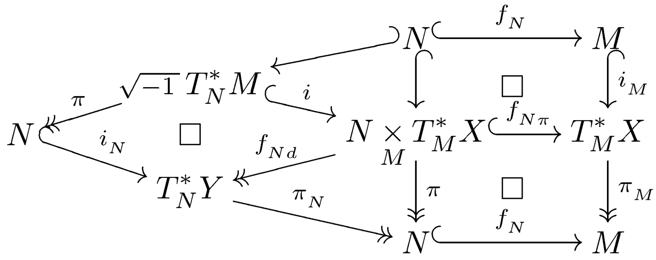
Then we identify $\mathscr{or}_{N/Y}$ with $f_N^{-1}\mathscr{or}_{M/L}$.

2.1. Remark. Let (z, τ) and $(\dot{z}, \dot{\tau})$ be admissible local coordinate systems, and $(\dot{z}, \dot{\tau}) = (\Psi(z,\tau), \psi(z,\tau))$ a holomorphic coordinate transformation. Then, since $\dot{t} := \operatorname{Re}\dot{\tau} = \psi(z,\tau)|_L = \psi(z,t)$ is real valued and holomorphic with respect to z-variables, we have $\dot{t} = \psi(t)$, hence we can show that $\dot{\tau} = \psi(\tau)$.

Let $\tau_N \colon T_N M \to N$ and $\pi_N \colon T_N^* M \to N$ be the *normal* and the *conormal bundles* to N in M respectively. By an admissible local coordinate system, we often identify normal bundles with base spaces; for example, $T_Y X = X$, $T_M X = X$, $T_N M = M$ etc. (i.e. we identify $(x;t) \in T_N M$ with $(x,t) \in M$). We denote by

$$
\begin{aligned}
(\tilde{z}; \tilde{z}^*) &= (z, \tau; z^*, \tau^*) = (\tilde{x} + \sqrt{-1}\,\tilde{y}; \tilde{x}^* + \sqrt{-1}\,\tilde{y}^*) \\
&= (x + \sqrt{-1}\,y, t + \sqrt{-1}\,s; x^* + \sqrt{-1}\,y^*, t^* + \sqrt{-1}\,s^*)
\end{aligned} \tag{2.2}
$$

the associated local coordinate system of T^*X with the local coordinate system in (2.1). The mapping f induces mappings:



where π_N, π_M and π are canonical projections, i_N, i_M and i are zero-section embeddings, and $\square$ means that the square is *Cartesian*. Assume that $N = \varphi^{-1}(0)$ for an analytic function φ such that we may choose that $\varphi(\tilde{x}) = t$. We use the same symbol $\varphi\colon X \to \mathbb{C}$ to stand for the complexification, and we may assume that $\varphi(\tilde{z}) = \tau$. Then $d\varphi$ induces $\tilde{\varphi}\colon T_Y X \to \mathbb{C}$, and we denote by $\hat{\sigma}\colon Y \to \dot{T}_Y X$ the section of $T_Y X \to \mathbb{C}$ given by $\tilde{\varphi}^{-1}(1)$, and by ${}^*\hat{\sigma}\colon Y \to \dot{T}^*_Y X$ the section of $T^*_Y X \to \mathbb{C}$ given by $d\varphi$. In the same way, $d\varphi$ induces $\tilde{\varphi}\colon T_N M \to \mathbb{R}$, and we can define mappings $\hat{s}\colon N \to \dot{T}_N M$ and ${}^*\hat{s}\colon N \to \sqrt{-1}\,\dot{T}^*_N M = \dot{T}^*_M X \cap \dot{T}^*_Y X$. Under the local coordinate system in (2.1), we have

$$\hat{\sigma}(z) = (z, 1), \quad {}^*\hat{\sigma}(z) = (z; 1 \cdot d\tau), \quad \hat{s}(x) = (x, 1), \quad {}^*\hat{s}(x) = (x; \sqrt{-1}\, dt).$$

We set

$$\begin{aligned} &\dot{T}_N M^+ := \mathbb{R}_{>0}\hat{s}(N) \simeq \{(x,t);\, t > 0\} \subset T_N M^+ := \dot{T}_N M^+ \cup T_N N \simeq \{(x,t);\, t \geqslant 0\}, \\ &\dot{T}^*_N M^+ := \frac{1}{\sqrt{-1}}\, \mathbb{R}_{>0}{}^*\hat{s}(N) \simeq \{(x;t^*);\, t^* > 0\}. \end{aligned}$$

By (1.1), for any $\mathscr{F} \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}_{>0}}(T_N M)$ we can obtain

$$\boldsymbol{R}\tau_{N!}\mathscr{F} \otimes \mathit{or}_{N/M} \to \boldsymbol{R}\tau_{N*}\boldsymbol{R}\Gamma_{T_N M^+}(\mathscr{F}) \otimes \mathit{or}_{N/M} \to \hat{s}^{\,-1}\mathscr{F} \xrightarrow{+1} . \tag{2.3}$$

As usual, let ν_* and μ_* be *specialization* and *microlocalization functors* respectively. We write $M \smallsetminus N = \Omega_+ \sqcup \Omega_-$, where each $\Omega_\pm$ is an open subset and $\partial\Omega_\pm = N$. We set $M_+ := \Omega_+ \sqcup N$. By an admissible local coordinate system, we can write

$$\Omega_+ = \{(x,t) \in M;\, t > 0\} \subset M_+ = \{(x,t) \in M;\, t \geqslant 0\}.$$

By (2.3), we can prove:

2.2. Proposition ([11, Proposition 1.2.3], [15]). *For any $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$, there exists the following morphism between distinguished triangles*:

$$\begin{array}{ccccccc}
\boldsymbol{R}\Gamma_N(\mathcal{F}) & \longrightarrow & \boldsymbol{R}\Gamma_{M_+}(\mathcal{F})|_N & \longrightarrow & \boldsymbol{R}\Gamma_{\Omega_+}(\mathcal{F})|_N \otimes \mathit{or}_{N/M} & \xrightarrow{+1} & \\
\| & & \| & & \| & & \\
\boldsymbol{R}\Gamma_N(\mathcal{F}) & \to & \boldsymbol{R}\tau_{N*}\boldsymbol{R}\Gamma_{T_N M^+}(\nu_N(\boldsymbol{R}\Gamma_M(\mathcal{F}))) & \longrightarrow & \hat{s}^{\,-1}\nu_N(\boldsymbol{R}\Gamma_M(\mathcal{F})) \otimes \mathit{or}_{N/M} & \xrightarrow{+1} & \\
\| & & \downarrow & & \downarrow & & \\
\boldsymbol{R}\Gamma_N(\mathcal{F}) & \longrightarrow & \boldsymbol{R}\tau_{N*}\boldsymbol{R}\Gamma_{T_N M^+}(\nu_Y(\mathcal{F})) & \longrightarrow & \hat{s}^{\,-1}\boldsymbol{R}\Gamma_{T_N M}(\nu_Y(\mathcal{F})) \otimes \mathit{or}_{N/M} & \xrightarrow{+1} & .
\end{array}$$

2.3. Proposition. *Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$, and assume that $\nu_Y(\mathcal{F}) \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}^\times}(T_Y X)$. Then there exist the following distinguished triangles*:

$$\begin{aligned} &\boldsymbol{R}\Gamma_Y(\mathcal{F}) \to {}^*\hat{\sigma}^{-1}\mu_Y(\mathcal{F}) \to \hat{\sigma}^{-1}\nu_Y(\mathcal{F})[-1] \xrightarrow{+1}, \\ &f^{-1}\mathcal{F} \longrightarrow \hat{\sigma}^{-1}\nu_Y(\mathcal{F}) \longrightarrow {}^*\hat{\sigma}^{-1}\mu_Y(\mathcal{F})[1] \xrightarrow{+1} . \end{aligned}$$

2.4. Theorem. *Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$, and assume that $\nu_Y(\mathcal{F}) \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}^\times}(T_Y X)$. Then there exist the following isomorphisms of distinguished triangles*:

$$\boldsymbol{R}\Gamma_{\underline{\underline{N}}}(\mathcal{F}) \longrightarrow \boldsymbol{R}\tau_{N*}\boldsymbol{R}\Gamma_{T_N\underline{\underline{M}}^+}(\nu_Y(\mathcal{F})) \longrightarrow \hat{s}^{-1}\boldsymbol{R}\Gamma_{T_N M}(\nu_{\underline{\underline{Y}}}(\mathcal{F})) \otimes or_{N/M} \xrightarrow{+1}$$

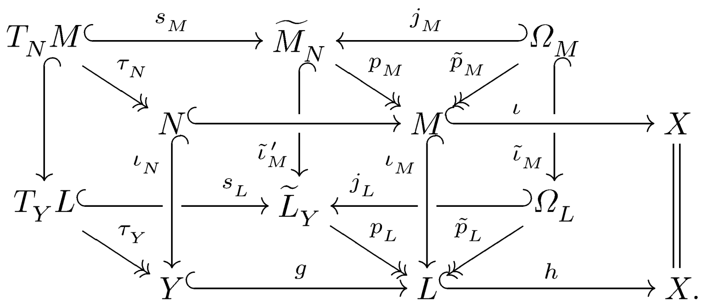
$$\boldsymbol{R}\Gamma_{\underline{\underline{N}}}(\mathcal{F}) \longrightarrow {}^*\hat{s}^{-1}\boldsymbol{R}\Gamma_{\sqrt{-1}\,T^*_N \underline{\underline{M}}}(\mu_Y(\mathcal{F})) \otimes \omega^{\otimes -1}_{N/M} \longrightarrow \hat{s}^{-1}\boldsymbol{R}\Gamma_{T_N M}(\nu_{\underline{\underline{Y}}}(\mathcal{F})) \otimes or_{N/M} \xrightarrow{+1}$$

$$\boldsymbol{R}\Gamma_N(\mathcal{F}) \longrightarrow \boldsymbol{R}\Gamma_N({}^*\hat{\sigma}^{-1}\mu_Y(\mathcal{F})) \longrightarrow \boldsymbol{R}\Gamma_N(\hat{\sigma}^{-1}\nu_Y(\mathcal{F}))[-1] \xrightarrow{+1} \cdot$$

In particular we obtain

$$\boldsymbol{R}\Gamma_{\Omega_+}(\mathcal{F})|_N \otimes \omega^{\otimes -1}_{M/X} = \hat{s}^{-1}\nu_N(\boldsymbol{R}\Gamma_M(\mathcal{F})) \otimes \omega^{\otimes -1}_{M/X} \to \boldsymbol{R}\Gamma_N(\hat{\sigma}^{-1}\nu_Y(\mathcal{F})) \otimes \omega^{\otimes -1}_{N/Y}. \tag{2.4}$$

Next, we denote by $\widetilde{M}_N$ and $\widetilde{L}_Y$ the normal deformations of N and Y in M and L respectively and regard $\widetilde{M}_N$ as a closed submanifold of $\widetilde{L}_Y$. We have the following commutative diagram:



Using an admissible local coordinate system, we can write:

$$p_L \colon \widetilde{L}_Y = \{(z,t;r);\, r \in \mathbb{R},\, (z,rt) \in L\} \ni (z,t;r) \mapsto (z,rt) \in L,$$
$$p_M \colon \widetilde{M}_N = \{(x,t;r);\, r \in \mathbb{R},\, (x,rt) \in M\} \ni (x,t;r) \mapsto (x,rt) \in M,$$

$$T_Y L = \widetilde{L}_Y \cap \{(z,t;r);\, r = 0\}, \qquad \Omega_L = \widetilde{L}_Y \cap \{(z,t;r);\, r > 0\},$$
$$T_N M = \widetilde{M}_N \cap \{(x,t;r);\, r = 0\}, \quad \Omega_M = \widetilde{M}_N \cap \{(x,t;r);\, r > 0\}.$$

The mappings $\tilde{\tau} \colon T_Y L \to Y$, $p_L \colon \widetilde{L}_Y \to L$, $s_L \colon T_Y L \to \widetilde{L}_Y$ and $g \colon Y \to L$ induce natural mappings:

$$\begin{array}{ccccccc}
N \underset{M}{\times} T^*_M L & \xrightarrow{g_{Nd}} & T^*_N Y & \xleftarrow{\tilde{\tau}_\pi} & T_N M \underset{N}{\times} T^*_N Y & \underset{\tilde{\tau}_d}{\xrightarrow{\sim}} & T^*_{T_N M} T_Y L \\
\downarrow{\scriptstyle g_{N\pi}} & & & & & & \uparrow{\scriptstyle s_{Ld}}\,\wr \\
T^*_M L & \xleftarrow{p_{L\pi}} & \widetilde{M}_N \underset{M}{\times} T^*_M L & \underset{p_{Ld}}{\xrightarrow{\sim}} & T^*_{\widetilde{M}_N} \widetilde{L}_Y & \xleftarrow{s_{L\pi}} & T_N M \underset{\widetilde{M}_N}{\times} T^*_{\widetilde{M}_N} \widetilde{L}_Y\,,
\end{array}$$

and by these mappings we use the following identifications:

$$T_N M \underset{N}{\times} T^*_N Y = T^*_{T_N M} T_Y L = T_N M \underset{\widetilde{M}_N}{\times} T^*_{\widetilde{M}_N} \widetilde{L}_Y, \quad \widetilde{M}_N \underset{M}{\times} T^*_M L = T^*_{\widetilde{M}_N} \widetilde{L}_Y,$$

and we denote by

$$\pi_{N|M}\colon T^*_{T_NM}T_YL = T_NM \underset{N}{\times} T^*_NY = T^*_{T_NM}T_YL \to T_NM,$$
$$\pi_{N,M}\colon T^*_{\widetilde{M}_N}\widetilde{L}_Y = \widetilde{M}_N \underset{M}{\times} T^*_ML \to \widetilde{M}_N,$$

the natural projections. $T_YL \smallsetminus T_YY$ has two components with respect to its fiber. We denote by $\dot{T}_YL^+$ one of them as $\dot{T}_NM^+ = \dot{T}_YL^+ \cap T_NM$ and represent by fixing a local coordinate system

$$\dot{T}_YL^+ = \{(z,t) \in T_YL;\, t > 0\}$$

(in this case we choose $\varphi(\tilde{z}) = \tau$). Define open embeddings i_+ and i_{N+} by:

$$\begin{array}{ccc} \dot{T}_YL^+ & \overset{i_+}{\hookrightarrow} & T_YL \\ \cup & & \cup \\ \dot{T}_NM^+ & \overset{i_{N+}}{\hookrightarrow} & T_NM. \end{array}$$

We regard $\dot{T}_NM^+ \underset{N}{\times} T^*_NY$ as an open set of $T^*_{T_NM}T_YL$. Moreover i_+ induces mappings:

$$\begin{array}{ccccc} T^*_{\dot{T}_NM^+}\dot{T}_YL^+ & \overset{\sim}{\longleftarrow} & \dot{T}_NM^+ \underset{T_NM}{\times} T^*_{T_NM}T_YL & \overset{i_{\pi+}}{\hookrightarrow} & T^*_{T_NM}T_YL \\ & & \wr\| & & \|\wr \\ & & \dot{T}_NM^+ \underset{N}{\times} T^*_NY & \overset{i_{N+}\times 1\!\!1}{\hookrightarrow} & T_NM \underset{N}{\times} T^*_NY. \end{array}$$

Hence we identify $T^*_{\dot{T}_NM^+}\dot{T}_YL^+$ with $\dot{T}_NM^+ \underset{N}{\times} T^*_NY$, and $i_{\pi+}$ with $i_{N+} \times 1\!\!1$. We set

$$\tilde{\tau}_{\pi+} := \tilde{\tau}_\pi \circ i_{\pi+}\colon \dot{T}_NM^+ \underset{N}{\times} T^*_NY \twoheadrightarrow T^*_NY.$$

Then we recall

2.5. Theorem ([24, Theorem 2.2]). *Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$, and assume that $\nu_Y(\mathcal{F}) \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}^\times}(T_YX)$. Then there exists the following natural isomorphism*:

$$i^{-1}_{\pi+}\,\mu_{T_NM}(\nu_Y(\boldsymbol{R}\Gamma_L(\mathcal{F})) \otimes \omega^{\otimes -1}_{M/X} \simeq \tilde{\tau}^{-1}_{\pi+}\mu_N(\hat{\sigma}^{-1}\nu_Y(\mathcal{F})) \otimes \omega^{\otimes -1}_{N/Y}\,.$$

2.6. Definition. Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$, and assume that $\nu_Y(\mathcal{F}) \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}^\times}(T_YX)$. By virtue of Theorem 2.5 we define:

$$(2.5)\quad \gamma_+\colon i^{-1}_{\pi+}\, s^{-1}_{L\pi}\mu_{\widetilde{M}_N}(\boldsymbol{R}\Gamma_{\Omega_L}(p^{-1}_L\boldsymbol{R}\Gamma_L(\mathcal{F}))) \otimes \omega^{\otimes -1}_{M/X}$$
$$\to i^{-1}_{\pi+}\,\mu_{T_NM}(\nu_Y(\boldsymbol{R}\Gamma_L(\mathcal{F}))) \otimes \omega^{\otimes -1}_{M/X} \simeq \tilde{\tau}^{-1}_{\pi+}\,\mu_N(\hat{\sigma}^{-1}\nu_Y(\mathcal{F})) \otimes \omega^{\otimes -1}_{N/Y}\,.$$

We can see that the restriction of (2.5) to T_NM^+ coincides with (2.4).

Next, we recall the definition of the near-hyperbolicity condition:

2.7. Definition ([11, Definition 1.3.1]). Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$. Then we say that $\mathcal{F}$ is *near-hyperbolic* at $\mathring{x} \in N$ in the ϵdt-codirection ($\epsilon = \pm$) if there exist positive constants C and ε_1 such that

$$\begin{aligned}\mathrm{SS}(\mathcal{F}) \cap \{(z, \tau; z^*, \tau^*) \in T^*X;\ |z - \mathring{x}| < \varepsilon_1,\ |\tau| < \varepsilon_1, \epsilon t > 0\} \\ \subset \{(z, \tau; z^*, \tau^*) \in T^*X;\ |t^*| \leqslant C(((|y| + |s|)|y^*| + |x^*|)\}\end{aligned}$$

holds by the local coordinate system $(z, \tau; z^*, \tau^*)$ of T^*X in (2.2).

2.8. Theorem. *Let $\mathcal{F} \in \mathbf{D}^{\mathrm{b}}(X)$. Assume that $\nu_Y(\mathcal{F}) \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}^\times}(T_Y X)$ and $\mathcal{F}$ is near-hyperbolic at $\mathring{x} \in N$ in the dt-codirection. Then, for any $p^* = (\hat{s}(\mathring{x}); \sqrt{-1}\,\mathring{y}^*) \in T^*_{\dot{T}_N M^+}\dot{T}_Y L^+$ and $p_0 := (\mathring{x}; \sqrt{-1}\,\mathring{y}^*) \in T^*_N Y$, the morphism γ_+ induces isomorphisms*:

$$\begin{aligned}\gamma_+ \colon s_{L\pi}^{-1}\mu_{\widetilde{M}_N}(\boldsymbol{R}\varGamma_{\varOmega_L}(p_L^{-1}\boldsymbol{R}\varGamma_L(\mathcal{F})))_{p^*} \otimes \omega_{M/X}^{\otimes -1} \simeq \mu_{T_N M}(\nu_Y(\boldsymbol{R}\varGamma_L(\mathcal{F})))_{p^*} \otimes \omega_{M/X}^{\otimes -1} \\ \simeq \mu_N(\hat{\sigma}^{-1}\nu_Y(\mathcal{F}))_{p_0} \otimes \omega_{N/Y}^{\otimes -1}.\end{aligned}$$

2.9. Corollary. *Under the assumption of Theorem 2.8, the morphism γ_+ induces isomorphisms*:

$$\begin{aligned}\gamma_+ \colon \boldsymbol{R}\varGamma_{\varOmega_+}(\mathcal{F})_{\mathring{x}} \otimes \omega_{M/X}^{\otimes -1} = \nu_N(\boldsymbol{R}\varGamma_M(\mathcal{F}))_{\hat{s}(\mathring{x})} \otimes \omega_{M/X}^{\otimes -1} \\ \simeq \boldsymbol{R}\varGamma_{T_N M}(\nu_Y(\boldsymbol{R}\varGamma_L(\mathcal{F})))_{\hat{s}(\mathring{x})} \otimes \omega_{M/X}^{\otimes -1} \simeq \boldsymbol{R}\varGamma_N(\sigma^{-1}\nu_Y(\mathcal{F}))_{\mathring{x}} \otimes \omega_{N/Y}^{\otimes -1}.\end{aligned}$$

§ 3. Operators of Infinite Order

We inherit the notation from the preceding section. For a set (or a sheaf) S with a suitable algebraic structure, we denote by $\mathrm{Mat}_{m,n}(S)$ the set of matrices of size $m \times n$ whose components belong to S. We set $\mathrm{Mat}_m(S) := \mathrm{Mat}_{m,m}(S)$, and denote by $\mathbb{1}_m$ the identity matrix of size m. For the theory of $\mathscr{D}$-Modules, we refer to Björk [2], Kashiwara [3]. We denote by $\mathscr{O}_X$ and $\mathscr{D}_X$ the Rings of *holomorphic functions* and *holomorphic partial differential operators* on X. Let $\varOmega_X$ be the sheaf of the *holomorphic forms with maximal degree* on X, and $\varOmega_X^{\otimes -1} := \mathscr{H}om_{\mathscr{O}_X}(\varOmega_X, \mathscr{O}_X)$. Let $\mathscr{D}_{Y\to X} := \mathscr{O}_Y \underset{f^{-1}\mathscr{O}_X}{\otimes} f^{-1}\mathscr{D}_X$ and $\mathscr{D}_{X\leftarrow Y} := \varOmega_Y \underset{\mathscr{O}_Y}{\otimes} \mathscr{D}_{Y\to X} \underset{f^{-1}\mathscr{O}_X}{\otimes} f^{-1}\varOmega_X^{\otimes -1}$ be the *transfer* $(\mathscr{D}_Y \otimes f^{-1}\mathscr{D}_X^{\mathrm{op}})$- and $(f^{-1}\mathscr{D}_X^{\mathrm{op}} \otimes \mathscr{D}_Y)$*-Modules* associated with $f \colon Y \hookrightarrow X$ respectively. We denote by

$$\boldsymbol{D}f^*\mathscr{N} := \mathscr{D}_{Y\to X} \overset{\boldsymbol{L}}{\underset{f^{-1}\mathscr{D}_X}{\otimes}} f^{-1}\mathscr{N},$$

$$\boldsymbol{D}f^{!}\mathscr{N} := \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y^{\mathrm{op}}}(f^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{N}, \mathscr{D}_X) \overset{\boldsymbol{L}}{\underset{f^{-1}\mathscr{D}_X}{\otimes}} \mathscr{D}_{X\leftarrow Y}, \mathscr{D}_Y)[-1],$$

the *inverse image* and the *extraordinary inverse image* respectively in $\mathscr{D}$-Module theory. Under the local coordinate system in (2.1), we set $\vartheta := \tau\partial_\tau$ (or $t\partial_t$ in real case).

3.1. Definition. Let $\mathscr{M} \in \mathfrak{Coh}(\mathscr{D}_X)$. We say that $\mathscr{M}$ is *near-hyperbolic* at $\mathring{x} \in N$ in the ϵdt-codirection if so is $\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ in the sense of Definition 2.7.

3.2. Definition. Let $m \in \mathbb{N}$ and $w \in \mathbb{N}_0$ with $w \leqslant m$. Then we say that P is a *Fuchsian partial differential operator* of weight (m, w) in the sense of Baouendi-Goulaouic [1] if P can be written in the following form:

$$P(z,\tau,\partial_z,\partial_\tau) = \tau^{m-w}\partial_\tau^m + \sum_{i=w}^{m-1} P_i(z,\tau,\partial_z)\,\tau^{i-w}\partial_\tau^i + \sum_{i=0}^{w-1} P_i(z,\tau,\partial_z)\,\partial_\tau^i\,,$$

where $P_i \in \mathscr{D}_{X^{(m-i)}}$ with $[P_i, \tau] = 0$ $(0 \leqslant i \leqslant m)$, and $P_i(z, 0, \partial_z) \in \mathscr{O}_Y$ $(w \leqslant i \leqslant m)$.

We say that P is Fuchsian hyperbolic in the sense of Tahara [22] if the principal symbol is written as $\sigma_m(P)(z,\tau,z^*,\tau^*) = \tau^{m-w}p(z,\tau,z^*,\tau^*)$, and $p(z,\tau,z^*,\tau^*)$ satisfies the following:

$$\text{(3.1)}\quad \begin{cases} \text{If } (x,t;x^*) \text{ are real, all the roots to the equation } p(x,t,x^*,\tau^*) = 0 \text{ with respect} \\ \text{to } \tau^* \text{ are real.} \end{cases}$$

Then $\mathscr{D}_X/\mathscr{D}_X P$ is near-hyperbolic in the $\pm dt$-codirections (see [11, Lemma 1.3.2]).

Note that a Fuchsian partial differential operator of weight $(m, 0)$ is called *an operator with regular singularity along* Y *in a weak sense* in Kashiwara-Oshima [6], and if the weight of P is (m, m), then Y is non-characteristic for $\mathscr{D}_X/\mathscr{D}_X P$.

3.3. Definition. We call a matrix $P = \vartheta - A(z,\tau,\partial_z) \in \mathrm{Mat}_m(\mathscr{D}_X)$ is a *Fuchsian Volevič system* of size m due to Tahara [22] if the following hold: Let $A_{ij}(z,\tau,\partial_z)$ be the (i,j)-component of $A(z,\tau,\partial_z)$.

(1) There exists $\{n_i\}_{i=1}^m \subset \mathbb{Z}$ such that $A_{ij}(z,\tau,\partial_z) \leqslant \mathscr{D}_{X^{(n_i-n_j+1)}}$ for any $1 \leqslant i, j \leqslant m$.

(2) $[A_{ij}, \tau] = 0$ and $A_{ij}(z, 0, \partial_z) \in \mathscr{O}_Y$ for any $1 \leqslant i, j \leqslant m$.

Moreover we say that P is Fuchsian hyperbolic in the sense of Tahara [22] if

$$\det[\tau\tau^*\mathbb{1}_m - \sigma(A)(z,\tau,z^*)] = \tau^m p(z,\tau,z^*,\tau^*),$$

and $p(z,\tau,z^*,\tau^*)$ satisfies the condition (3.1). Then $\mathscr{D}_X^m/\mathscr{D}_X^m P$ satisfies the near-hyperbolicity condition. Here we set $\sigma(A)(z,\tau,z^*) := (\sigma_{n_i-n_j+1}(A_{ij})(z,\tau,z^*))_{i,j=1}^m$

Let $\mathcal{F}_Y(\mathscr{D}_X) \subset \mathfrak{Coh}(\mathscr{D}_X)$ denote the subcategory of Fuchsian $\mathscr{D}_X$-Modules along Y due to Laurent-Monteiro Fernandes [10].

3.4. Example. (1) If P is a Fuchsian partial differential operator, we have $\mathscr{D}_X/\mathscr{D}_X P \in \mathcal{F}_Y(\mathscr{D}_X)$.

(2) If P is a Fuchsian Volevič system of size m, then $\mathscr{D}_X^m/\mathscr{D}_X^m P \in \mathcal{F}_Y(\mathscr{D}_X)$.

3.5. Proposition. *Let $\mathscr{M} \in \mathfrak{Coh}(\mathscr{D}_X)$. Then the following conditions are equivalent:*

(1) $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$.

(2) *Locally, there exists an epimorphism* $\bigoplus_{i=1}^{I} \mathscr{D}_X/\mathscr{D}_X P_i \twoheadrightarrow \mathscr{M}$, *where each P_i is a Fuchsian differential operator with weight $(m_i, 0)$.*

3.6. Proposition. *Let $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ be an exact sequence in $\mathfrak{Coh}(\mathscr{D}_X)$. Then $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$ if and only if $\mathscr{M}', \mathscr{M}'' \in \mathcal{F}_Y(\mathscr{D}_X)$.*

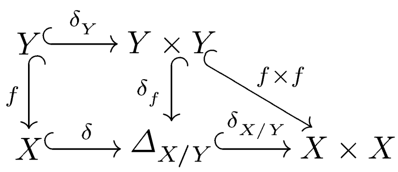
3.7. Proposition. *Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. Then*

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_Y(\mathscr{O}_X)) \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}^\times}(T_Y X).$$

3.8. Definition. We take the admissible local coordinate system in (2.1), and write $X \times X = \{(z, \tau, w, \tau')\}$ on a neighborhood of $Y \times Y = \{(z, 0, w, 0) \in X \times X\}$. We set (see [18])

$$\Delta_{X/Y} := \{(z, \tau, w, \tau') \in X \times X;\, \tau = \tau'\} = \{(z, w, \tau)\}.$$

Then we regard $Y \times Y$ as a closed subset of $\Delta_{X/Y}$. Let $\Delta_Y \subset Y \times Y$ be the diagonal set. We have closed embeddings



where $\delta\colon X \ni (z, \tau) \mapsto (z, z, \tau) \in \Delta_{X/Y}$, $\delta_{X/Y}\colon \Delta_{X/Y} \ni (z, w, \tau) \mapsto (z, \tau, w, \tau) \in X \times X$ etc.

3.9. Remark. $\Delta_{X/Y}$ does not depend on the choice of admissible local coordinate systems on a neighborhood of $Y \times Y$. Indeed, let (z, τ) and $(\dot{z}, \dot{\tau})$ be admissible local coordinate systems. By Remark 2.1, we write $\dot{\tau} = \psi(\tau)$. Set $\dot{\tau}' := \psi(\tau')$. We may write

$$\dot{\tau}' - \dot{\tau} = \frac{d\psi}{d\tau}(\tau)(\tau' - \tau) + \psi''(\tau', \tau)(\tau' - \tau)^2$$

with $\dfrac{d\psi}{d\tau}(\tau) \neq 0$ for $|\tau| \ll 1$. Hence if moreover $0 < |\tau' - \tau| \ll 1$, then $\dot{\tau}' \neq \dot{\tau}$. Thus we have $\{(\dot{z}, \dot{\tau}, \dot{w}, \dot{\tau}') \in X \times X;\, \dot{\tau} = \dot{\tau}'\} \subset \{(z, \tau, w, \tau') \in X \times X;\, \tau = \tau'\}$ on a neighborhood of $Y \times Y$. For the same reason, we have $\{(z, \tau, w, \tau') \in X \times X;\ \tau = \tau'\} \subset \{(\dot{z}, \dot{\tau}, \dot{w}, \dot{\tau}') \in X \times X;\ \dot{\tau} = \dot{\tau}'\}$ on a neighborhood of $Y \times Y$. In the same way, we can show that $\Delta_{X/Y} \cap (M \times M)$ does not depend on the choice of admissible local coordinate systems on a neighborhood of $N \times N$.

We set $\mathscr{O}^{(0,n+1)}_{X\times X} := \mathscr{O}_{X\times X} \underset{q_2^{-1}\mathscr{O}_X}{\otimes} q_2^{-1}\Omega_X = \Omega_{X\times X} \underset{q_1^{-1}\mathscr{O}_X}{\otimes} q_1^{-1}\Omega_X^{\otimes -1}$, where $q_i\colon X \times X \to X$ is the i-th projection, and set $\mathscr{O}^{(0,n)}_{Y\times Y}$ in the same way. Further we set

$$\mathscr{O}^{(0,n)}_{\Delta_{X/Y}} := \Omega_{\Delta_{X/Y}} \underset{p_1^{-1}\mathscr{O}_X}{\otimes} p_1^{-1}\Omega_X^{\otimes -1},$$

where $p_1 := q_1 \circ \delta_{X/Y}\colon \Delta_{X/Y} \to X$. Under the admissible local coordinate system, we see that $\mathscr{O}^{(0,n+1)}_{X\times X} = \mathscr{O}_{X\times X}\, dw\, d\tau'$, $\mathscr{O}^{(0,n)}_{Y\times Y} = \mathscr{O}_{Y\times Y}\, dw$ and $\mathscr{O}^{(0,n)}_{\Delta_{X/Y}} = \mathscr{O}_{\Delta_{X/Y}} dw$, where $dw := dw_1 \wedge \cdots \wedge dw_n$ etc. Let $\Delta_X \subset X \times X$ be the diagonal set. Then

$$\mathscr{D}^\infty_X = H^{n+1}_{\Delta_X}(\mathscr{O}^{(0,n+1)}_{X\times X}) \simeq \boldsymbol{R}\Gamma_{\Delta_X}(\mathscr{O}^{(0,n+1)}_{X\times X})[n+1]$$

is the Ring on X of *holomorphic partial differential operators of infinite order*. By the tangent mapping $\delta'\colon T_Y X \hookrightarrow T_{Y\times Y}\Delta_{X/Y}$ of $\delta\colon X \hookrightarrow \Delta_{X/Y}$, we regard $T_Y X$ as a closed subset of $T_{Y\times Y}\Delta_{X/Y}$.

3.10. Theorem. *The object $\boldsymbol{R}\Gamma_{T_Y X}(\nu_{Y\times Y}(\boldsymbol{R}\Gamma_{\Delta_{X/Y}}(\mathscr{O}_{X\times X})))$ is concentrated in degree $n+1$.*

For the proof, we use the abstract edge of the wedge theorem due to Kashiwara (see [5]).

3.11. Definition. We define

$$\begin{aligned}\widehat{\mathscr{D}}^\nu_{T_Y X} &:= \boldsymbol{R}\Gamma_{T_Y X}(\nu_{Y\times Y}(\boldsymbol{R}\Gamma_{\Delta_{X/Y}}(\mathscr{O}^{(0,n+1)}_{X\times X})))[n+1] \\ &= H^n_{T_Y X}(\nu_{Y\times Y}(H^1_{\Delta_{X/Y}}(\mathscr{O}^{(0,n+1)}_{X\times X}))).\end{aligned}$$

3.12. Remark. Let $\mathring{p} = (\mathring{z}, \mathring{\tau}) \in T_Y X \simeq \mathbb{C}^n \times \mathbb{C}$. For ρ, $\delta > 0$, we set

$$\mathbb{D}_\rho(\mathring{z}) := \bigcap_{i=1}^n \{z \in \mathbb{C}^n;\ |z_i - \mathring{z}_i| < \rho\}, \quad \mathbb{B}_\delta := \{\tau \in \mathbb{C};\ |\tau| < \delta\}.$$

Then $P = P(z, \tau, \partial_z, \partial_\tau) = \sum_{\alpha,i} a_{\alpha,i}(z,\tau)\partial_z^\alpha \partial_\tau^i \in \widehat{\mathscr{D}}^\nu_{T_Y X, \mathring{p}}$ is given as follows:

(a) Assume that $\mathring{\tau} = 0$. Then there exist ρ, $\delta_0 > 0$ such that each $a_{\alpha,i}(z,\tau)$ is holomorphic on $\mathrm{Cl}[\mathbb{D}_\rho(\mathring{z}) \times \mathbb{B}_{\delta_0}]$, and for any $\varepsilon > 0$, there exists $0 < \delta(\varepsilon) < \delta_0$ satisfying the following: for any $\varepsilon_0 > 0$, there exists $C_{\varepsilon_0,\varepsilon} > 0$ such that

$$\sup\{|a_{\alpha,i}(z,\tau)|;\ (z,\tau) \in \mathrm{Cl}[\mathbb{D}_\rho(\mathring{z}) \times \mathbb{B}_{\delta(\varepsilon)}]\} \leqslant \frac{C_{\varepsilon_0,\varepsilon}\, \varepsilon^{|\alpha|} \varepsilon_0^{\,i}}{\alpha!\, i!}\,.$$

(b) Assume that $\mathring{\tau} \neq 0$. Then there exist ρ, δ_0, $\delta > 0$ such that each $a_\alpha(z,\tau)$ is holomorphic on $\mathrm{Cl}\,\mathbb{D}_\rho(\mathring{z}) \times \{\tau \in \mathbb{C};\ 0 < |\tau| \leqslant \delta_0, |\arg(\tau/\mathring{\tau})| \leqslant \delta\}$, and for any $\varepsilon > 0$, there exists $0 < \delta(\varepsilon) < \delta_0$ satisfying the following: for any $\varepsilon_0 > 0$ and $0 < r < \delta(\varepsilon)$, there exists $C_{\varepsilon_0,\varepsilon,r} > 0$ such that

$$\sup\{|a_{\alpha,i}(z,\tau)|;\ z \in \mathbb{D}_\rho(\mathring{z}),\ r \leqslant |\tau| \leqslant \delta(\varepsilon),\ |\arg(\tau/\mathring{\tau})| \leqslant \delta\} \leqslant \frac{C_{\varepsilon_0,\varepsilon,r}\,\varepsilon^{|\alpha|}\varepsilon_0^{\,i}}{\alpha!\,i!}\,.$$

Set $\tau_{X,Y} := f \circ \tau_Y : T_Y X \to X$.

3.13. Remark. (1) $\widehat{\mathscr{D}}^{\nu}_{T_Y X}$ is a Ring with formal adjoints, and $\tau^{-1}_{X,Y}\mathscr{D}^\infty_X$ is a Subring of $\widehat{\mathscr{D}}^{\nu}_{T_Y X}$, compatible with formal adjoints.

(2) $\nu_Y(\mathscr{O}_X)$ is a $\widehat{\mathscr{D}}^{\nu}_{T_Y X}$-Module.

3.14. Definition (Tahara [22]). We take the admissible local coordinate system in (2.1). Let $\mathring{z} \in Y$. For $m \in \mathbb{N}$, we define $P(z,\tau,\partial_z) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha(z,\tau)\partial_z^\alpha \in \widehat{\mathscr{D}}_{X|Y,\,\mathring{z}}$ as follows:

(a) There exist ρ, $\delta_0 > 0$ such that $a_\alpha(z,\tau) \in \varGamma(\mathrm{Cl}[\mathbb{D}_\rho(\mathring{z}) \times \mathbb{B}_{\delta_0}]; \mathscr{O}_X)$,

(b) there exist A, $m > 0$ satisfying the following: for any $0 < \delta \leqslant \delta_0$, there exists $C_\delta > 0$ such that

$$\max\{|a_\alpha(z,\tau)|;\ (z,\tau) \in \mathrm{Cl}[\mathbb{D}_\rho(\mathring{z}) \times \mathbb{B}_\delta]\} \leqslant \frac{C_\delta(A\delta^{1/m})^{|\alpha|}}{\alpha!}.$$

We can see that $\widehat{\mathscr{D}}_{X|Y,\,\tau_Y(\mathring{p})} \subset \widehat{\mathscr{D}}^{\nu}_{T_Y X,\tau_Y(\mathring{p})} \subset \widehat{\mathscr{D}}^{\nu}_{T_Y X,\mathring{p}}$ for any $\mathring{p} \in T_Y X$.

3.15. Definition. We set

$$\widehat{\mathscr{D}}^{\nu}_{T_Y X\to Y} := H^n_{T_Y X}(\nu_{Y\times Y}(\mathscr{O}^{(0,n)}_{\Delta_{X/Y}})) = \boldsymbol{R}\varGamma_{T_Y X}(\nu_{Y\times Y}(\mathscr{O}^{(0,n)}_{\Delta_{X/Y}}))[n].$$

Then $\widehat{\mathscr{D}}^{\nu}_{T_Y X\to Y}$ is a $(\widehat{\mathscr{D}}^{\nu}_{T_Y X} \otimes \tau_Y^{-1}(\mathscr{D}^\infty_Y)^{\mathrm{op}})$-Module, and under an admissible local coordinate system we have an exact sequence $0 \to \widehat{\mathscr{D}}^{\nu}_{T_Y X} \xrightarrow{\cdot\partial_\tau} \widehat{\mathscr{D}}^{\nu}_{T_Y X} \to \widehat{\mathscr{D}}^{\nu}_{T_Y X\to Y} \to 0$.

3.16. Remark. (1) Let $\mathring{p} = (\mathring{z},\mathring{\tau}) \in T_Y X$. Then $P(z,\tau,\partial_z) = \sum_{\alpha\in\mathbb{N}_0^n} a_\alpha(z,\tau)\,\partial_z^\alpha \in \widehat{\mathscr{D}}^{\nu}_{T_Y X\to Y,\mathring{p}}$ is given as follows:

(a) Assume that $\mathring{\tau} = 0$. Then there exist ρ, $\delta_0 > 0$ such that each $a_\alpha(z,\tau)$ is holomorphic on $\mathrm{Cl}[\mathbb{D}_\rho(\mathring{z}) \times \mathbb{B}_{\delta_0}]$, and for any $\varepsilon > 0$, there exist $0 < \delta(\varepsilon) < \delta_0$ and $C_\varepsilon > 0$ such that

$$\sup\{|a_\alpha(z,\tau)|;\ (z,\tau) \in \mathrm{Cl}[\mathbb{D}_\rho(\mathring{z}) \times \mathbb{B}_{\delta(\varepsilon)}]\} \leqslant \frac{C_\varepsilon\,\varepsilon^{|\alpha|}}{\alpha!}\,.$$

(b) Assume that $\mathring{\tau} \neq 0$. Then there exist ρ, δ_0, $\delta > 0$ such that each $a_\alpha(z,\tau)$ is holomorphic on $\mathrm{Cl}\,\mathbb{D}_\rho(\mathring{z}) \times \{\tau \in \mathbb{C};\ 0 < |\tau| \leqslant \delta_0, |\arg(\tau/\mathring{\tau})| \leqslant \delta\}$, and for any $\varepsilon > 0$, there exists $0 < \delta(\varepsilon) < \delta_0$ satisfying the following: for any $0 < r < \delta(\varepsilon)$, there exists $C_{\varepsilon,r} > 0$ such that

$$\sup\{|a_\alpha(z,\tau)|;\ z \in \mathrm{Cl}\,\mathbb{D}_\rho(\mathring{z}),\ r \leqslant |\tau| \leqslant \delta(\varepsilon),\ |\arg(\tau/\mathring{\tau})| \leqslant \delta\} \leqslant \frac{C_{\varepsilon,r}\,\varepsilon^{|\alpha|}}{\alpha!}.$$

(2) $\widehat{\mathscr{D}}^{\nu}_{T_Y X \to Y}|_Y = \mathscr{O}\widetilde{\mathscr{D}}_{Y|L}$ is defined by Oaku [18, Definition 2.3].

3.17. Definition. (1) For any $\mathscr{F} \in \mathbf{D}^{\mathrm{b}}(\widehat{\mathscr{D}}^{\nu}_{T_Y X})$, we set

$$\widehat{\Psi}_Y(\mathscr{F}) := \boldsymbol{R}\mathscr{H}om_{\widehat{\mathscr{D}}^{\nu}_{T_Y X}}(\widehat{\mathscr{D}}^{\nu}_{T_Y X \to Y}, \mathscr{F}).$$

Then $\widehat{\Psi}_Y(\mathscr{F})$ is represented by $\mathscr{F} \xrightarrow{\partial_\tau \cdot} \mathscr{F}$ under an admissible local coordinate system.

(2) For any $\mathscr{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathscr{D}_X)$, we set

$$\widehat{\Psi}^{\mathscr{D}}_Y(\mathscr{N}) := \widehat{\Psi}_Y(\widehat{\mathscr{D}}^{\nu}_{T_Y X} \overset{\boldsymbol{L}}{\underset{\tau^{-1}_{X,Y}\mathscr{D}_X}{\otimes}} \tau^{-1}_{X,Y}\mathscr{N}), \quad \Psi^{\infty}_Y(\mathscr{N}) := \hat{\sigma}^{-1}\widehat{\Psi}^{\mathscr{D}}_Y(\mathscr{N}).$$

3.18. Proposition. *Let $\mathscr{N} \in \mathfrak{Coh}(\mathscr{D}_X)$. Then $H^i\Psi^{\infty}_Y(\mathscr{N}) = 0$ holds for $i \notin [-n, 1]$, and $\Psi^{\infty}_Y(\mathscr{N})$ is represented by a bounded complex of $\mathscr{D}^{\infty}_Y$-Modules.*

3.19. Example. (1) $\widehat{\Psi}_Y(\nu_Y(\mathscr{O}_X)) \simeq \tau_Y^{-1}\mathscr{O}_Y$.

(2) $\tau_Y^{-1}\mathscr{D}^{\infty}_Y \simeq \widehat{\Psi}_Y(\widehat{\mathscr{D}}^{\nu}_{T_Y X \to Y})$.

(3) $\Psi^{\infty}_Y(\mathscr{D}_X/\mathscr{D}_X\vartheta) \simeq \Psi^{\infty}_Y(\mathscr{D}_X/\mathscr{D}_X\partial_\tau) \simeq \mathscr{D}^{\infty}_Y$.

(4) If $\mathscr{M} \in \mathfrak{Coh}(\mathscr{D}_X)$ satisfies that $\operatorname{supp}\mathscr{M} \subset Y$, then $\Psi^{\infty}_Y(\mathscr{M}) = 0$.

§ 4. Holomorphic Solutions to Fuchsian Systems

We inherit the notation from the preceding section.

4.1. Theorem. *Let $P = \vartheta - A(z,\tau,\partial_z)$ be a Fuchsian Volevič system of size m. Then for any $\mathring{p} \in \dot{T}_Y X$, the following hold:*

$$\widehat{\Psi}^{\mathscr{D}}_Y(\mathscr{D}^m_X/\mathscr{D}^m_X P)_{\mathring{p}} \simeq \widehat{\Psi}_Y(\widehat{\mathscr{D}}^{\nu}_{T_Y X \to Y})^m_{\mathring{p}} \simeq (\mathscr{D}^{\infty}_{Y,\tau_Y(\mathring{p})})^m.$$

Idea of Proof. We set $A_0(z) := A(z,0,\partial_z) \in \mathrm{Mat}_m(\mathscr{O}_Y)$. Let $\{\alpha_i\}_{i=1}^m$ be the set of eigenvalues of $A_0(\tau_Y(\mathring{p}))$, and set

$$\upsilon_P := \begin{cases} \max\{\alpha_i - \alpha_j;\ \alpha_i - \alpha_j \in \mathbb{N}\} & (\text{if } \{\alpha_i - \alpha_j\}_{i,j=1}^m \cap (\mathbb{Z} \smallsetminus \{0\}) \neq \emptyset), \\ 0 & (\text{if } \{\alpha_i - \alpha_j\}_{i,j=1}^m \cap (\mathbb{Z} \smallsetminus \{0\}) = \emptyset). \end{cases}$$

We consider the following problem for $R(z,\tau,\partial_z) \in \mathrm{Mat}_m(\widehat{\mathscr{D}}_{X|Y,\,\tau_Y(\mathring{p})})$:

$$\tag{4.1} \begin{cases} (\vartheta - A(z,\tau,\partial_z))(\tau^{\upsilon_P}R)(z,\tau,\partial_z) = \tau^{\upsilon_P}R(z,\tau,\partial_z)(\vartheta - A_0(z)), \\ R(z,0,\partial_z) = \mathbb{1}_m\,. \end{cases}$$

By using a result of Tahara [22], we can obtain a unique invertible solution to (4.1). Then $\tilde{R}(z,\tau,\partial_z) := \tau^{\upsilon_P}R(z,\tau,\partial_z)\tau^{A_0(z)} \in \mathrm{Mat}_m(\widehat{\mathscr{D}}^{\nu}_{T_YX,\,\mathring{p}})$ is invertible and

$$(\vartheta - A(z,\tau,\partial_z))\tilde{R}(z,\tau,\partial_z) = \tilde{R}(z,\tau,\partial_z)\vartheta.$$

From this result, we can prove the theorem. □

4.2. Proposition. (1) *If P is a Fuchsian operator of weight (m,w), then locally $\Psi^{\infty}_Y(\mathscr{D}_X/\mathscr{D}_XP) \simeq (\mathscr{D}^{\infty}_Y)^m$.*

(2) *If $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$, then $H^i\Psi^{\infty}_Y(\mathscr{M}) = 0$ holds for $i \notin [-n,0]$.*

4.3. Remark. Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. Then $\Psi^{\infty}_Y(\mathscr{M})$ is represented by

$$0 \to (\mathscr{D}^{\infty}_Y)^{r_n}/(\mathscr{D}^{\infty}_Y)^{r_{n+1}}Q \to (\mathscr{D}^{\infty}_Y)^{r_{n-1}} \to \cdots \to (\mathscr{D}^{\infty}_Y)^{r_1} \to (\mathscr{D}^{\infty}_Y)^{r_0},$$

where $r_i \in \mathbb{N}$ and $Q \in \mathrm{Mat}_{r_{n+1},r_n}(\mathscr{D}^{\infty}_Y)$.

4.4. Proposition. (1) *For any $\mathscr{N} \in \mathfrak{Coh}(\mathscr{D}_X)$, there exists a natural morphism $\Psi^{\infty}_Y(\mathscr{N}) \to \mathscr{D}^{\infty}_Y \overset{\boldsymbol{L}}{\underset{\mathscr{D}_Y}{\otimes}} \boldsymbol{D}f^*\mathscr{N}$.*

(2) *For any $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$, there exists a natural morphism $\mathscr{D}^{\infty}_Y \overset{\boldsymbol{L}}{\underset{\mathscr{D}_Y}{\otimes}} \boldsymbol{D}f^!\mathscr{M} \to \Psi^{\infty}_Y(\mathscr{M})$.*

As usual, $\mathscr{C}^{\mathbb{R}}_{Y|X} := H^1\mu_Y(\mathscr{O}_X) = \mu_Y(\mathscr{O}_X)[1]$ denotes the sheaf of *holomorphic microfunctions* on T^*_YX. Then $\mathscr{B}^{\infty}_{Y|X} := \mathscr{C}^{\mathbb{R}}_{Y|X}|_Y = H^1_Y(\mathscr{O}_X) = \boldsymbol{R}\Gamma_Y(\mathscr{O}_X)[1]$ is the sheaf of *holomorphic hyperfunctions*.

4.5. Theorem. *For any $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$, there exist the following isomorphisms between distinguished triangles*:

$$\begin{array}{ccc}
f^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{O}_X) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M},\mathscr{O}_Y) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\hat{\sigma}^{-1}\nu_Y(\mathscr{O}_X)) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}^{\infty}_Y}(\Psi^{\infty}_Y(\mathscr{M}),\mathscr{O}_Y) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},{}^*\hat{\sigma}^{-1}\mathscr{C}^{\mathbb{R}}_{Y|X}) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},{}^*\hat{\sigma}^{-1}\mathscr{C}^{\mathbb{R}}_{Y|X}), \\
\downarrow{+1} & & \downarrow{+1}
\end{array}$$

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}^{\infty}_{Y|X}) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^!\mathscr{M},\mathscr{O}_Y)[-1] \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},{}^*\hat{\sigma}^{-1}\mathscr{C}^{\mathbb{R}}_{Y|X}) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},{}^*\hat{\sigma}^{-1}\mathscr{C}^{\mathbb{R}}_{Y|X}) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\hat{\sigma}^{-1}\nu_Y(\mathscr{O}_X)) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}^{\infty}_Y}(\Psi^{\infty}_Y(\mathscr{M}),\mathscr{O}_Y). \\
\downarrow{+1} & & \downarrow{+1}
\end{array}$$

4.6. Remark. Let $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X P$, where P is a Fuchsian partial differential operator of weight (m, w), or $\mathscr{M} = \mathscr{D}_X^m/\mathscr{D}_X^m P$, where P is a Fuchsian Volevič system of size m. Then locally $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \hat{\sigma}^{-1}\nu_Y(\mathscr{O}_X)) \simeq \mathscr{O}_Y^{\oplus m}$. This isomorphism follows from the results due to Mandai [12] or Mandai-Tahara [13].

Let $\mathcal{R}_Y(\mathscr{D}_X)$ be the subcategory of $\mathfrak{Coh}(\mathscr{D}_X)$ consisting of regular-specializable $\mathscr{D}_X$-Modules, and $\Psi_Y(\mathscr{M})$ (resp. $\Phi_Y(\mathscr{M})$) denotes the nearby cycle (resp. the vanishing cycle) of $\mathscr{M}$. We remark that $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$ if and only if the following holds: for any $u \in \mathscr{M}$, locally there exists $P \in \mathscr{D}_X$ such that $Pu = 0$, where P is of the following form:

$$P = \vartheta^m + \sum_{i=0}^{m-1} b_i \vartheta^i + \tau \sum_{|\alpha|+i \leqslant m} a_{\alpha,i}(z,\tau)\partial_z^\alpha \vartheta^i \quad (b_i \in \mathbb{C}).$$

For any $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$, we have the following distinguished triangles (see [9]):

$$\begin{array}{ccc}
f^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{O}_Y) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \hat{\sigma}^{-1}\nu_Y(\mathscr{O}_X)) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{O}_Y) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, {}^*\hat{\sigma}^{-1}\mathscr{C}^{\mathbb{R}}_{Y|X}) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{O}_Y), \\
\downarrow{+1} & & \downarrow{+1}
\end{array}$$

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}^{\infty}_{Y|X}) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^!\mathscr{M}, \mathscr{O}_Y)[-1] \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, {}^*\hat{\sigma}^{-1}\mathscr{C}^{\mathbb{R}}_{Y|X}) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Phi_Y(\mathscr{M}), \mathscr{O}_Y) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \hat{\sigma}^{-1}\nu_Y(\mathscr{O}_X)) & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\Psi_Y(\mathscr{M}), \mathscr{O}_Y). \\
\downarrow{+1} & & \downarrow{+1}
\end{array}$$

4.7. Theorem. *If $\mathscr{M} \in \mathcal{R}_Y(\mathscr{D}_X)$, then $\Psi_Y^\infty(\mathscr{M}) \simeq \mathscr{D}_Y^\infty \overset{\boldsymbol{L}}{\underset{\mathscr{D}_Y}{\otimes}} \Psi_Y(\mathscr{M})$. In particular, if Y is non-characteristic for $\mathscr{M}$, then $\Psi_Y^\infty(\mathscr{M}) \simeq \mathscr{D}_Y^\infty \overset{\boldsymbol{L}}{\underset{\mathscr{D}_Y}{\otimes}} \boldsymbol{D}f^*\mathscr{M}$.*

§ 5. Boundary Values for Hyperfunction Solutions

We denote by $\mathscr{B}_M$ and $\mathscr{C}_M$ the sheaves of *hyperfunctions* on M and of *microfunctions* on $T^*_M X$ respectively. For any $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$, we have

$$\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{B}_M)) \otimes \mathscr{or}_{N/M} = \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^!\mathscr{M}, \mathscr{B}_N)[-1].$$

5.1. Definition ([4], [5]). We define the sheaf on $\sqrt{-1}\,T^*_N M$ of second hyperfunctions by

$$\mathscr{B}^2_{\sqrt{-1}\,T^*_N M} := H^{n+1}_{\sqrt{-1}\,T^*_N M}(\mathscr{C}^{\mathbb{R}}_{Y|X}) \otimes \mathscr{or}_{N/Y} \simeq \boldsymbol{R}\Gamma_{\sqrt{-1}\,T^*_N M}(\mu_Y(\mathscr{O}_X)) \otimes \mathscr{or}_{N/Y}[n+2].$$

By Holmgren type theorem for hyperfunctions and [4], [5], we have monomorphisms

$$\Gamma_{M_+}(\mathscr{B}_M)|_N \rightarrowtail {}^*\hat{s}^{-1}\mathscr{C}_M \rightarrowtail {}^*\hat{s}^{-1}\mathscr{B}^2_{\sqrt{-1}\,T^*_N M}\,.$$

Hence we obtain

5.2. Theorem. *Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. Then there exists the following morphism between distinguished triangles*:

$$\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{B}_M)) \otimes \mathscr{or}_{N/M} & = & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^!\mathscr{M}, \mathscr{B}_N)[-1] \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{M_+}(\mathscr{B}_M))|_N \otimes \mathscr{or}_{N/M} & \to & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, {}^*\hat{s}^{-1}\mathscr{B}^2_{\sqrt{-1}\,T^*_N M}) \otimes \mathscr{or}_{N/M} \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N & \longrightarrow & \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y^\infty}(\Psi_Y^\infty(\mathscr{M}), \mathscr{B}_N). \\
\downarrow{+1} & & \downarrow{+1}
\end{array}$$

5.3. Definition. Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. By Theorem 5.2 we can define

$$\gamma_+ \colon \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N \to \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y^\infty}(\Psi_Y^\infty(\mathscr{M}), \mathscr{B}_N). \tag{5.1}$$

Taking cohomologies, we have

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{B}_M)) \otimes \mathscr{or}_{N/M} & = & R^{-1}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^!\mathscr{M}, \mathscr{B}_N) \\
\downarrow & & \downarrow \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{M_+}(\mathscr{B}_M))|_N \otimes \mathscr{or}_{N/M} & \rightarrowtail & \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, {}^*\hat{s}^{-1}\mathscr{B}^2_{\sqrt{-1}\,T^*_N M}) \otimes \mathscr{or}_{N/M} \\
\downarrow & & \downarrow \\
\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N & \longrightarrow & \mathscr{H}om_{\mathscr{D}_Y^\infty}(H^0\Psi_Y^\infty(\mathscr{M}), \mathscr{B}_N) \\
\downarrow & & \downarrow \\
\mathscr{E}xt^1_{\mathscr{D}_X}(\mathscr{M}, \Gamma_N(\mathscr{B}_M)) \otimes \mathscr{or}_{N/M} & = & R^0\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^!\mathscr{M}, \mathscr{B}_N).
\end{array}$$

Therefore

5.4. Proposition. *Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. Then* (5.1) *induces a monomorphism*

$$\gamma_+^0 \colon \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))|_N \rightarrowtail \mathscr{H}om_{\mathscr{D}_Y^\infty}(H^0\Psi_Y^\infty(\mathscr{M}), \mathscr{B}_N).$$

Next, we recall definitions of several sheaves attached to the boundary due to Oaku [18]. Note that in Oaku [18] these sheaves are defined on cosphere bundles, so we shall present definitions on cotangent bundles along the line of Oaku-Yamazaki [19]. We refer to Oaku [18] or Oaku-Yamazaki [19] for the proofs. Although only the higher-codimensional case is treated in [19], the same proofs also work as in the one-codimensional case.

5.5. Definition. We set:

$$\begin{aligned}\mathscr{C}_{N|M} &:= s_{L\pi}^{-1}\,\mu_{\widetilde{M}_N}(\boldsymbol{R}\Gamma_{\Omega_L}(p_L^{-1}\boldsymbol{R}\Gamma_L(\mathscr{O}_X)))\otimes \mathscr{or}_{M/X}[n],\\ \widetilde{\mathscr{C}}_{N|M} &:= \mu_{T_N M}(\nu_Y(\boldsymbol{R}\Gamma_L(\mathscr{O}_X)))\otimes \mathscr{or}_{M/X}[n],\\ \widetilde{\mathscr{B}}_{N|M} &:= \widetilde{\mathscr{C}}_{N|M}|_{T_N M}\,.\end{aligned}$$

Then $\mathscr{C}_{N|M}$ and $\widetilde{\mathscr{C}}_{N|M}$ are concentrated in degree zero, and $\nu_N(\mathscr{B}_M) = \mathscr{C}_{N|M}|_{T_N M}$.

5.6. Proposition ([18]). *(1) $\mathscr{C}_{N|M}$ and $\widetilde{\mathscr{C}}_{N|M}$ are concentrated in degree zero; that is, $\widetilde{\mathscr{C}}_{N|M}$ and $\widetilde{\mathscr{C}}_{N|M}$ are regarded as sheaves on $T^*_{T_N M}T_Y L$.*

(2) There exists a canonical monomorphism $s^*_{N|M}\colon \mathscr{C}_{N|M} \to \widetilde{\mathscr{C}}_{N|M}$.

(3) $\nu_N(\mathscr{B}_M) = \mathscr{C}_{N|M}|_{T_N M}$, and there exists the following commutative diagram with exact rows on $T_N M$:

$$\begin{array}{ccccccccc} 0 \to & \nu_Y(\mathscr{BO}_L)|_{T_N M} & \to & \nu_N(\mathscr{B}_M) & \to & \dot\pi_{N|M*}\mathscr{C}_{N|M} & \to 0\\ & \| & & \downarrow & & \downarrow & \\ 0 \to & \nu_Y(\mathscr{BO}_L)|_{T_N M} & \longrightarrow & \widetilde{\mathscr{B}}_{N|M} & \longrightarrow & \dot\pi_{N|M*}\widetilde{\mathscr{C}}_{N|M} & \to 0.\end{array}$$

Here $\mathscr{BO}_L := H^1_L(\mathscr{O}_X)\otimes \mathscr{or}_{L/X} \simeq \boldsymbol{R}\Gamma_L(\mathscr{O}_X)\otimes \mathscr{or}_{L/X}[1]$. Note that $\nu_Y(\mathscr{BO}_L)$ is concentrated in degree zero.

5.7. Definition. Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. By Definition 2.6, Proposition 3.7, Theorem 4.1 and Proposition 5.6, we define the morphism γ_+:

$$\begin{aligned}\gamma_+\colon i_{\pi+}^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}_{N|M}) &\to i_{\pi+}^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\widetilde{\mathscr{C}}_{N|M})\\ &\simeq \tilde\tau_{\pi+}^{-1}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y^\infty}(\Psi_Y^\infty(\mathscr{M}),\mathscr{C}_N).\end{aligned}$$

The restriction of γ_+ to the zero-section $T_N M^+$ of $T^*_{T_N M^+}T_Y L^+$ coincides with the boundary value morphism (5.1).

We can obtain the following Holmgren type theorem:

5.8. Theorem. *Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. Then the morphism γ_+ gives a monomorphism*

$$\gamma_+^0\colon i_{\pi+}^{-1}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}_{N|M}) \rightarrowtail \tilde\tau_{\pi+}^{-1}\mathscr{H}om_{\mathscr{D}_Y^\infty}(H^0\Psi_Y^\infty(\mathscr{M}),\mathscr{C}_N).$$

5.9. Remark. Theorem 5.8 gives another proof of Proposition 5.4.

5.10. Theorem. *Let $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$. Assume that $\mathscr{M}$ is near-hyperbolic at $\mathring{x} \in N$ in the dt-codirection. Then, for any $p^* = (\hat s(\mathring{x}); \sqrt{-1}\,\mathring{y}^*) \in T^*_{T_N M^+}T_Y L^+$, there exists an isomorphism*

$$\gamma_+\colon \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}_{N|M})_{p^*} \simeq \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y^\infty}(\Psi_Y^\infty(\mathscr{M}),\mathscr{C}_N)_{p_0}.$$

Here $p_0 := \tilde{\tau}_\pi(p^*) = (\mathring{x}; \sqrt{-1}\,\mathring{y}^*) \in T^*_N Y$. *In particular, there exists an isomorphism*

$$\gamma_+ \colon \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \Gamma_{\Omega_+}(\mathscr{B}_M))_{\mathring{x}} = \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \nu_N(\mathscr{B}_M))_{\hat{s}(\mathring{x})} \xrightarrow{\sim} \boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y^\infty}(\Psi_Y^\infty(\mathscr{M}), \mathscr{B}_N)_{\mathring{x}}.$$

We consider the mappings:

$$\begin{array}{ccccc} T^*_M X & \xleftarrow{f_{N\pi}} & N \underset{M}{\times} T^*_M X & \xrightarrow{f_{Nd}} & T^*_N Y \\ {\scriptstyle k}\big\downarrow & \square & {\scriptstyle k}\big\downarrow & \square & {\scriptstyle k}\big\downarrow \\ T^*X & \xleftarrow{f_\pi} & Y \underset{X}{\times} T^*X & \xrightarrow{f_d} & T^*Y. \end{array}$$

Then the sheaf of *microfunction with a real analytic parameter* t on $T^*_N Y$ is defined by

$$\mathscr{C}^A_{N|M} := f_{Nd!}\, f^{-1}_{N\pi}\mathscr{C}_M \simeq H^{n+1}(k^{-1}\boldsymbol{R}f_{d!}\, f_\pi^{-1}\mu hom(\mathbb{C}_M, \mathscr{O}_X) \otimes or_{M/X}).$$

The sheaf $\mathring{\mathscr{C}}_{N|M_+}$ of *mild microfunctions* on $T^*_N Y$ is defined by Kataoka [8], and reformulated by Schapira-Zampieri as [21]

$$\mathring{\mathscr{C}}_{N|M_+} = H^{n+1}(\boldsymbol{R}f_{d!}\, f_\pi^{-1}\mu hom(\mathbb{C}_{\Omega_+}, \mathscr{O}_X) \otimes or_{M/X}).$$

Then we have natural monomorphisms ([17], [19]):

$$\tilde{\tau}^{-1}_{\pi+}\mathring{\mathscr{C}}^A_{N|M_+} \hookrightarrow \tilde{\tau}^{-1}_{\pi+}\mathring{\mathscr{C}}_{N|M_+} \hookrightarrow i^{-1}_{\pi+}\mathscr{C}_{N|M},$$

and restricting to N, we have natural monomorphisms

$$\mathscr{B}^A_{N|M} \hookrightarrow \mathring{\mathscr{B}}_{N|M_+} \hookrightarrow \hat{s}^{-1}\nu_N(\mathscr{B}_M) = \Gamma_{\Omega_+}(\mathscr{B}_M)|_N.$$

Here $\mathring{\mathscr{B}}_{N|M_+}$ denotes the sheaf of mild hyperfunctions. Setting $Df^*\mathscr{M} := H^0\boldsymbol{D}f^*\mathscr{M}$, we can obtain a monomorphism

$$\mathscr{H}om_{\mathscr{D}_Y}(Df^*\mathscr{M}, \mathscr{C}_N) \rightarrowtail \mathscr{H}om_{\mathscr{D}_Y^\infty}(H^0\Psi_Y^\infty(\mathscr{M}), \mathscr{C}_N).$$

For any $\mathscr{M} \in \mathcal{F}_Y(\mathscr{D}_X)$, by construction and [25], we obtain the following:

(1) There exist the following commutative diagrams:

$$\begin{array}{ccc} \tilde{\tau}^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}^A_{N|M}) & \xrightarrow{\gamma^A} & \\ \big\downarrow & & \\ \tilde{\tau}^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathring{\mathscr{C}}_{N|M_+}) & \xrightarrow{\mathring{\gamma}} & \tilde{\tau}^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M}, \mathscr{C}_N) \\ \big\downarrow & & \big\downarrow \\ i^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{C}_{N|M}) & \xrightarrow{\gamma_+} & \tilde{\tau}^{-1}_{\pi+}\boldsymbol{R}\mathscr{H}om_{\mathscr{D}_Y^\infty}(\Psi_Y^\infty(\mathscr{M}), \mathscr{C}_N) \end{array} \tag{5.2}$$

$$
\begin{array}{ccc}
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}^A_{N|M}) & \xrightarrow{\qquad\gamma^A\qquad} & \\
\downarrow & & \searrow \\
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\mathring{\mathscr{B}}_{N|M_+}) & \xrightarrow{\ \mathring{\gamma}\ } & \boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_Y}(\boldsymbol{D}f^*\mathscr{M},\mathscr{B}_N) \\
\downarrow & & \downarrow \\
\boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\Gamma_{\Omega_+}(\mathscr{B}_M))|_N & \xrightarrow{\ \gamma_+\ } & \boldsymbol{R}\mathscr{H}\!om_{\mathscr{D}_Y^\infty}(\Psi_Y^\infty(\mathscr{M}),\mathscr{B}_N).
\end{array}
\tag{5.3}
$$

Moreover (5.2) and (5.3) induce the following monomorphisms:

$$
\begin{array}{ccc}
\tilde{\tau}_{\pi+}^{-1}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}^A_{N|M}) & \xrightarrow{\qquad\gamma^{A,0}\qquad} & \\
\downarrow & & \searrow \\
\tilde{\tau}_{\pi+}^{-1}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\mathring{\mathscr{C}}_{N|M_+}) & \xrightarrow{\ \mathring{\gamma}^0\ } & \tilde{\tau}_{\pi+}^{-1}\mathscr{H}\!om_{\mathscr{D}_Y}(Df^*\mathscr{M},\mathscr{C}_N) \\
\downarrow & & \downarrow \\
i_{\pi+}^{-1}\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\mathscr{C}_{N|M}) & \xrightarrow{\ \gamma_+^0\ } & \tilde{\tau}_{\pi+}^{-1}\mathscr{H}\!om_{\mathscr{D}_Y^\infty}(H^0\Psi_Y^\infty(\mathscr{M}),\mathscr{C}_N)
\end{array}
$$

$$
\begin{array}{ccc}
\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\mathscr{B}^A_{N|M}) & \xrightarrow{\qquad\gamma^{A,0}\qquad} & \\
\downarrow & & \searrow \\
\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\mathring{\mathscr{B}}_{N|M_+}) & \xrightarrow{\ \mathring{\gamma}^0\ } & \mathscr{H}\!om_{\mathscr{D}_Y}(Df^*\mathscr{M},\mathscr{B}_N) \\
\downarrow & & \downarrow \\
\mathscr{H}\!om_{\mathscr{D}_X}(\mathscr{M},\Gamma_{\Omega_+}(\mathscr{B}_M))|_N & \xrightarrow{\ \gamma_+^0\ } & \mathscr{H}\!om_{\mathscr{D}_Y^\infty}(H^0\Psi_Y^\infty(\mathscr{M}),\mathscr{B}_N).
\end{array}
$$

(2) Let $p^* = (\hat{s}(\mathring{x});\sqrt{-1}\,\mathring{y}^*) \in T^*_{\dot{T}_N M^+}\dot{T}_Y L^+$. Assume that $\mathscr{M}$ is near-hyperbolic at $\mathring{x} \in N$ in the $\pm dt$-codirections. Then γ^A, $\mathring{\gamma}$ and γ_+ are isomorphisms at p^* in (5.2) (resp. at $\mathring{x}$ in (5.3)).

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