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A study of pinch points and cusps in the Landau-Nakanishi geometry

Dedicated to Professor Masafumi Yoshino on his sixtieth birthday

By

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Abstract

We study the detailed singularity structure of the nonzero-α LN surface associated with the truss bridge diagram T_n of lower degree ($n = 1, 2, 3$) and that with the complemented truss bridge diagram $\widetilde{T}_3$, which is obtained by addition of an external line to the non-external vertex of T_3.

§ 1. Introduction

This is a sequel of our previous papers ([3] and [5]), which aim at some better understanding of Sato's postulates on the S-matrix. ([9]; see also [6], [7], [8] and references cited therein for this subject.) We report on, in this article, singularity structure of the nonzero-α LN surface, i.e., the projection of the nonzero-α Landau-Nakanishi variety to the base space, associated with the truss bridge diagram T_n of lower degree ($n = 1, 2, 3$) and that with the complemented truss bridge diagram $\widetilde{T}_3$ which is obtained by addition of an external line to the non-external vertex (i.e., a vertex without an external line) of T_3. Singular points of these surfaces are classified into two kinds: a cusp and a pinch point. Further, a common shape so called the "Whitney umbrella" is observed near these singular points (cf. Fig.5), where a pinch point appears as a tip of the umbrella at which parametrization of the surface degenerates and a cusp forms a shank of the

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umbrella which is a self-intersection point of the surface. We give precise description of these singular points for diagrams listed above.

Concrete understanding of pinch points and cusps is believed to be useful for further study of the Landau-Nakanishi geometry. As such an evidence, in the last two sections, we see that an acnode, i.e., an isolated point, found by R. J. Eden et al. [1] in the Landau-Nakanishi geometry of T_2 and the higher codimensional component appearing in the nonzero-α LN surface of T_3 (cf. [3] and [5]) can be well understood from the viewpoint of singularity structure.

§ 2. A LN surface: a review

We briefly recall the notations and terminologies used in this paper, which are basically the same as those in [3] and [5].

Let G be a Feynman graph. Then G consists of, by definition, finitely many points V_1, V_2, …, $V_{n'}$ (called vertices), finitely many line segments L_1, L_2, …, L_N (called internal lines) and finitely many half-lines L^e_1, L^e_2, …, L^e_n (called external lines), where each of the end-points W^+_ℓ and W^-_ℓ of L_ℓ ($\ell = 1, 2, \ldots, N$) coincides with some V_j ($j = 1, 2, \ldots, n'$) with $W^+_\ell \neq W^-_\ell$ and the (unique) end-point of L^e_r ($r = 1, 2, \ldots, n$) coincides with some V_j ($j = 1, 2, \ldots, n'$).

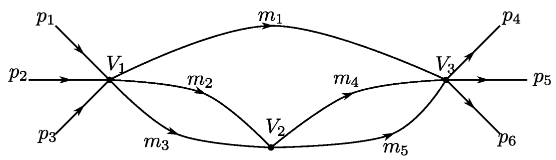

Figure 1. An example of a Feynman graph.

In this article we assume that each internal line and each external line are oriented (and specified with an arrow like "→—" if necessary). Using this orientation we define the incidence number $[j : \ell]$ for a pair of a vertex V_j and an internal line L_ℓ by the following rule:

$$(2.1) \qquad [j : \ell] = \begin{cases} +1 & \text{when the internal line } L_\ell \text{ ends at the vertex } V_j, \\ -1 & \text{when } L_\ell \text{ starts from } V_j, \\ 0 & \text{neither of the end-points of } L_\ell \text{ coincides with } V_j. \end{cases}$$

The incidence number $[j : r]$ for a pair of a vertex V_j and an external line L^e_r is defined in a similar manner. Furthermore, for an oriented closed loop C in G, we set

$$\sigma(C, \ell) = \begin{cases} +1 & \text{if } L_\ell \subset C \text{ and if } L_\ell \text{ and } C \text{ have the same orientation,} \\ -1 & \text{if } L_\ell \subset C \text{ and if } L_\ell \text{ and } C \text{ have different orientations,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

We also assume that a ν-dimensional real (or complex if so specified) vector $p_r = (p_{r,0}, \ldots, p_{r,\nu-1})$ is assigned to each external line L_r^e, and strictly positive number m_ℓ and vector $k_\ell = (k_{\ell,0}, \ldots, k_{\ell,\nu-1})$ are assigned to each internal line L_ℓ.

Definition 2.1. The LN surface $L(G)$ associated with a Feynman graph G is, by definition, the totality of external vectors $(p_1, \ldots, p_n)$ in $\mathbb{R}^{\nu n}$ that satisfies the following equations for some $(\alpha_1, \ldots, \alpha_N; k_1, \ldots, k_N) \in \mathbb{R}^N \times \mathbb{R}^{\nu N}$:

$$\begin{cases} \displaystyle\sum_{r=1}^{n} [j : r] p_r + \sum_{\ell=1}^{N} [j : \ell] k_\ell = 0 & (j = 1, 2, \ldots, n'), \\ \displaystyle\sum_{\ell=1}^{N} \sigma(C, \ell)\ \alpha_\ell k_\ell = 0 & (\text{any closed loop } C \text{ in } G), \\ \alpha_\ell (k_\ell^2 - m_\ell^2) = 0,\ \ k_{\ell,0} > 0 & (\ell = 1, 2, \ldots, N), \\ \displaystyle\sum_{\ell=1}^{N} |\alpha_\ell| > 0. \end{cases} \tag{2.3}$$

Here, for $k_\ell = (k_{\ell,0}, k_{\ell,1}, \ldots, k_{\ell,\nu-1})$, we set $k_\ell^2 = k_{\ell,0}^2 - k_{\ell,1}^2 - \cdots - k_{\ell,\nu-1}^2$.

We also obtain several variants of $L(G)$ by modifying Definition 2.1 as follows:

1. The positive-α LN surface $L^+(G)$ of G is defined by (2.3) with the additional conditions $\alpha_\ell \geq 0$ for all ℓ.
2. The leading positive-α LN surface $L^\oplus(G)$ of G is defined by (2.3) with the additional conditions $\alpha_\ell > 0$ for all ℓ.
3. The nonzero-α LN surface $L^\otimes(G)$ of G is defined by (2.3) with the additional conditions $\alpha_\ell \neq 0$ for all ℓ.

Since $L^\oplus(G)$ and $L^\otimes(G)$ are generally neither open nor closed, we often study their (topological) closures instead of themselves, which are denoted by $[L^\oplus(G)]$ and $[L^\otimes(G)]$, respectively.

A hooked 3-lines h_q with q hooks consists of 3 lines, the upper line, the middle line and the lower line, such that the middle line moves in a zigzag between the upper line and the lower line forming q hooks labeled by u (a hook formed by the upper line and the middle line) or d (a hook formed by the lower line and middle line) as shown below as an example in Figure 2.

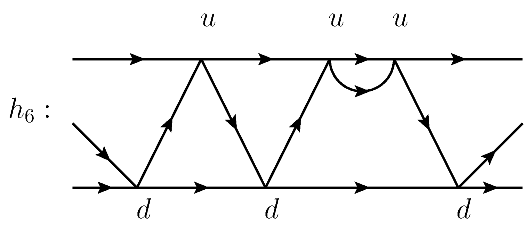

Figure 2. A hooked 3-lines associated with "*duduud*".

As a special case of a hooked 3-lines diagram, we have a truss bridge diagram:

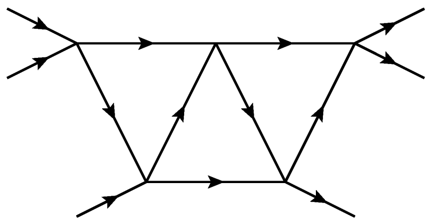

Figure 3. The truss bride T_3 (*ududu*).

Definition 2.2. A hooked 3-lines diagram generated by the sequence $udud \dots ud$ or $udud \dots udu$ is called a "truss bridge diagram". We denote by T_n the truss bridge diagram with n-trusses.

Hereafter, we always assume the following two conditions.

(H1) The space-time dimension is 2, i.e., $\nu = 2$.

(H2) The masses assigned to internal lines are all equal to $m > 0$.

§ 3. Pinch points and cusps of T_1, T_2 and T_3

We study, in this section, singularity structure of the nonzero-α LN surface $L^{\otimes} \subset \mathbb{R}^{\nu n}$ associated with the truss bridge diagram T_n of lower degree ($n = 1, 2, 3$). Here a singular point of $L^{\otimes}$ is, by definition, a point in the (topological) closure $[L^{\otimes}]$ of $L^{\otimes}$ at which $[L^{\otimes}]$ is not real analytic smooth, and we denote by $[L^{\otimes}]_{\text{sing}}$ the set of singular points of $L^{\otimes}$. To make its structure easily understood, we present several pictures of $[L^{\otimes}]$ drawn by a computer, in which one can observe a specific shape so called a "Whitney umbrella" near $[L^{\otimes}]_{\text{sing}}$. As a matter of fact, $[L^{\otimes}]_{\text{sing}}$ consists of the following two kinds of singularity: Let Ω be an open subset in $\mathbb{R}^{\dim_{\mathbb{R}} L^{\otimes}}$, and let $\varphi : \Omega \to \mathbb{R}^{\nu n}$ be a real analytic map such that $\varphi(\Omega) = [L^{\otimes}]$ and φ gives an isomorphism between $\Omega \setminus \varphi^{-1}([L^{\otimes}]_{\text{sing}})$ and $[L^{\otimes}] \setminus [L^{\otimes}]_{\text{sing}}$. Such an analytic map φ is sometimes called parametrization of $L^{\otimes}$.

- A pinch point: the image of a critical point of φ. That is, $p \in [L^{\otimes}]$ is a pinch point if and only if there exists $q \in \Omega$ with $p = \varphi(q)$ such that the rank of $d\varphi$ becomes less than $\dim_{\mathbb{R}} L^{\otimes}$ at q.
- A cusp: a self-intersection point of $[L^{\otimes}]$. That is, a point $p \in [L^{\otimes}]$ is called a cusp if and only if there exist distinct points q_1 and q_2 in Ω with $p = \varphi(q_1) = \varphi(q_2)$.

Note that subsequent computations are performed with the following conventions.

1. For a vector $v = (v_0, v_1) \in \mathbb{R}^2$, we apply the linear transformation

$$\widetilde{v}_0 = v_0 + v_1, \quad \widetilde{v}_1 = v_0 - v_1 \tag{3.1}$$

to internal and external vectors. All the computations are performed with the new coordinates.

2. Hence Lorentz metric $v^2 = v_0^2 - v_1^2$ for $v = (v_0, v_1)$ becomes $\widetilde{v}_0\widetilde{v}_1$ in the new coordinates $(\widetilde{v}_0, \widetilde{v}_1)$.

3. All the masses assigned to internal lines are assumed to be 1.

§ 3.1. The singularity structure of the nonzero-α LN surface of T_1

Let us first consider the truss bridge diagram T_1. We have essentially 3-external lines which emanate from the vertices A, B and C, and hence, the dimension of the space of external vectors is 6. However, since the sum of the external vectors must be zero by the energy-momentum conservation laws and since the LN surface is Lorentz invariant, we can regard the nonzero-α LN surface $L^{\otimes}(T_1)$ as a surface in $\mathbb{R}^3$.

We specify the coordinates (x, y, z) of the external vectors as described in Fig.4, that is, the external vector on the line from A is (x, x) and that from C is (y, z).

In what follows, for example, the symbol AB denotes not only the line AB itself but also an internal vector on this line. Set $AC = (s, 1/s)$ and $BC = (t, 1/t)$ for positive real numbers $s > 0$ and $t > 0$. Then, by the energy-momentum conservation laws at

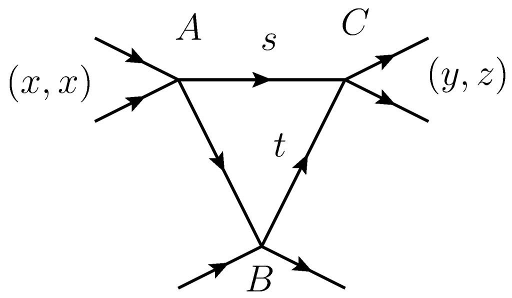

Figure 4. The truss bridge diagram T_1.

the vertex A, we obtain $AB = (1/s,\, s)$. Hence, by using parameters $s > 0$ and $t > 0$, each internal vector is expressed by

$$AB = (1/s,\, s)\,, \quad AC = (s,\, 1/s)\,, \quad BC = (t,\, 1/t)\,, \tag{3.2}$$

and we define the analytic map $\varphi : \mathbb{R}^2_{>0} \to \mathbb{R}^3$ by

$$\begin{cases} x = s + 1/s, \\ y = s + t, \\ z = 1/s + 1/t. \end{cases} \tag{3.3}$$

The map φ gives the required parametrization of $L^{\otimes}(T_1)$ as we will see.

We first compute a pinch point of T_1. Note that we have

$$dx = (1 - 1/s^2)ds, \quad dy = ds + dt, \quad dz = -1/s^2 ds - 1/t^2 dt. \tag{3.4}$$

If $s \neq 1$, clearly we get $dx \wedge dy \neq 0$. Furthermore, when $s = 1$ and $t \neq 1$, we have $dy \wedge dz \neq 0$. Therefore $d\varphi$ degenerates only at $s = t = 1$, and hence, the pinch point of T_1 is just one point given by

$$(s,\, t) = (1,\, 1), \quad \text{that is,} \quad (x,\, y,\, z) = (2,\, 2,\, 2). \tag{3.5}$$

Remark 3.1. We have the equivalence

$$(s,\, t) = (1,\, 1) \iff AB = AC = BC. \tag{3.6}$$

Hence we can conclude that the external vectors are located at a pinch point of T_1 when all the internal vectors coincide.

Now let us compute cusps of T_1. It suffices to determine points where φ is not injective. Note

$$\begin{cases} x = s + 1/s, \\ y = s + t, \\ z = 1/s + 1/t, \end{cases} \iff \begin{cases} s\,x - s^2 - 1 = 0, \\ y - t - s = 0, \\ s\,t\,z - t - s = 0. \end{cases} \tag{3.7}$$

Suppose that (s_1, t_1) and (s_2, t_2) give the same x, y and z. Then we may assume either $s_1 \neq s_2$ or $t_1 \neq t_2$. We first consider the case $t_1 \neq t_2$. By eliminating the variable s of the first and the second equations in (3.7), we obtain

$$-t^2 + (2\,y - x)\;t - y^2 + x\,y - 1 = 0,$$

and, by the first and the third ones in (3.7), we have

$$\left(-z^2 + x\,z - 1\right)\;t^2 + (2\,z - x)\;t - 1 = 0.$$

Then, as both the equations share two roots t_1 and t_2, by employing Euclidean algorithm (in this case, it is enough to divide the first equation by the second one and get its remainder since the second equation is of the second order), we have, for any t,

$$\begin{aligned} &\left((-2\,y + x)\;z^2 + \left(2\,x\,y - x^2 + 2\right)\;z - 2\,y\right)\;t + \\ &\qquad \left(y^2 - x\,y + 1\right)\;z^2 + \left(-x\,y^2 + x^2\,y - x\right)\;z + y^2 - x\,y = 0, \end{aligned}$$

that is,

$$\begin{cases} (-2\,y + x)\;z^2 + \left(2\,x\,y - x^2 + 2\right)\;z - 2\,y = 0, \\ \left(y^2 - x\,y + 1\right)\;z^2 + \left(-x\,y^2 + x^2\,y - x\right)\;z + y^2 - x\,y = 0. \end{cases}$$

By putting (3.3) into these equations, we have

$$\begin{cases} \dfrac{(s-1)\;(s+1)\;(t+s)\;(s\,t-1)}{s^2\,t^2} = 0, \\ \dfrac{(s-1)\;(s+1)\;(t+s)\;(s\,t-1)}{s^2\,t} = 0. \end{cases}$$

Therefore either $s = 1$ or $st = 1$ holds. Suppose $s = 1$. Then we have $x = 2$, and thus, we get $s_1 = s_2 = 1$, which contradicts $t_1 \neq t_2$ because of $y - t - s = 0$ in (3.7). Hence we exclude $s = 1$. On the other hand, on $\{st = 1\}$, φ is not injective because $(s, t) = (s^*, 1/s^*)$ and $(s, t) = (1/s^*, s^*)$ give the same (x, y, z). By applying the same argument to the case $s_1 \neq s_2$, we obtain the same set $\{st = 1\}$ on which φ is not injective.

Summing up, φ is injective on $\{st \neq 1\}$, that is, the restriction

$$\varphi : \mathbb{R}^2_{>0} \setminus \{st = 1\} \to \mathbb{R}^3 \tag{3.8}$$

becomes an embedding. Furthermore, the image of $\{st = 1\}$ by φ is the half line $\{x = y = z\}$ with $x \geq 2$, and it is doubly covered by φ. Hence the cusps of $L^{\otimes}(T_1)$ form an open half line

$$\{(x, y, z) \in \mathbb{R}^3;\ x = y = z,\ x > 2\} \tag{3.9}$$

whose end-point is the pinch point (2, 2, 2).

Remark 3.2. As a pinch point is characterized by a configuration of internal vectors, we can also characterize cusps in terms of a configuration of the internal vectors. Since we have

$$st = 1 \iff AB = BC, \tag{3.10}$$

the external vectors are located at a cusp when the internal vectors AB and BC coincide.

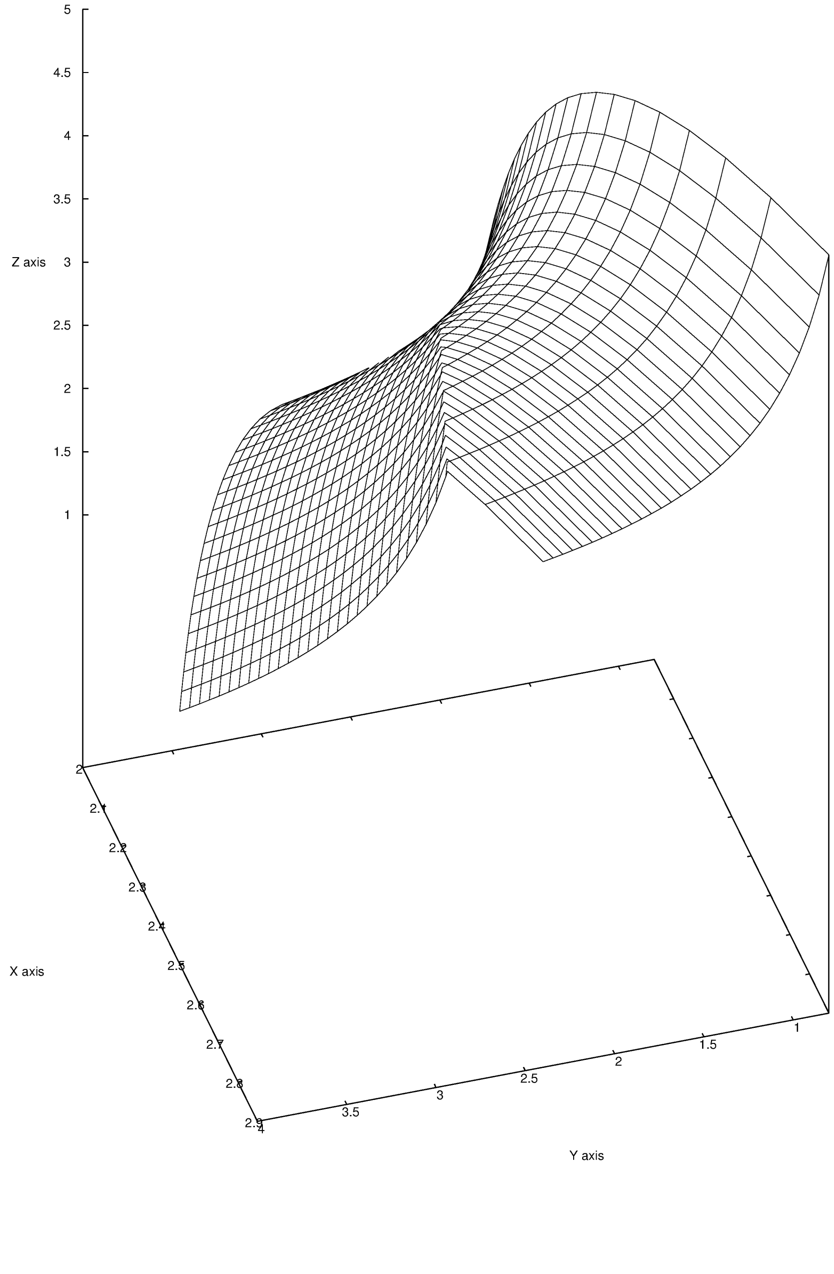

Figure 5. $L^{\otimes}(T_1)$ viewed from $(116, 287)$.

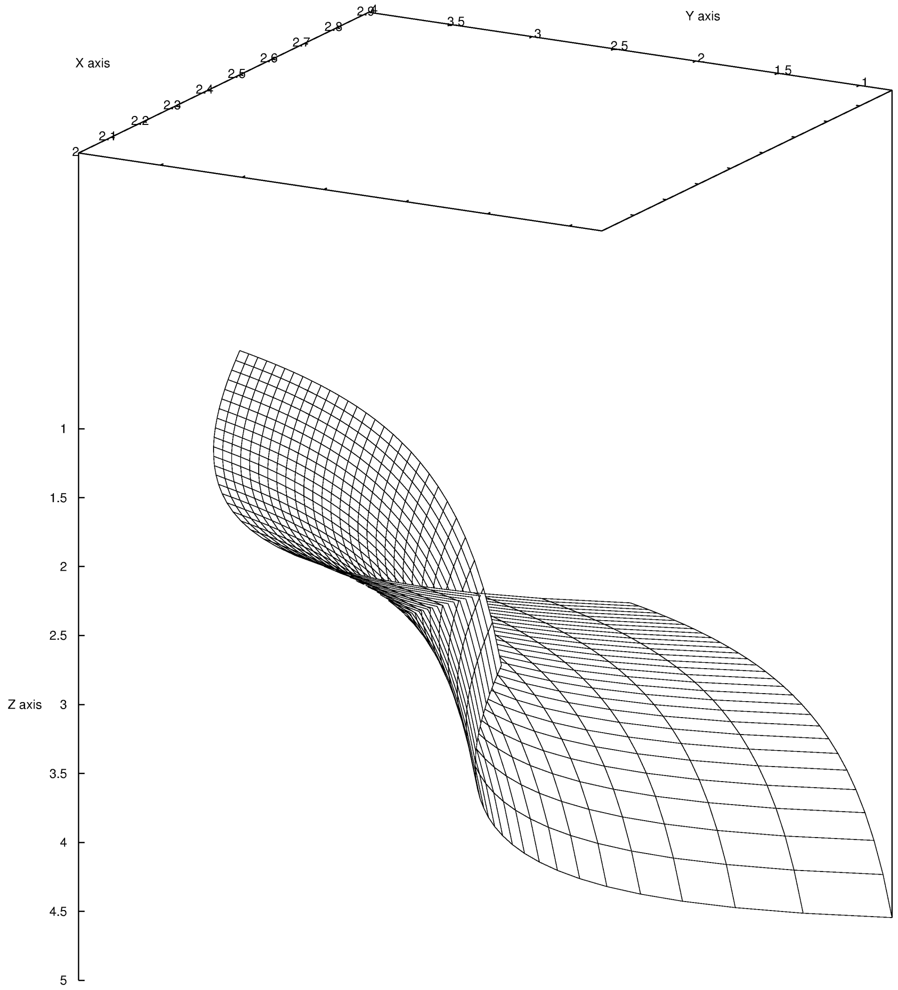

Figure 6. $L^{\otimes}(T_1)$ viewed from $(281, 289)$.

§ 3.2. The singularity structure of the nonzero-α LN surface of T_2

Let us consider the truss bridge diagram T_2. We have essentially 4-external lines which emanate from the vertices A, B, C and D, and hence, the dimension of the space of external vectors is 8. By the same reasoning as that in T_1, we can regard the nonzero-α LN surface $L^{\otimes}(T_2)$ of T_2 as a hypersurface in $\mathbb{R}^5$. However the dimension of the ambient space is still too big to understand $L^{\otimes}(T_2)$ visually. Therefore, instead of studying the hypersurface in $\mathbb{R}^5$ directly, we consider a family of slices of $L^{\otimes}(T_2)$ cut with 3-dimensional linear subspaces, where we choose each subspace so that it transversally intersects with the singular points of $L^{\otimes}(T_2)$.

Let a and b be real numbers. Then we specify the coordinates (x, y, z) and the parameters (a, b) of the external vectors as described in Fig.7, that is, the external vector on the line from A is (x, x), that from B is (y, a) and that from C is (z, b).

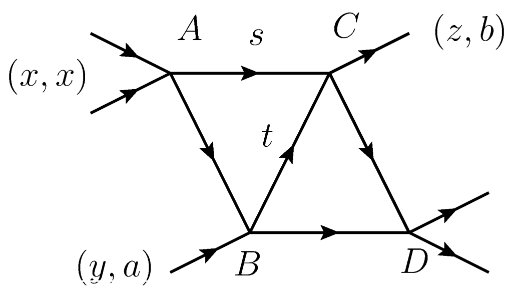

Figure 7. The truss bridge diagram T_2.

Let s and t be positive real numbers. We set

$$AC = (s,\, 1/s), \quad BC = (t,\, 1/t). \tag{3.11}$$

Then, by considering the energy-momentum conservation laws at each vertex A, B and C, we obtain

$$\begin{aligned} &AB = (1/s,\, s)\,, \quad AC = (s,\, 1/s)\,, \quad BC = (t,\, 1/t)\,, \\ &BD = \left(\frac{t}{st + at - 1},\, \frac{st + at - 1}{t}\right), \\ &CD = \left(-\frac{st}{bst - t - s},\, -\frac{bst - t - s}{st}\right). \end{aligned} \tag{3.12}$$

Note that, since an internal vector is located in the future light cone, i.e., the region $\{(v_0,\, v_1);\, v_0 > 0,\, v_1 > 0\}$, $(s,\, t)$ belongs to

$$\Omega := \{(s,\, t) \in \mathbb{R}^2_{>0};\, st + at - 1 > 0,\, t + s - bst > 0\}. \tag{3.13}$$

Then the real analytic map $\varphi : \Omega \to \mathbb{R}^3$ is defined by

$$\left\{\begin{aligned} x &= s + 1/s, \\ y &= \frac{s^2\,t^2 + a\,s\,t^2 - s\,t - a\,t + 1}{s\,(s\,t + a\,t - 1)}, \\ z &= \frac{b\,s\,t^2 - t^2 + b\,s^2\,t - s\,t - s^2}{b\,s\,t - t - s}. \end{aligned}\right. \tag{3.14}$$

This φ gives the required parametrization of a slice of $L^{\otimes}(T_2)$. Note that the slice is a surface in $\mathbb{R}^3$ and it is also called the nonzero-α LN surface for simplicity.

We can compute pinch points and cusps by the same arguments as those for T_1. The pinch points of T_2 are the following 3-points:

(P1) $(s,\, t) = (1,\, 1)$, i.e.,

$$(x, y, z) = \left(2,\ \frac{1}{a},\ \frac{2\,b - 3}{b - 2}\right). \tag{3.15}$$

The external vectors are located at this pinch point if and only if the configuration of internal vectors becomes

$$AB = BC = AC. \tag{3.16}$$

(P2) $(s,\, t) = \left(\frac{1}{b},\ \frac{2b}{ab+1}\right)$, i.e.,

$$(x, y, z) = \left(\frac{b^2 + 1}{b},\ -\frac{b\,(a\,b - 3)}{a\,b + 1},\ \frac{1}{b}\right). \tag{3.17}$$

The external vectors are located at this pinch point if and only if the configuration of internal vectors becomes

$$BC = CD = BD. \tag{3.18}$$

(P3) $(s,\, t) = \left(1,\ \frac{2}{a+b}\right)$, i.e.,

$$(x, y, z) = \left(2,\ -\frac{b^2 - 2\,b - a^2 + 2\,a + 4}{(b - a - 2)\,(b + a)},\ \frac{b^2 + 2\,b - a^2 - 2\,a - 4}{(b - a - 2)\,(b + a)}\right). \tag{3.19}$$

The external vectors are located at this pinch point if and only if the configuration of internal vectors becomes

$$AB = AC, \quad BD = CD. \tag{3.20}$$

The cusps of T_2 are given as follows. Note that, outside these cusps, φ becomes an embedding.

(C1) The image of $\{st = 1\}$, which is the half curve defined by

$$(x - z)(x - b) = 1, \quad y = 1/a \qquad (x > 2). \tag{3.21}$$

Note that its end-point is the pinch point (P1). Furthermore, this half curve is realized by the configuration of internal vectors with

$$AB = BC. \tag{3.22}$$

(C2) The image of $\{s = 1/b\}$, which is the half line defined by

$$x = \frac{b^2 + 1}{b}, \quad z = 1/b \qquad \left(y > -\frac{b\,(a\,b - 3)}{a\,b + 1}\right). \tag{3.23}$$

Note that its end-point is the pinch point (P2). This half line is realized by the configuration of internal vectors with

$$BC = CD. \tag{3.24}$$

(C3) This is a half portion of some analytic curve $\mathcal{C}$ whose end-point is the pinch point (P3). The defining equation of $\mathcal{C}$ is very complicated and long. See also the following remarks.

Remark 3.3. The cusps (C3) is the image by φ of the subset in the $(s,\, t)$-space

$$\begin{aligned} \{ &(b + a)\ s\ (s + a)\ (b\,s - 1)\ t^4 \\ &+ (s + a)\ (b\,s - 1)\ \left(b\,s^2 + a\,s^2 - 2\,s - b - a\right)\ t^3 \\ &- (s - 1)\ (s + 1)\ \left(2\,b\,s^2 + a\,s^2 + b^2\,s + 3\,a\,b\,s + a^2\,s - 2\,s - b - 2\,a\right)\ t^2 \\ &+ (s - 1)\ (s + 1)\ \left(s^2 + 2\,b\,s + 2\,a\,s - 1\right)\ t + s\ \left(1 - s^2\right) = 0 \}. \end{aligned} \tag{3.25}$$

Remark 3.4. Clear description of a configuration of internal vectors which realizes a point in the cusps (C3) is not yet known. For a specific (a, b), however, we have simple description of these cusps as follows: Suppose $a = b$. Then we can easily confirm that the distinct points

$$(s,\, t) = (s^*,\, 1/a) \quad \text{and} \quad (s,\, t) = (1/s^*,\, 1/a) \qquad (s^* > 0,\, s^* \neq 1)$$

give the same (x, y, z) by φ. The image of these points is defined by

$$y = z = 1/a \qquad (x > 2), \tag{3.26}$$

and it is the cusps (C3) when $a = b$.

Furthermore, when a is sufficiently close to b, we can find the cusps (C3) in the following way: Set $s = s(x) := 2^{-1}(x + \sqrt{x^2 - 4})$ $(x > 2)$ and

$$
\begin{aligned}
F(\rho, \sigma;\, x, a, b) &:= \left(\rho + \frac{1}{s + a - \rho^{-1}} - s^{-1}\right) - \left(\sigma + \frac{1}{s^{-1} + a - \sigma^{-1}} - s\right), \\
G(\rho, \sigma;\, x, a, b) &:= \left(s + \rho - \frac{1}{s^{-1} + \rho^{-1} - b}\right) - \left(s^{-1} + \sigma - \frac{1}{s + \sigma^{-1} - b}\right).
\end{aligned} \tag{3.27}
$$

Define the subspace $H \subset \mathbb{R}^5$ by

$$\{(\rho, \sigma, x, a, b) \in \mathbb{R}^5;\ a = b,\ \rho = \sigma = a^{-1}\}. \tag{3.28}$$

We can easily confirm

$$F|_H = G|_H = 0. \tag{3.29}$$

Since

$$
\begin{aligned}
\partial_\rho F|_H &= 1 - a^2 s^{-2}, \quad \partial_\sigma F|_H = -1 + a^2 s^2, \\
\partial_\rho G|_H &= 1 - a^2 s^2, \quad \partial_\sigma G|_H = -1 + a^2 s^{-2}
\end{aligned} \tag{3.30}
$$

hold, we get

$$J|_H = \det \left.\begin{pmatrix} \partial_\rho F & \partial_\sigma F \\ \partial_\rho G & \partial_\sigma G \end{pmatrix}\right|_H = 2a^2(s^{-2} - s^2) + a^4(s^4 - s^{-4}). \tag{3.31}$$

Then, by noticing

$$s^2 - s^{-2} = x(s - s^{-1}), \quad s^4 - s^{-4} = x(x^2 - 2)(s - s^{-1}), \tag{3.32}$$

we have

$$J|_H = a^2 x(s - s^{-1})(a^2(x^2 - 2) - 2), \tag{3.33}$$

from which $J|_H \neq 0$ follows if $x > 2$ and $x \neq \sqrt{2/a^2 + 2}$. Hence we can find an open subset D in $\{(x,\, a,\, b) \in \mathbb{R}^3; x > 2\}$ such that it is an open neighborhood of the locally closed subset

$$\{(x,\, a,\, b) \in \mathbb{R}^3;\ x > 2,\ x \neq \sqrt{2/a^2 + 2},\ a = b\} \tag{3.34}$$

and there exist real analytic functions $\rho = \rho(x; a, b)$ and $\sigma = \sigma(x; a, b)$ defined on D satisfying

$$
\begin{aligned}
&F(\rho(x; a, b),\, \sigma(x; a, b);\, x,\, a,\, b) = 0 && ((x, a, b) \in D), \\
&G(\rho(x; a, b),\, \sigma(x; a, b);\, x,\, a,\, b) = 0 && ((x, a, b) \in D), \\
&\rho(x; a, a) = \sigma(x; a, a) = 1/a && ((x, a, a) \in D).
\end{aligned} \tag{3.35}
$$

Now define 2-points

$$q_1 = (s^*,\, \rho(x^*; a, b)), \quad q_2 = (1/s^*,\, \sigma(x^*; a, b)), \tag{3.36}$$

where $x^* := s^* + 1/s^*$ and s^* is a positive real number satisfying $(x^*, a, b) \in D$. Then, since ρ and σ satisfy $F = G = 0$, the points q_1 and q_2 give the same (x, y, z) by φ. The (x, y, z) thus obtained belongs to the cusps (C3) because $q_1 \neq q_2$ and $\rho(x; a, a) = \sigma(x; a, a) = 1/a$ hold.

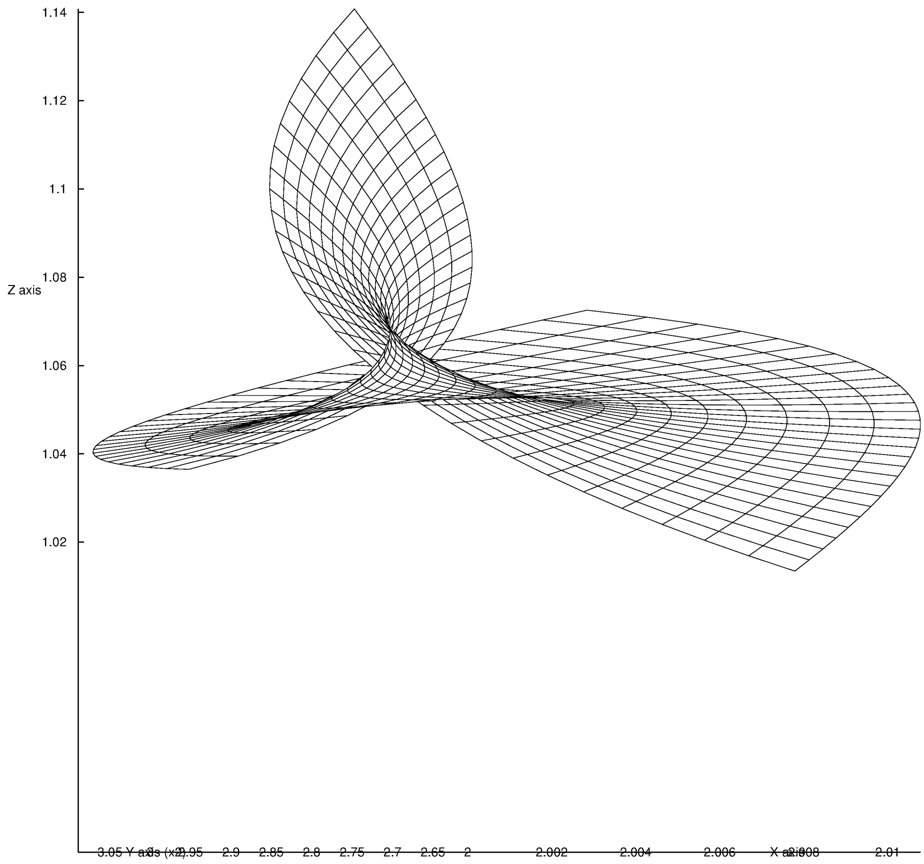

Figure 8. $L^{\otimes}(T_2)$ with $(a, b) = (0.7,\, 0.95)$ viewed from $(90, 320)$.

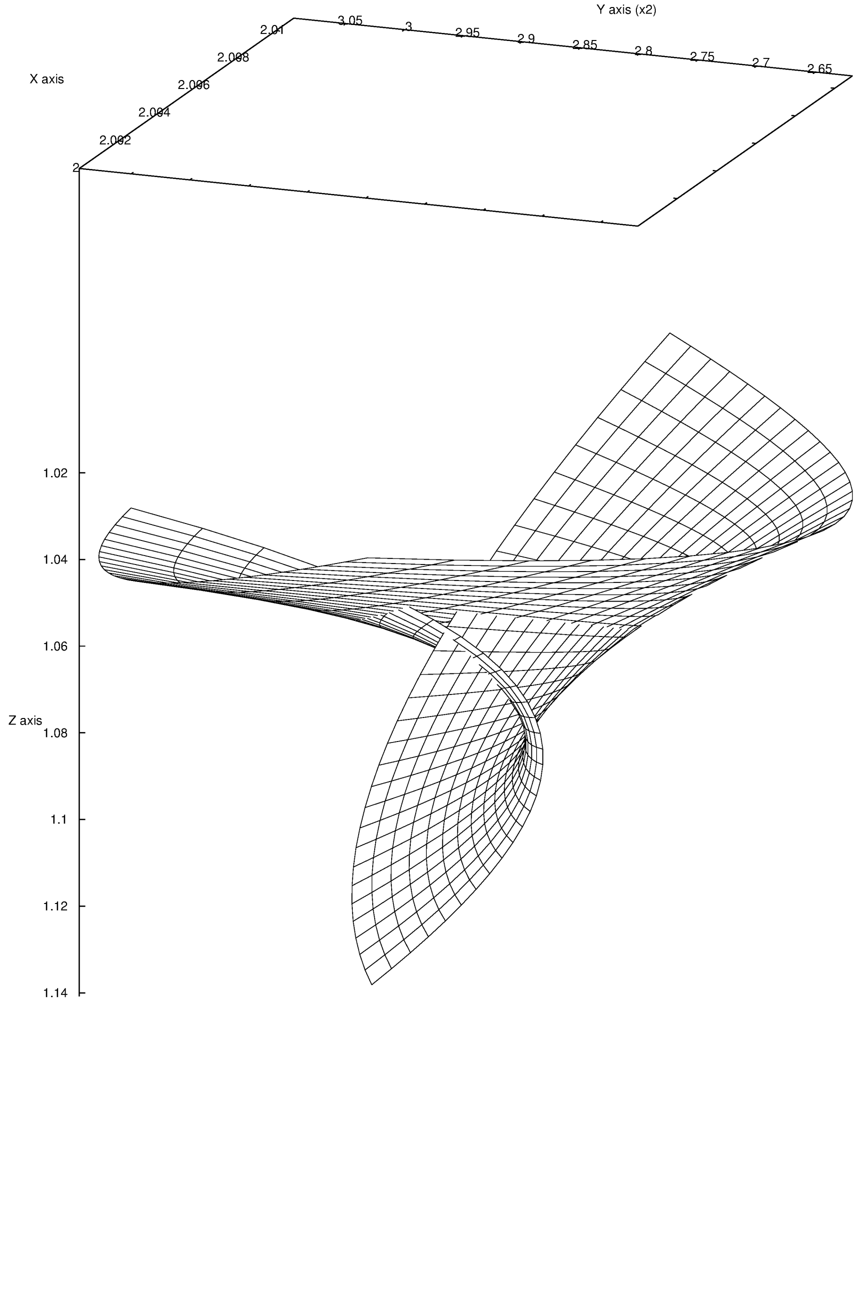

Figure 9. $L^{\otimes}(T_2)$ with $(a, b) = (0.7,\, 0.95)$ viewed from $(281, 291)$.

§ 3.3. The singularity structure of the LN surface of T_3

Let us now consider the truss bridge diagram T_3. By the same reasoning as that for T_2, the nonzero-α LN surface $L^{\otimes}(T_3)$ of T_3 can be regarded as a subset in $\mathbb{R}^5$, and hence, we need to consider a family of slices of $L^{\otimes}(T_3)$ cut with 3-dimensional linear subspaces. Let a and b be real numbers. Then we specify the coordinates $(x,\, y,\, z)$ and the parameters $(a,\, b)$ of the external vectors in the same way as that for T_2 (see Fig.10 also).

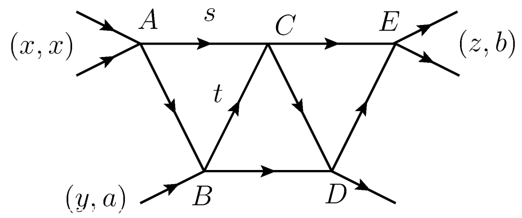

Figure 10. The truss bridge diagram T_3.

The biggest difference between T_2 and T_3 is existence of the "non-external vertex" C, that strongly constrains a configuration of internal vectors. As a matter of fact, it follows from Lemma 3.2 [5] that internal vectors satisfy either (A) or (B) below:

$$(A)\ \ AC = CD \ \text{ and } \ BC = CE \qquad (B)\ \ AC = CE \ \text{ and } \ BC = CD. \tag{3.37}$$

Therefore we have a different component of $L^{\otimes}(T_3)$ corresponding to either (A) or (B).

3.3.1. The component of $L^{\otimes}(T_3)$ with the configuration (A)

We first study the component of $L^{\otimes}(T_3)$ where internal vectors satisfy the configuration (A). Note that, in this case, each slice of the component becomes a surface. Set

$$AC = (s,\, 1/s) \quad \text{and} \quad BC = (t,\, 1/t). \tag{3.38}$$

Then, it follows from the energy-momentum conservation laws and the configuration (A) that we have

$$\begin{aligned} &AC = CD = (s,\, 1/s)\,, \quad AB = (1/s,\, s)\,, \quad BC = CE = (t,\, 1/t)\,,\\ &BD = \left(\frac{t}{st + at - 1},\, \frac{st + at - 1}{t}\right),\\ &DE = \left(\frac{t}{bt - 1},\, \frac{bt - 1}{t}\right). \end{aligned} \tag{3.39}$$

Define

$$\Omega := \{(s,\, t) \in \mathbb{R}^2_{>0};\, bt - 1 > 0,\, st + at - 1 > 0\},$$

and the analytic map $\varphi : \Omega \to \mathbb{R}^3$ by

$$\begin{cases} x = s + 1/s, \\ y = \dfrac{s^2\,t^2 + a\,s\,t^2 - s\,t - a\,t + 1}{s\,(s\,t + a\,t - 1)}, \\ z = \dfrac{b\,t^2}{b\,t - 1}. \end{cases} \tag{3.40}$$

This φ gives the required parametrization of a slice of the component of $L^{\otimes}(T_3)$ with the configuration (A). The slice becomes a surface in the 3-dimensional linear space and it is often called the "surface component" of T_3.

The pinch points are the following 3-points:

(P1) $(s,\, t) = \left(1,\, \dfrac{1}{a}\right)$, i.e.,

$$(x, y, z) = \left(2,\, 1/a,\, \frac{b}{a\,(b-a)}\right). \tag{3.41}$$

The external vectors are located at this pinch point if and only if the configuration of internal vectors becomes

$$AB = BD = AC\ (= CD). \tag{3.42}$$

(P2) $(s,\, t) = \left(b - a,\, \dfrac{2}{b}\right)$, i.e.,

$$(x, y, z) = \left(b - a + \frac{1}{b-a},\, \frac{3\,b - 4\,a}{b\,(b-a)},\, 4/b\right). \tag{3.43}$$

The external vectors are located at this pinch point if and only if the configuration of internal vectors becomes

$$(BC =)\ CE = BD = DE. \tag{3.44}$$

(P3) $(s,\, t) = \left(1,\, \dfrac{2}{b}\right)$, i.e.,

$$(x, y, z) = \left(2,\, -\frac{b^2 - 2\,a\,b - 2\,b + 4\,a + 4}{b\,(b - 2\,a - 2)},\, 4/b\right). \tag{3.45}$$

The external vectors are located at this pinch point if and only if the configuration of internal vectors becomes

$$(CD =)\ AC = AB, \quad (BC =)\ CE = DE. \tag{3.46}$$

The cusps of T_3 are given as follows. Note that, outside these cusps, φ becomes an embedding.

(C1) The image of $\{t = 1/a\}$, which is the half line defined by

$$y = \frac{1}{a}, \quad z = \frac{b}{a\,(b-a)} \qquad (x > 2). \tag{3.47}$$

Note that its end-point is the pinch point (P1). Furthermore, this half line is realized by the configuration of internal vectors with

$$AB = BD. \tag{3.48}$$

(C2) The image of $\{s = b - a\}$, which is the half line defined by

$$x = b - a + \frac{1}{b-a}, \quad y = z - \frac{1}{b-a} \qquad (z > 4/b). \tag{3.49}$$

Note that its end-point is the pinch point (P2). Furthermore, this half line is realized by the configuration of internal vectors with

$$BD = DE. \tag{3.50}$$

(C3) This is a half portion of some analytic curve $\mathcal{C}$ whose end-point is the pinch point (P3). The defining equation of $\mathcal{C}$ is very complicated and long. See also Remark 3.5 below. The situation is quite similar to the one for the cusps (C3) of T_2.

Remark 3.5. The cusps (C3) is the image by φ of the subset in the $(s,\,t)$-space

$$\begin{aligned}
&\big\{b\,s\,(s+a)\,(b\,s - a\,s - 1)\,t^4 \\
&+ (s+a)\,(b\,s - a\,s - 1)\,\big(b\,s^2 - 2\,s - b\big)\,t^3 \\
&- (s-1)\,(s+1)\,\big(2\,b\,s^2 - a\,s^2 + b^2\,s + a\,b\,s - a^2\,s - 2\,s - b - a\big)\,t^2 \\
&+ (s-1)\,(s+1)\,\big(s^2 + 2\,b\,s - 1\big)\,t + s\,\big(1 - s^2\big) = 0\big\}.
\end{aligned} \tag{3.51}$$

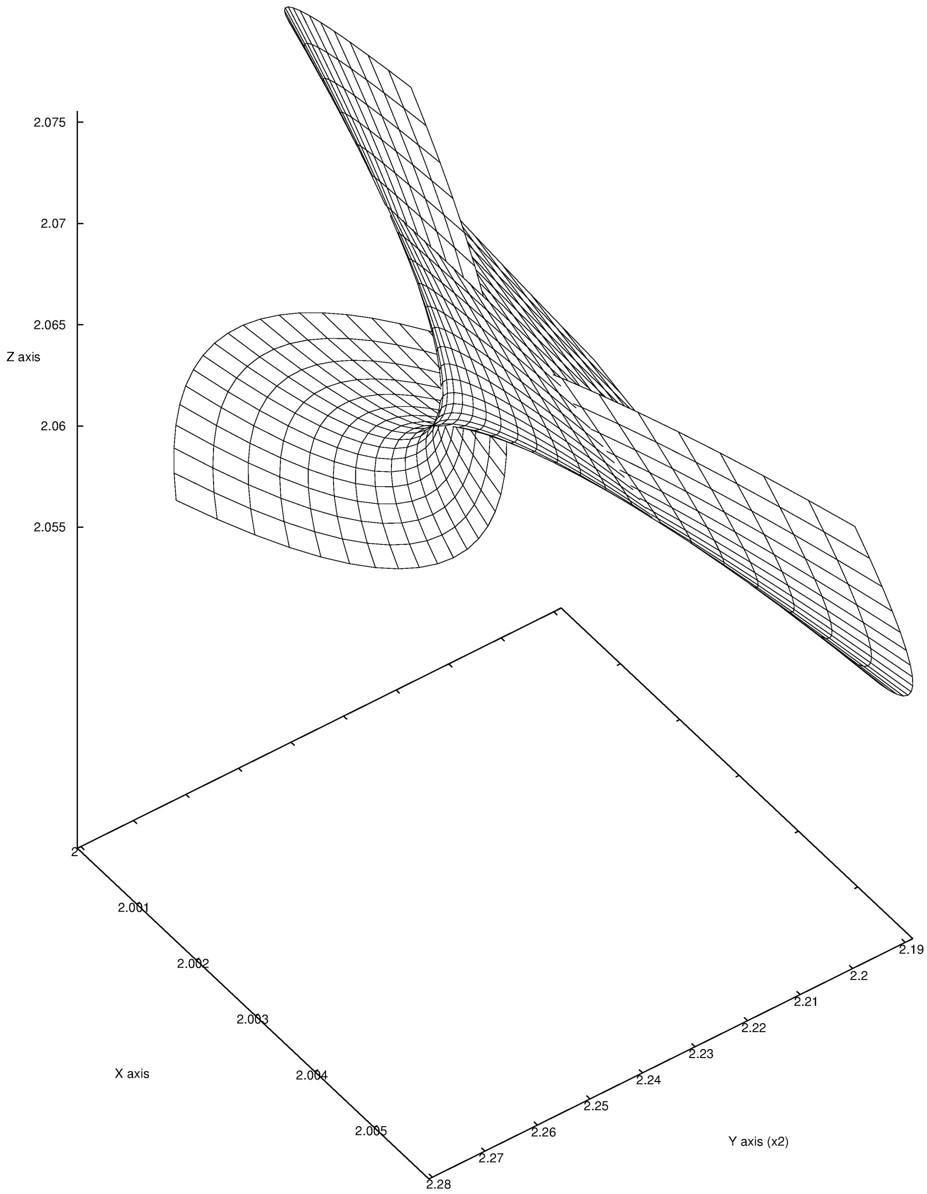

Figure 11. $L^{\otimes}(T_3)$ with (A) and $(a, b) = (0.9,\ 1.95)$ viewed from $(119, 306)$.

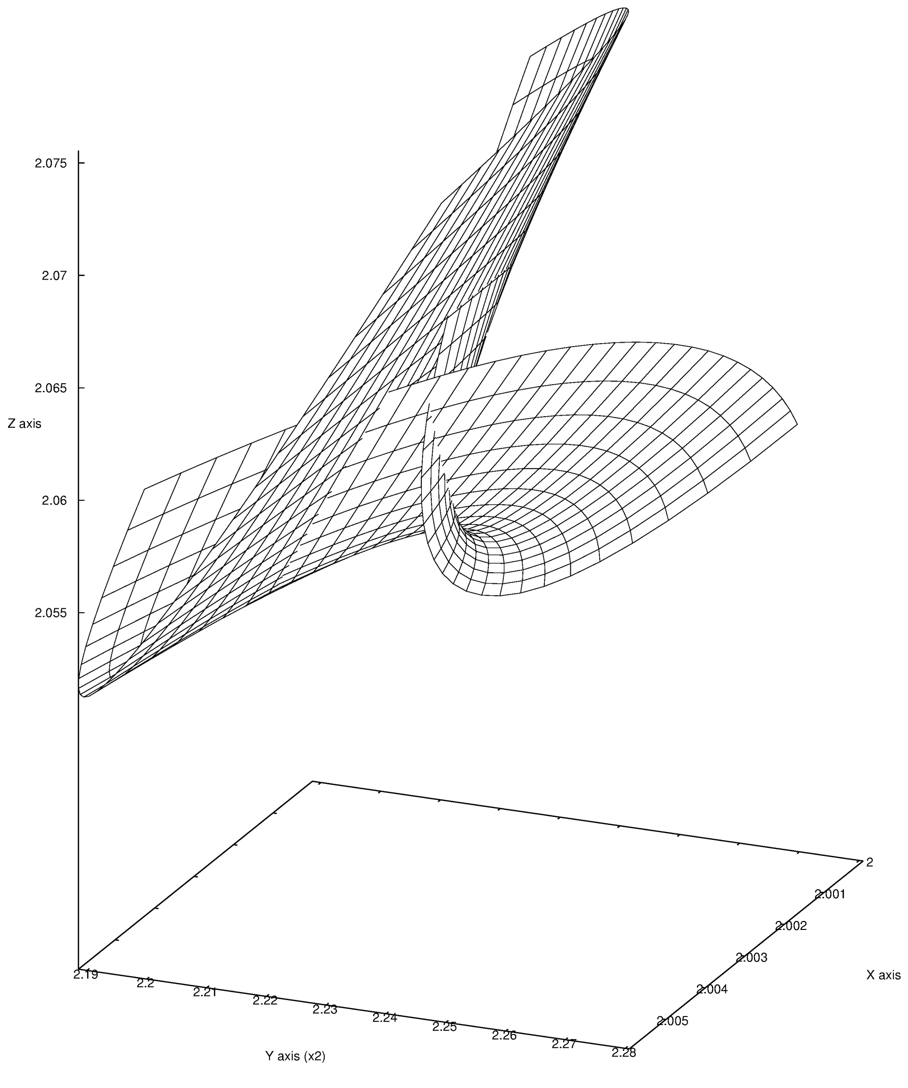

Figure 12. $L^{\otimes}(T_3)$ with (A) and $(a, b) = (0.9,\ 1.95)$ viewed from $(76, 113)$.

3.3.2. The component of $L^{\otimes}(T_3)$ with the configuration (B)

We now study the component of $L^{\otimes}(T_3)$ where internal vectors satisfy the configuration (B). Note that, in this case, each slice of the component becomes a curve in the 3-dimensional linear space and it is often called the “non-surface component” or the “higher codimensional component”.

Let $s > 0$, and we set $AC = (s,\, 1/s)$. It follows from the configuration (B) and the closed loop condition for the triangle $\triangle BCD$ that the internal vectors satisfy

$$AC = CE, \quad BC = BD = CD. \tag{3.52}$$

Then, the internal vectors are uniquely determined by the condition (3.52) and the energy-momentum conservation laws at each vertex as follows.

$$\begin{aligned} &AC = CE = (s,\, 1/s)\,, \quad AB = (1/s,\, s)\,, \\ &BC = CD = BD = \left(\frac{2}{s+a},\, \frac{s+a}{2}\right), \\ &DE = \left(\frac{s}{b\,s-1},\, \frac{b\,s-1}{s}\right). \end{aligned} \tag{3.53}$$

Remark 3.6. When a Feynman graph has a non-external vertex, generally speaking, its nonzero-α LN surface may have a higher codimensional component.

Define

$$\Omega = \{s \in \mathbb{R}_{>0};\; s+a > 0,\; bs-1 > 0\}, \tag{3.54}$$

and the analytic map $\varphi : \Omega \to \mathbb{R}^3$ by

$$\begin{cases} x = s + 1/s, \\ y = \dfrac{3\,s-a}{s\,(s+a)}, \\ z = \dfrac{b\,s^2}{b\,s-1}. \end{cases} \tag{3.55}$$

Then the map φ gives the required parametrization of the higher codimensional component of T_3. Note that this curve is smooth if $(a,\, b) \neq (1,\, 2)$, and it has only one singular point $s = 1$, i.e., $(x,\, y,\, z) = (2,\, 2,\, 2)$ if $(a,\, b) = (1,\, 2)$.

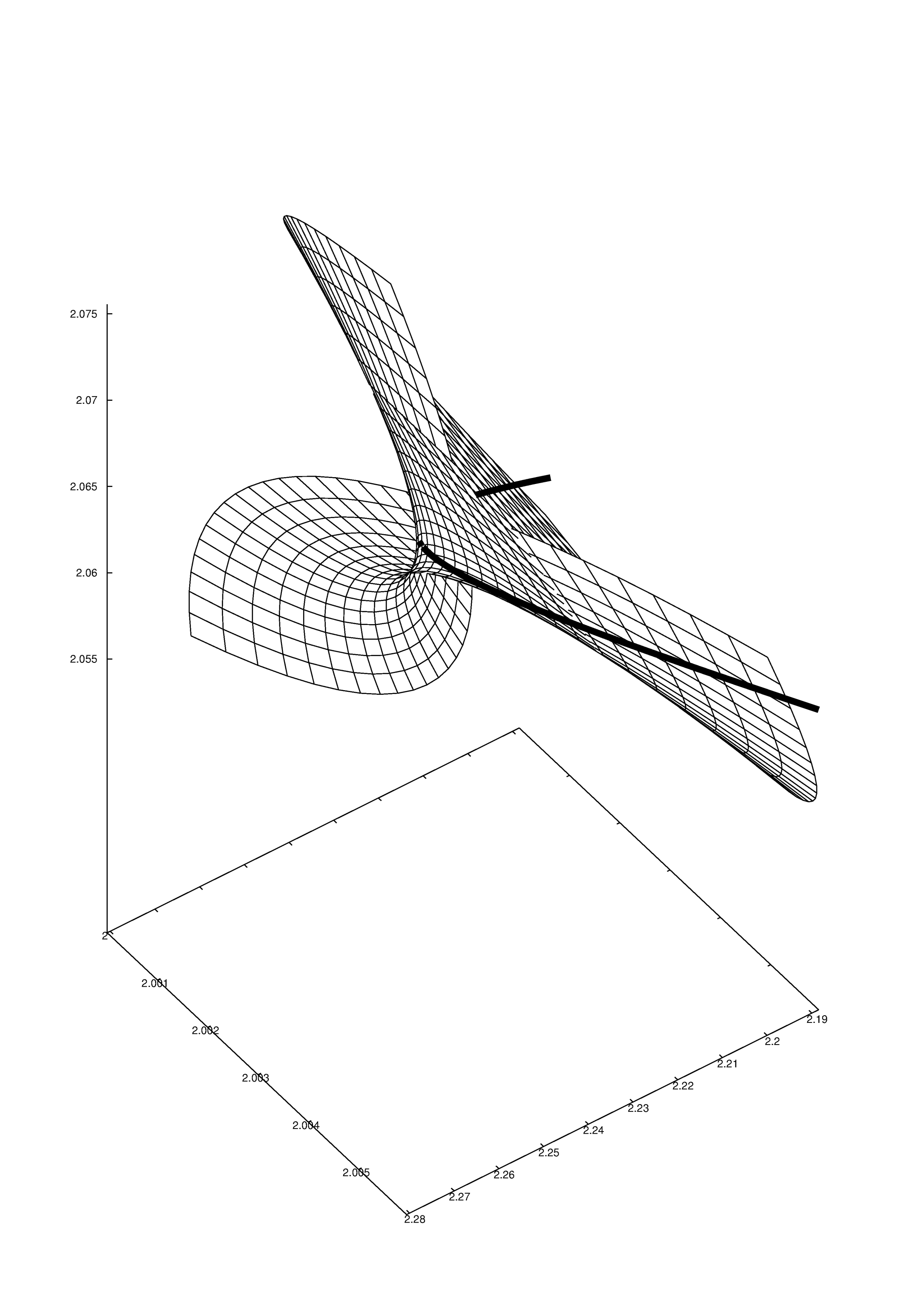

Figure 13. $L^{\otimes}(T_3)$ with (B) and $(a, b) = (0.9,\ 1.95)$: The curve which crosses the surface is the non-surface component, viewed from $(119, 306)$.

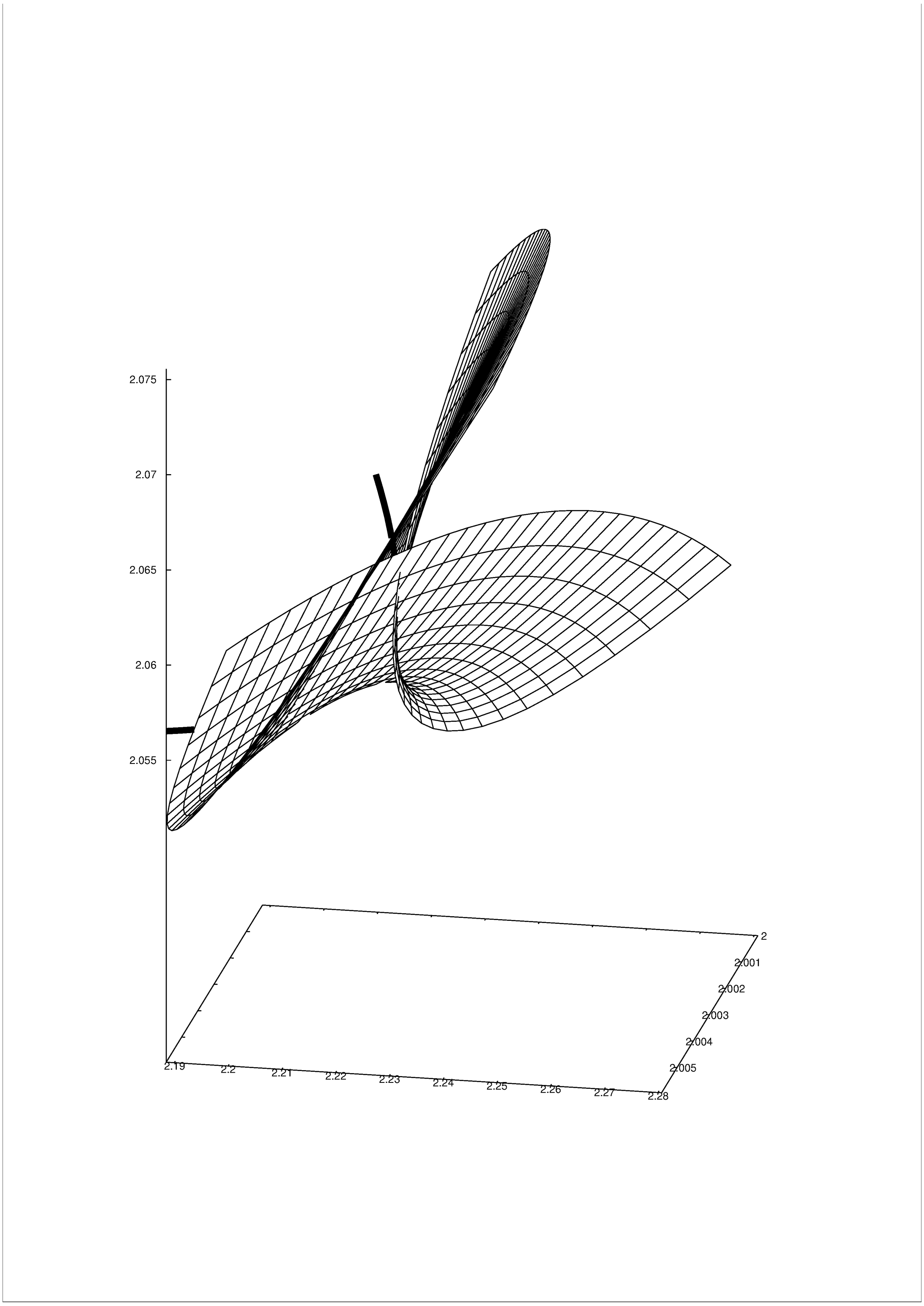

Figure 14. $L^{\otimes}(T_3)$ with (B) and $(a, b) = (0.9,\ 1.95)$: The curve which crosses the surface is the non-surface component, viewed from $(77, 101)$.

§ 4. An acnode in the Landau-Nakanishi geometry of T_2

An acnode (i.e., an isolated point) appearing in the Landau-Nakanishi geometry of T_2 was first found by R. J. Eden et al. [1] (see also [2]). We study, in this section, its origin from the viewpoint of singularity structure.

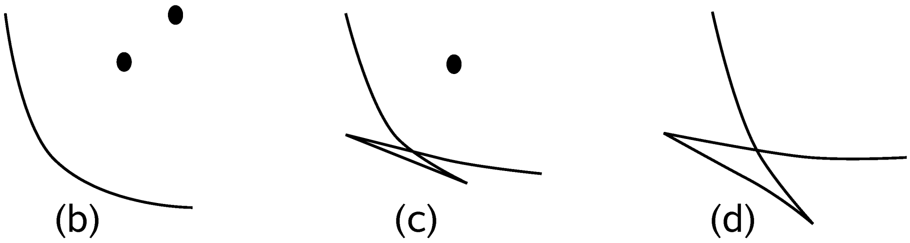

Figure 15. A rough picture of acnodes described in p.106 [2].

In the book [2], R. J. Eden et al. had studied a family of LN curves of T_2, which is, by definition, the intersection of $L^{\oplus}(T_2)$ and a parameterized family of 2-dimensional subspaces. They first change a parameter of the family to the complex domain, and then, put it back to the real domain. After these changes of a parameter, they found an isolated point (acnode) apart from the curve as it is shown in Fig.15. By further continuous changes of a parameter (from (b) to (d) in Fig.15), the acnode continuously moves and finally disappears after it hits on the curve.

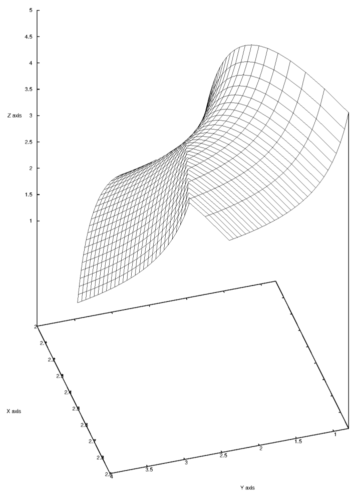

Figure 16. $L^{\otimes}(T_1)$.

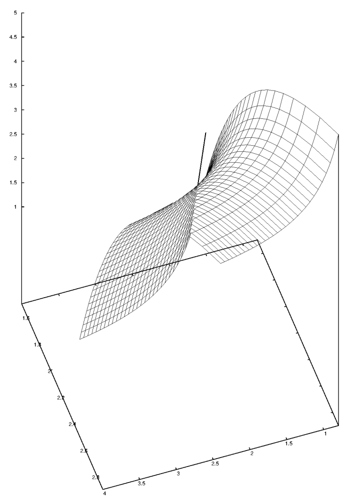

Figure 17. The analytic closure of $L^{\otimes}(T_1)$.

At first glance, existence of such an acnode seems strange because the nonzero-α LN surface of T_2 generically has a shape like the one drawn in Fig.8, and thus, it is impossible to obtain an isolated point possessed with such a behavior when we cut the surface with a suitable family of 2-dimensional subspaces in $\mathbb{R}^3$. Singularity structure studied in the previous section, however, can explain an origin of the acnode. In fact, the cusps in T_1, T_2 and T_3 are half portions of real analytic curves. Hence, if we take the analytic closures of these surfaces, the whole parts of these curves appear. For example, if we take the analytic closure of the nonzero-α LN surface of T_1, then we get the original surface with the whole line and a half portion of this line is located far from the surface as Fig.17 shows.

Furthermore, their change of a parameter to the complex domain entails complexification of the nonzero-α LN surface of T_2. As a consequence, when they put a parameter back to the real domain, the resulting surface contains the analytic closure of the original surface. Hence the acnode they observed can be understood as a point in a portion of an analytic extension of some cusps which is located far from the surface itself.

We give, in Fig.18, some slices of the analytic closure of the nonzero-α LN surface of T_2 cut with the 2-dimensional linear subspace $\{(x, y, z) \in \mathbb{R}^3; x + y = k\}$. One can surely find an isolated point (acnode) which continuously moves and disappears after it hits on the curve. We note that, if we consider a region which is much wider than that shown below, we find another acnode as in Fig.15 (b) (see Fig. 19).

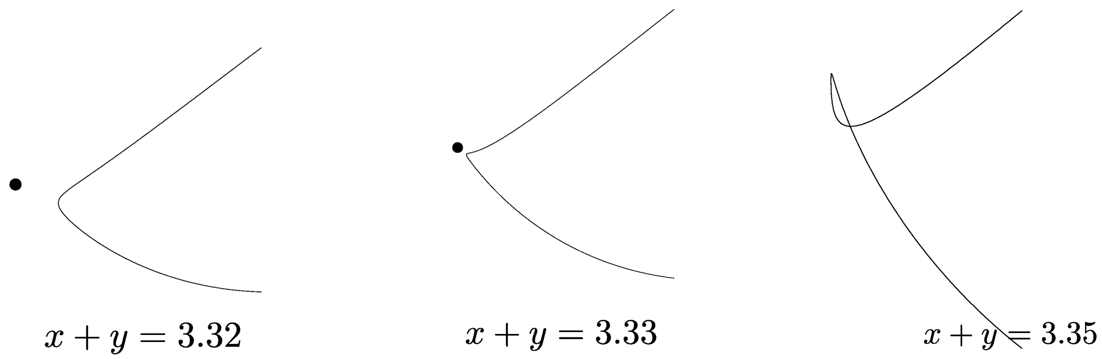

Figure 18. Slices by $\{x + y = k\}$.

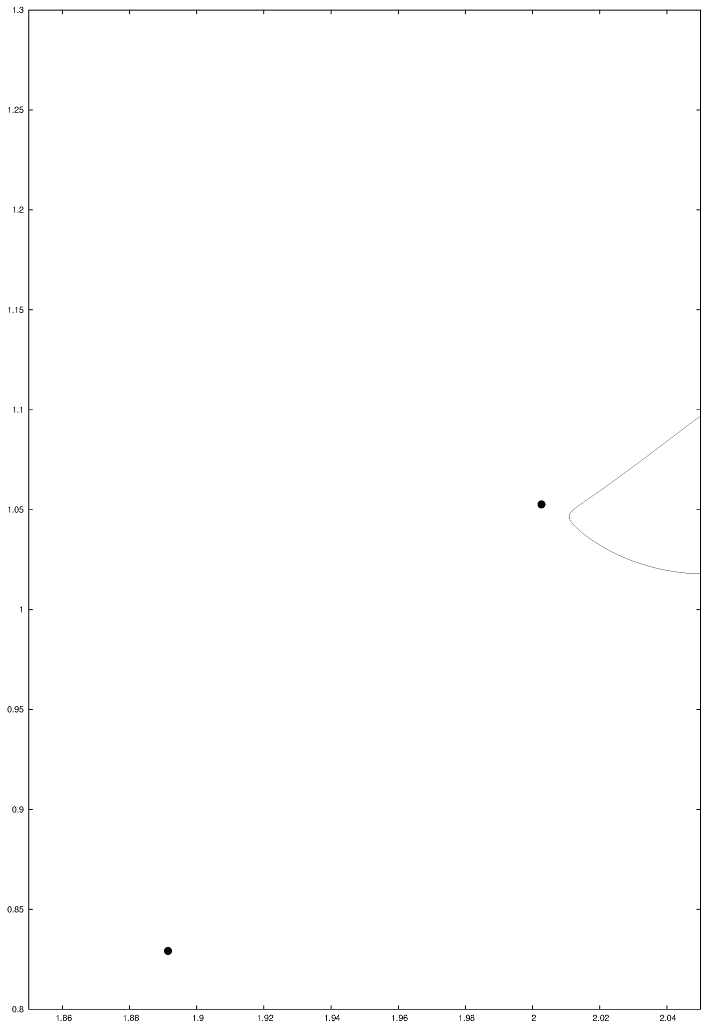

Figure 19. Slices by $\{x + y = 3.32\}$ in the wider region.

§ 5. The non-surface component of T_3

As we pointed out in [3] (see Section 3 of this article also), $L^{\oplus}(T_3)$ contains two components; one a hypersurface and the other with codimension 2. This is a phenomenon which is not observed in $L^{\oplus}(G)$ if G has no non-external vertex. In [3] and [5] we studied the non-surface component (i.e., the codimension 2 component) of $L^{\oplus}(T_3)$ by using the property that it coincides with the intersection of two LN surfaces associated with ice-cream cone diagrams I_L and I_R, which are obtained by contracting some parts of T_3 ([5, Section 5]). Here we present another approach that we understand the non-surface component from the viewpoint of singularity structure of a surface component. We first introduce the complemented truss bridge diagram $\widetilde{T}_3$ by adding an extra external line to the non-external vertex of T_3 and compute its pinch points and cusps. Then we investigate correspondence between the non-surface component of T_3 and the restriction of pinch points and cusps of $\widetilde{T}_3$ to the ambient space of T_3.

§ 5.1. The complemented truss bridge diagram $\widetilde{T}_3$

The complemented truss bridge diagram $\widetilde{T}_3$ is described in Fig. 20, which is obtained by addition of the external line to the vertex C of T_3. It has 5-external lines, and hence, the nonzero-α LN surface $L^{\otimes}(\widetilde{T}_3)$ of $\widetilde{T}_3$ is regarded as a hypersurface in $\mathbb{R}^7$. In this case, we consider a family of slices of $L^{\otimes}(\widetilde{T}_3)$ cut with 4-dimensional linear subspaces. Let a, b and c be real numbers. We specify the coordinates (x, y, z, w) and the parameters (a, b, c) of the external vectors as described in Fig.20, that is, the external vector on the line from A is (x, x), that from B is (y, a), that from C is (w, c) and that from E is (z, b). Note that the ambient space of T_3 is identified with the subspace $\{w = c = 0\}$ in this situation.

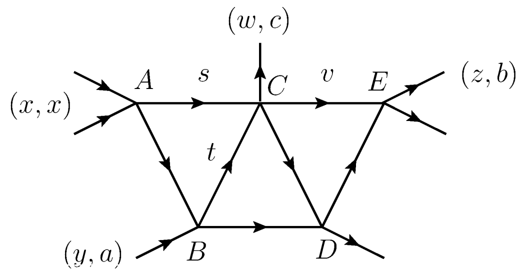

Figure 20. The complemented truss bridge diagram $\widetilde{T}_3$.

Remark 5.1. The reason why we consider slices cut with 4-dimensional subspaces instead of 3-dimensional ones is as follows: As we will see later, there exists a connected component of pinch points of $\widetilde{T}_3$ which has non-transversal intersection with the ambient space of T_3, i.e., the subspace $\{w = c = 0\}$. Hence we need, at least, a 4-dimensional linear subspace so that it contains the ambient space of T_3 and it intersects transversally with each component of singular points of $\widetilde{T}_3$.

Let s, t and v be positive real numbers, and set

$$AC = (s,\, 1/s), \quad BC = (t,\, 1/t), \quad CE = (v,\, 1/v). \tag{5.1}$$

Then it follows from the energy-momentum conservation laws that the internal vectors are given by

$$\begin{aligned} &AC = (s,\, 1/s)\,, \quad AB = (1/s,\, s)\,, \quad BC = (t,\, 1/t)\,, \quad CE = (v,\, 1/v)\,, \\ &CD = \left(-\frac{s\,t\,v}{c\,s\,t\,v - t\,v - s\,v + s\,t},\, -\frac{c\,s\,t\,v - t\,v - s\,v + s\,t}{s\,t\,v}\right), \\ &BD = \left(\frac{t}{s\,t + a\,t - 1},\, \frac{s\,t + a\,t - 1}{t}\right), \\ &DE = \left(\frac{v}{b\,v - 1},\, \frac{b\,v - 1}{v}\right). \end{aligned} \tag{5.2}$$

Set

$$\Omega := \{(s,\, t,\, v) \in \mathbb{R}^3_{>0};\, bv - 1 > 0,\ st + at - 1 > 0,\ -cstv + tv + sv - st > 0\}. \tag{5.3}$$

Then the analytic map $\varphi : \Omega \to \mathbb{R}^4$ is defined by

$$\begin{cases} x = s + 1/s, \\ y = \dfrac{s^2\,t^2 + a\,s\,t^2 - s\,t - a\,t + 1}{s\,(s\,t + a\,t - 1)}, \\ z = \dfrac{b\,v^2}{b\,v - 1}, \\ w = -\dfrac{c\,s\,t\,v^2 - t\,v^2 - s\,v^2 - c\,s\,t^2\,v + t^2\,v - c\,s^2\,t\,v + 2\,s\,t\,v + s^2\,v - s\,t^2 - s^2\,t}{c\,s\,t\,v - t\,v - s\,v + s\,t}. \end{cases} \tag{5.4}$$

This φ gives parametrization of a slice of $L^{\otimes}(\widetilde{T}_3)$.

§ 5.2. Pinch points of the complemented truss bridge diagram $\widetilde{T}_3$

Let us compute $d\varphi$. Since x (resp. z) depends only on s (resp. v) and y depends only on s and t, the 3×4 Jacobian matrix of φ takes a form as

$$J := \begin{pmatrix} \dfrac{\partial x}{\partial s} & 0 & \dfrac{\partial y}{\partial s} & \dfrac{\partial w}{\partial s} \\ 0 & \dfrac{\partial z}{\partial v} & 0 & \dfrac{\partial w}{\partial v} \\ 0 & 0 & \dfrac{\partial y}{\partial t} & \dfrac{\partial w}{\partial t} \end{pmatrix}. \tag{5.5}$$

We also have

$$\frac{\partial x}{\partial s} = \frac{(s-1)\,(s+1)}{s^2}, \quad \frac{\partial z}{\partial v} = \frac{b\,v\,(b\,v-2)}{(b\,v-1)^2}, \tag{5.6}$$

from which

$$\frac{\partial x}{\partial s} = 0 \iff s = 1 \quad \text{and} \quad \frac{\partial z}{\partial v} = 0 \iff v = 2/b \tag{5.7}$$

follows. By taking these observations into account, we compute a pinch point at which $\mathrm{Rank}(J) < 3$ holds.

Case I: $s \neq 1$ and $v \neq 2/b$.

In this case, we have

$$\mathrm{Rank}(J) < 3 \iff \frac{\partial y}{\partial t} = 0 \text{ and } \frac{\partial w}{\partial t} = 0.$$

Since

$$\frac{\partial y}{\partial t} = \frac{(s+a)\;t\;(s\,t + a\,t - 2)}{(s\,t + a\,t - 1)^2}$$

and

$$\frac{\partial w}{\partial t} = \frac{t\;(c\,s\,v - v + s)\;(c\,s\,t\,v - t\,v - 2\,s\,v + s\,t)}{(c\,s\,t\,v - t\,v - s\,v + s\,t)^2}$$

hold, we conclude that $\mathrm{Rank}(J) < 3$ if and only if

$$\begin{cases} (s\,t + a\,t - 2) = 0, \\ (c\,s\,v - v + s) = 0. \end{cases}$$

Here we use the facts $s + a > 0$ and $-cstv + tv + 2sv - st > 0$ which follow from the conditions $st + at - 1 > 0$ and $-cstv + tv + sv - st > 0$. Summing up, we have one component of pinch points: Here we note that we find $CD = -BD$ by (5.2) if $cstv - tv - 2sv + st = 0$ and hence the point in question is located outside the region of our concern in this paper.

(I.a) $t = \dfrac{2}{s+a}$ and $v = -\dfrac{s}{c\,s-1}$ (s free). For these parameters, we have

$$w = \frac{c\,s^2}{c\,s-1}. \tag{5.8}$$

We can understand the pinch points (I.a) through a configuration of internal vectors. First note that

$$t = \frac{2}{s+a} \iff BC = BD \quad \text{and} \quad v = -\frac{s}{c\,s-1} \iff BC = CD. \tag{5.9}$$

Therefore the external vectors are located at a pinch point (I.a) if and only if the configuration of internal vectors satisfy

$$BC = BD = CD. \tag{5.10}$$

In particular, when $c = 0$, we have $w = 0$ by (5.8) which implies $v = s$, i.e., $AC = CE$. Summing up, if $c = 0$, we get

$$BC = BD = CD, \quad AC = CE, \tag{5.11}$$

and thus, it follows from (3.52) that the restriction of pinch points (I.a) to $\{w = c = 0\}$ coincides with the non-surface component of T_3.

<u>**Case II**: $s = 1$ and $v \neq 2/b$.</u>

In this case,

$$\mathrm{Rank}(J) < 3 \iff \frac{\partial y}{\partial s}\frac{\partial w}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial w}{\partial s} = 0.$$

As we have

$$\left(\frac{\partial y}{\partial s}\frac{\partial w}{\partial t} - \frac{\partial y}{\partial t}\frac{\partial w}{\partial s}\right)\bigg|_{s=1} = -\frac{(t-1)\ t\ (t+1)\ (c\,v - a\,v - 2\,v + 1)\ (c\,t\,v + a\,t\,v - 2\,v + t)}{(a\,t + t - 1)^2\,(c\,t\,v - t\,v - v + t)^2},$$

we obtain 3-components of pinch points:

(II.a) $s = 1$ and $t = 1$ (v free). For these parameters, we have

$$w = -\frac{c\,v^2 - 2\,v^2 - 2\,c\,v + 4\,v - 2}{c\,v - 2\,v + 1} \tag{5.12}$$

and, in particular, when $c = 0$,

$$w = -\frac{2\,(v-1)^2}{2\,v - 1} \tag{5.13}$$

holds.

(II.b) $s = 1$ and $v = -\dfrac{1}{c - a - 2}$ (t free). For these parameters, we have

$$w = \frac{a\,c\,t^2 + c\,t^2 - a^2\,t^2 - 3\,a\,t^2 - 2\,t^2 + a\,c\,t + c\,t - a^2\,t - 2\,a\,t - t - c + a + 1}{(c - a - 2)\ (a\,t + t - 1)} \tag{5.14}$$

and, in particular, when $c = 0$,

$$w = \frac{(a+1)\ (t+1)\ (a\,t + 2\,t - 1)}{(a+2)\ (a\,t + t - 1)}. \tag{5.15}$$

In this case, (5.2) entails $CD = -BD$; hence the point in question is located outside the region of our concern in this paper.

(II.c) $s = 1$ and $v = -\dfrac{t}{(c+a)\,t-2}$ (t free). For these parameters, we have

(5.16)
$$w = \frac{a\,c\,t^3 + c\,t^3 + a^2\,t^3 + a\,t^3 + a\,c\,t^2 - c\,t^2 + a^2\,t^2 - 2\,a\,t^2 - t^2 - c\,t - 3\,a\,t + t + 2}{(a\,t+t-1)\,(c\,t+a\,t-2)}$$

and, in particular, when $c = 0$,

(5.17)
$$w = \frac{(t+1)\,(a\,t-1)\,(a\,t+t-2)}{(a\,t-2)\,(a\,t+t-1)}.$$

Note that this component passes through a pinch point of T_3 when $t = 1/a$ and it also intersects with the non-surface component of T_3 when $t = 2/(a+1)$.

Case III: $s \neq 1$ and $v = 2/b$.

In this case,

$$\mathrm{Rank}(J) < 3 \iff \frac{\partial y}{\partial t}\frac{\partial w}{\partial v} = 0.$$

By noticing

$$\left.\frac{\partial y}{\partial t}\frac{\partial w}{\partial v}\right|_{v=2/b} = \frac{4\,(s+a)\,t\,(s\,t+a\,t-2)\,(c\,s\,t-t-s)\,(c\,s\,t+b\,s\,t-t-s)}{(s\,t+a\,t-1)^2\,(2\,c\,s\,t+b\,s\,t-2\,t-2\,s)^2},$$

we have 3-components of pinch points:

(III.a) $v = 2/b$ and $t = \dfrac{2}{s+a}$ (s free). For these parameters, we have

(5.18)
$$\begin{aligned} w =& \big(b\,s^4 - 2\,b\,c\,s^3 - b^2\,s^3 + 2\,a\,b\,s^3 - 2\,s^3 - 2\,a\,b\,c\,s^2 + 4\,c\,s^2 - a\,b^2\,s^2 + a^2\,b\,s^2 \\ &+ 4\,b\,s^2 - 4\,a\,s^2 - 4\,b\,c\,s + 4\,a\,c\,s - 2\,b^2\,s + 4\,a\,b\,s - 2\,a^2\,s - 4\,s + 4\,b - 4\,a\big) \\ &/ \big(b\,(s+a)\,\big(s^2 - 2\,c\,s - b\,s + a\,s + 2\big)\big) \end{aligned}$$

and, in particular, when $c = 0$,

(5.19)
$$w = \frac{(s-b+a)\,(b\,s-2)\,\big(s^2+a\,s+2\big)}{b\,(s+a)\,(s^2-b\,s+a\,s+2)}.$$

Note that this component passes through a pinch point of T_3 when $s = b - a$ and it also intersects with the non-surface component of T_3 when $s = 2/b$.

(III.b) $v = 2/b$ and $t = \dfrac{s}{c\,s - 1}$ (s free). For these parameters, we have

$$w = \frac{c\,s^2}{c\,s - 1}. \tag{5.20}$$

In this case, (5.2) entails $CD = -CE$, and hence, the point in question is located outside the region of our concern in this paper.

(III.c) $v = 2/b$ and $t = \dfrac{s}{(c + b)\ s - 1}$ (s free). For these parameters, we have

$$w = \frac{b\,c\,s^2 + b^2\,s^2 - 4\,c\,s - 4\,b\,s + 4}{b\ (c\,s + b\,s - 1)} \tag{5.21}$$

and, in particular, when $c = 0$,

$$w = \frac{(b\,s - 2)^2}{b\ (b\,s - 1)}. \tag{5.22}$$

Case IV: $s = 1$ and $v = 2/b$.

Always $\mathrm{Rank}(J) < 3$ in this case. Hence we have one component of pinch points:

(IV.a) $s = 1$ and $v = 2/b$ (t free). For these parameters, we have

$$w = \frac{2\,b\,c\,t^2 + b^2\,t^2 - 2\,b\,t^2 + 2\,b\,c\,t - 4\,c\,t + b^2\,t - 4\,b\,t + 4\,t - 2\,b + 4}{b\ (2\,c\,t + b\,t - 2\,t - 2)} \tag{5.23}$$

and, in particular, when $c = 0$,

$$w = \frac{(b - 2)\ (t + 1)\ (b\,t - 2)}{b\ (b\,t - 2\,t - 2)}. \tag{5.24}$$

Note that this component passes through a pinch point of T_3 when $t = 2/b$.

Summing up, for the restriction of pinch points of $\widetilde{T}_3$ to the ambient space of T_3, i.e., the subspace $\{w = c = 0\}$, we have observed the following two facts: The 3-components (II.c), (III.a) and (IV.a) of pinch points of $\widetilde{T}_3$ transversally intersect with the ambient space of T_3 for generic parameters. Their intersection contains all pinch points of T_3.

The component (I.a) of pinch points of $\widetilde{T}_3$, however, has non-transversal intersection with the ambient space of T_3. The important fact is that their intersection is nothing but the non-surface component of T_3.

§ 5.3. Cusps of $\widetilde{T}_3$ emanating from the pinch points (I.a)

Let us consider the surface $\widetilde{S}$ generated by external vectors when internal vectors of $\widetilde{T}_3$ satisfy the conditions

$$AC = CE, \quad BC = CD \tag{5.25}$$

with the energy-momentum conservation laws at each vertex and $k_\ell^2 = 1$ for any internal vector k_ℓ. Here we forget a closed loop condition, in particular, the one for the triangle $\triangle BCD$. Note that the conditions $AC = CE$ and $BC = CD$ imply $(w, c) = (0, 0)$, and hence, $\widetilde{S}$ is contained in the ambient space of T_3.

It is easy to see that parametrization of $\widetilde{S}$ is given by (5.4) with $v = s$ and $c = 0$, i.e.,

$$\begin{cases} x = s + 1/s. \\ y = \dfrac{s^2\,t^2 + a\,s\,t^2 - s\,t - a\,t + 1}{s\,(s\,t + a\,t - 1)}, \\ z = \dfrac{bs^2}{bs-1}, \\ w = 0. \end{cases} \tag{5.26}$$

Since x and z depend only on the variable s, by eliminating the variable s in the expressions of x and z, we obtain the surface

(5.27)

$$\begin{aligned} S &:= \{(x,y,z,w) \in \mathbb{R}^4;\, b\,x\,z^2 - b^2\,z^2 - z^2 - b\,x^2\,z + b^2\,x\,z + 2\,b\,z - b^2 = 0,\;\; w = 0\} \\ &\simeq \{(x,y,z) \in \mathbb{R}^3;\, b\,x\,z^2 - b^2\,z^2 - z^2 - b\,x^2\,z + b^2\,x\,z + 2\,b\,z - b^2 = 0\}. \end{aligned}$$

For a fixed s (as a consequence, x and z are also fixed), we regard the second equation in (5.26) as the following algebraic equation of t.

$$(-s^2 - as)t^2 + (s^2y + asy + s + a)t - sy - 1 = 0. \tag{5.28}$$

The discriminant of the above equation is

$$D := (s + a)\;(s\,y + 1)\;\left(s^2\,y + a\,s\,y - 3\,s + a\right), \tag{5.29}$$

and hence, the equation (5.28) has a real root if and only if

$$y \geq \frac{3s - a}{s^2 + as}. \tag{5.30}$$

As a matter of fact, we have

$$\frac{3s-a}{s^2+as} \geq -1/s$$

because

$$\frac{3s-a}{s^2+as}+1/s=4/(s+a)>0$$

hold by the future light cone condition $st+at-1>0$. We also have $y>-1/s$ because of

$$y+1/s=\frac{(s+a)\,t^2}{s\,t+a\,t-1}>0.$$

Hence we conclude that $D\geq 0$ if and only if (5.30) holds.

As a consequence of the above observation, when $y=\dfrac{3s-a}{s^2+as}$, we have the double root $t=2/(s+a)$ in (5.28). In addition, it follows from (5.2) that $t=2/(s+a)$ is equivalent to $BC=BD$. Hence, by noticing the fact

$$(x,y,z,w)\in\widetilde{S}\iff(x,y,z,w)\in S\text{ and }y\geq\frac{3s-a}{s^2+as},$$

we know that the boundary $\partial\widetilde{S}$ of $\widetilde{S}$ coincides with the non-surface component of T_3 because $AC=CE$ and $BC=CD=BD$ hold on $\partial\widetilde{S}$. Furthermore, if $y>\dfrac{3s-a}{s^2+as}$, (5.28) has two distinct roots t_1 and t_2 which give the same x, y and z by (5.26). Therefore the parametrization (5.26) doubly covers the subset $\widetilde{S}\setminus\partial\widetilde{S}$, which entails that $\widetilde{S}\setminus\partial\widetilde{S}$ are cusps of $\widetilde{T}_3$. Summing up, we have obtained the following theorem.

Theorem 5.2. *The restriction of $[L^{\otimes}(\widetilde{T}_3)]$ to the ambient space of T_3 coincides with the union of the surface and non-surface components of T_3 and $\widetilde{S}$. Furthermore, the non-surface component of T_3 is obtained by the restriction of pinch points (I.a) of $\widetilde{T}_3$ to the ambient space of T_3, and $\widetilde{S}$ is the union of cusps of $\widetilde{T}_3$ emanating from the non-surface component of T_3.*

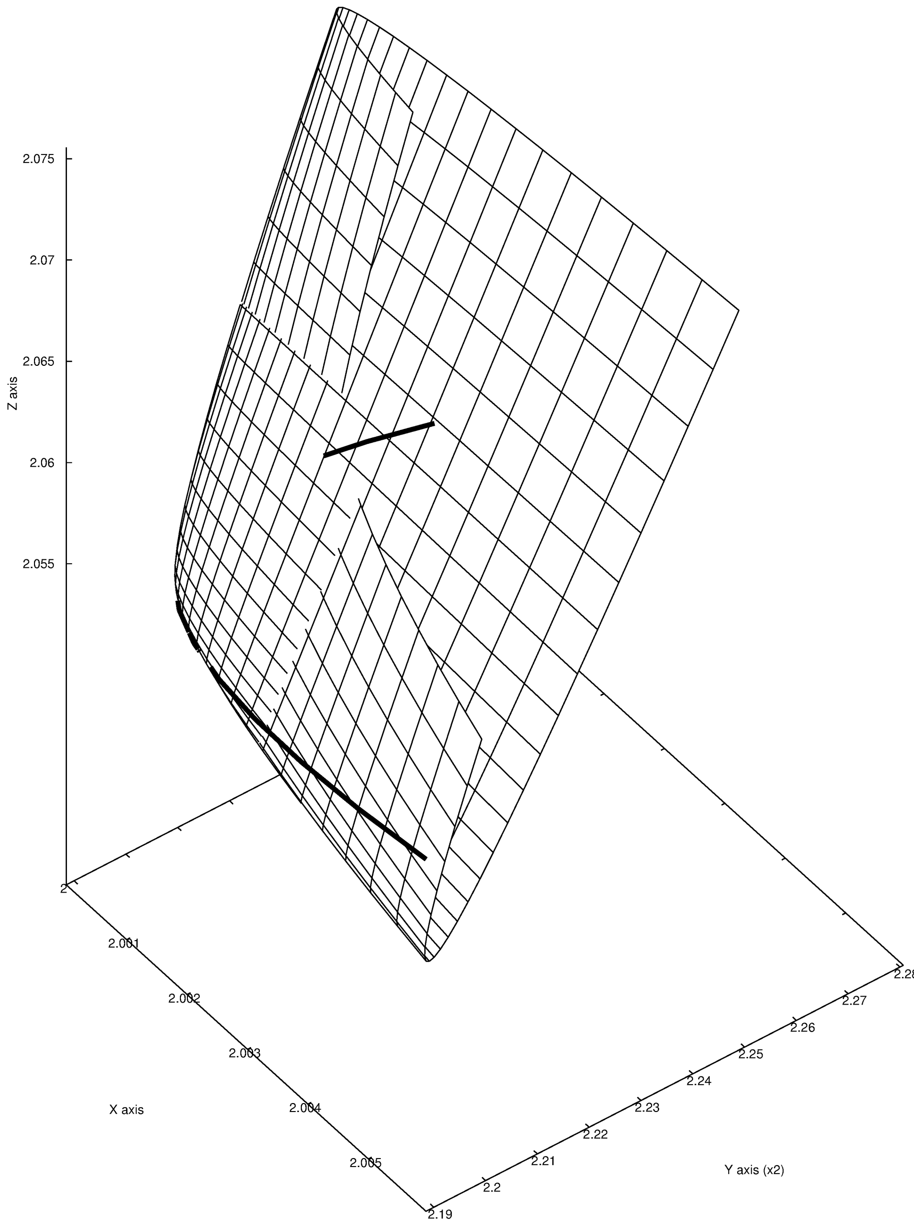

Figure 21. The surface and non-surface components of $L^{\otimes}(T_3)$ with $(a, b) = (0.9,\ 1.95)$, viewed from $(61, 53)$.

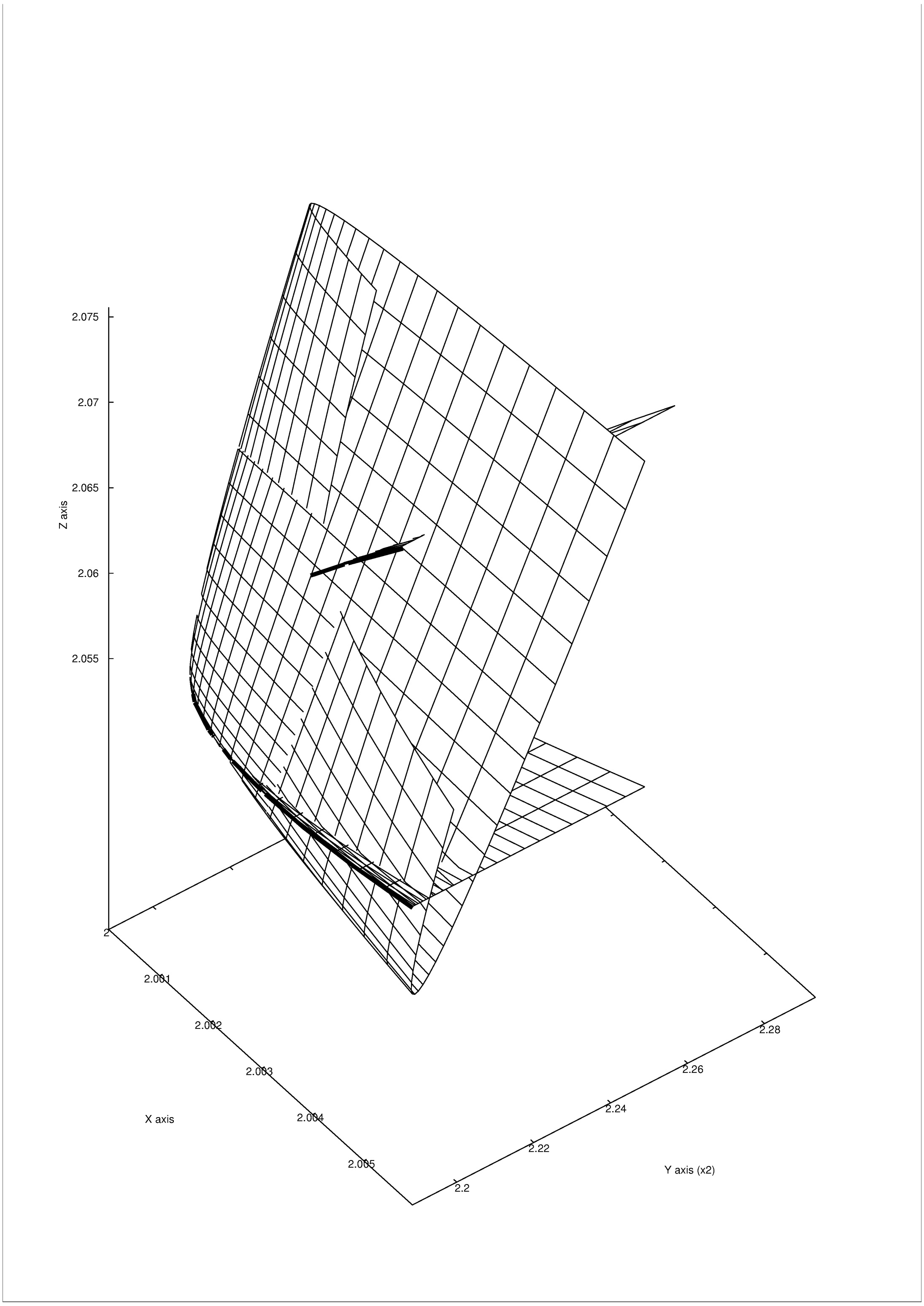

Figure 22. $L^{\otimes}(\widetilde{T}_3)$ with $(w, a, b, c) = (0,\ 0.9,\ 1.95,\ 0)$, viewed from $(61, 53)$.

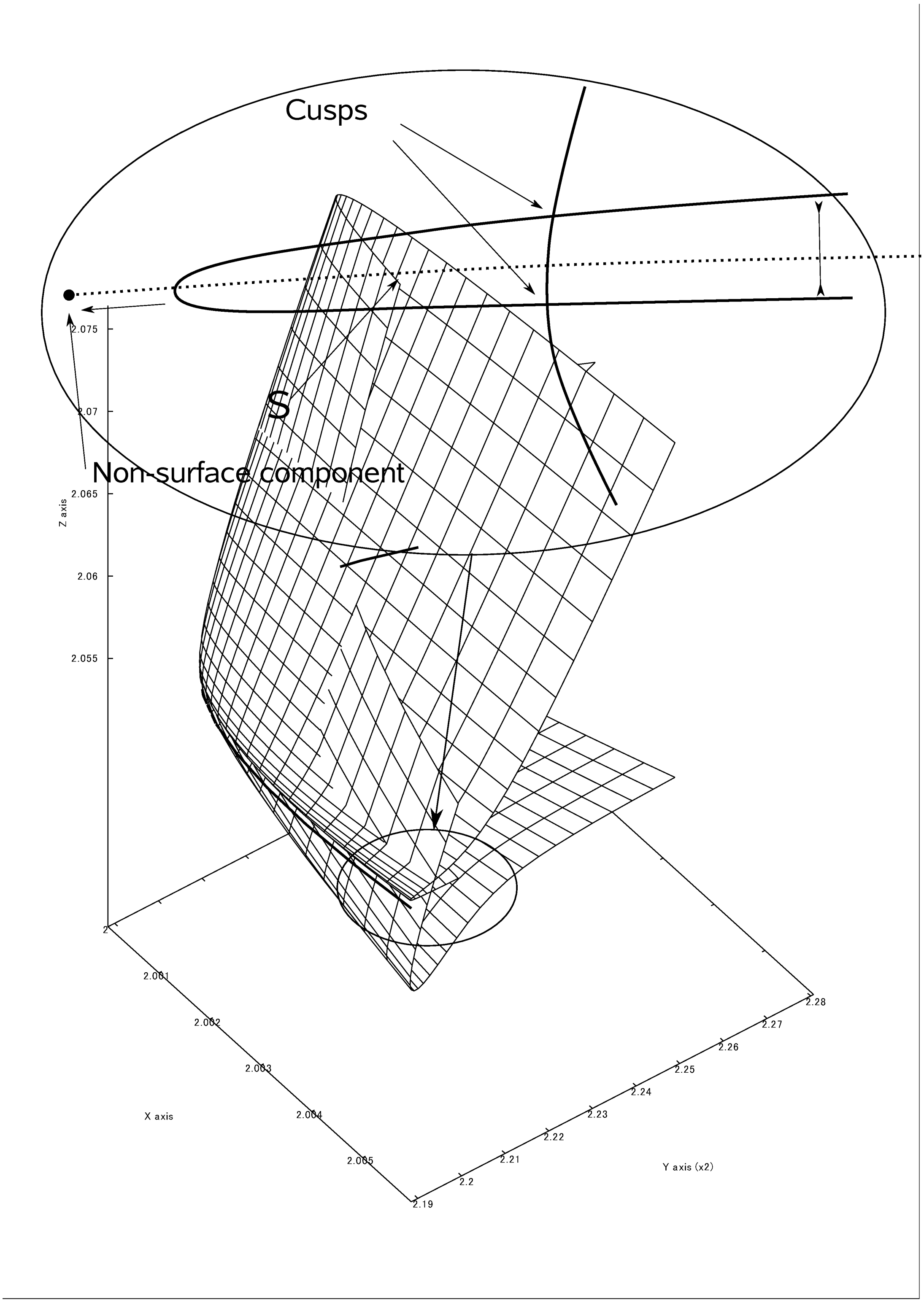

Figure 23. $L^{\otimes}(\widetilde{T}_3)$ with $(w,\, a,\, b, c) = (0.0003,\, 0.9,\, 1.95,\, 0)$ viewed from $(61, 53)$.

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