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Some remarks on integrability and normal forms for vector fields

By

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Abstract

We consider normalization of a holomorphic vector field and discuss relationship between existence of a convergent normalization and Liouville integrabilty of the given vector field. In particular, an alternative proof is given for the results by Zung and Stolovitch in the framework of vector fields near a center-type equilibrium point.

§ 1. Introduction

Normal form theory is a basic tool for studying solutions near an equilibrium point of a vector field. It is a classical topic developed by Poincaré. The simplest normal form is achieved by linearization of a holomorphic vector field. Actually, the linearization is always possible formally under the non-resonance condition (see (2.2) below), while analytic linearization is obtained by adding appropriate conditions on the linear term of a given vector field. Those conditions are so-called Poincaré condition, Siegel condition and Brjuno condition. In the Poincaré case, one can prove convergence of the normalizing transformation by using the majorant method. On the other hand, Siegel's and Brjuno's conditions, which are number theoretical conditions on those eigenvalues of the linear term, were invented to overcome the small divisor difficulty which arises in proving convergence of the normalizing transformation. As is well known, the latter two conditions gave rise to analytical techniques having wide variety of applications, such as rapidly convergent iteration method in KAM theory and renormalization method in the theory of (complex) dynamical systems.

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This normalization problem is a typical example of *conjugacy problem* which is one of the main topics in the modern theory of dynamical systems. However, from the viewpoint of real dynamical systems, the non-resonance condition makes strong restrictions on the study of behavior of solutions. For example, the existence of a purely imaginary eigenvalue violates the non-resonance condition because purely imaginary eigenvalues occur in (plus-minus) pairs for real systems. In such cases, the normal form becomes a non-linear vector field possessing the so-called "resonant terms" and is called *Poincaré-Dulac normal form.* The Birkhoff normal form for Hamiltonian systems is a kind of such normal forms.

In this note, we study the problem on existence of a convergent Poincaré-Dulac normalization. In this case, the situation is different from linearization case, and the existence of a convergent normalization is closely related to integrability of a given vector field. Here the integrability implies the existence of additional vector fields commuting with the given one and that of integrals of those vector fields. This is called *complete integrability* or *integrability in the sense of Liouville.* The relationship between existence of a convergent normalization and complete integrability was studied by Stolovitch [9, 10] and Zung [13, 14]. They obtained their results by generalizing the result for Hamiltonian case [3] as well as generalizing the notion of Liouville integrability following Bogoyavlesnki [1].

More recently, the author studied the relationship between existence of a convergent Birkhoff normalization and superintegrablity of Hamiltonian systems [4]. Here the superinterability means that the system possesses more than n integrals, where n is the number of degrees of freedom. The result of [4] shows that one can find a convergent Birkhoff transformation if the number of integrals excessing the degree of freedom is equal to the resonance degree and that the Birkhoff normal form turns out to be functions of smaller number of variables. The aim of this note is to show that the results by Stolovitch and Zung mentioned above can be proved in special cases by extending the idea of the proof of [4]. After giving preliminaries about normal forms in §2, we will state the result (Theorem 3.2) in §3 and give a sketch of its proof in §4 and §5. Throughout this note, we consider only complex normal form although it is possible to formulate the result for real analytic vector fields. It is because the description of real normal forms would be complicated, depending on the type of the equilibrium point. The analyticity assumption is crucial for our arguments.

§ 2. Poincaré-Dulac normal form

Before stating the result, we give preliminaries about normal forms. Let X be an m-dimensional holomorphic vector field in a neighborhood of the origin $z = 0 \in \mathbb{C}^m$. It

can be written as

$$X(z) = \sum_{i=1}^{m} X_i(z)\frac{\partial}{\partial z_i}, \qquad X_i(z) = X_i^0(z) + X_i^1(z) + \cdots + X_i^d(z) + \cdots .$$

Here $X_i(z)$ are holomorphic functions of $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ satisfying $X_i(0) = 0$, and $X_i^d(z)$ $(d = 0, 1, \ldots)$ denotes a homogenous polynomial of degree $d+1$ in z. The vector field X can be identified with a system of differential equations, which we write as follows:

$$\dot{z} = X(z) = X^0(z) + X^1(z) + X^d(z) + \cdots , \quad X^0(z) \not\equiv 0, \tag{2.1}$$

where $X^d(z)$ denotes the vector function whose components are homogeneous polynomials $X_1^d(z), \ldots, X_m^d(z)$.

Suppose that the linear part $X^0(z)$ is in diagonal form, namely

$$X^0(z) = \Lambda z, \qquad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m).$$

The vector field X is said to be in *Poincaré-Dulac normal form* or simply in *normal form* if it commutes with its linear part, namely

$$[X, X^0] = 0 \quad \text{for} \quad X^0 = \sum_{i=1}^{m} \lambda_i z_i \frac{\partial}{\partial z_i} .$$

Here the bracket $[\cdot, \cdot]$ denotes the Lie bracket defined by $[X, Y] = XY - YX$ as a differential operator. If we consider the system of differential equations (2.1) determined by X, it can also be written as

$$[X(z), Y(z)] = DX(z)Y(z) - DY(z)X(z).$$

We write i-th component $X_i(z)$ of X as

$$X_i(z) = \sum_{\alpha \in \mathbb{Z}_+^m} c_{i\alpha} z^\alpha \quad (c_{i\alpha} \in \mathbb{C}),$$

where $\alpha = (\alpha_1, \ldots, \alpha_m)$, $z^\alpha = z_1^{\alpha_1} \cdots z_m^{\alpha_m}$ (multi-index notation) and $\mathbb{Z}_+$ is the set of nonnegative integers. Then it is easy to see

Proposition 2.1. *The vector field X is in normal form if and only if*

$$c_{i\alpha} = 0 \quad \textit{for all } \alpha \in \mathbb{Z}_+^m \textit{ such that } \quad \langle \alpha, \lambda \rangle - \lambda_i \neq 0 \quad (i = 1, \ldots, m),$$

where $\langle \alpha, \lambda \rangle = \sum_{j=1}^{m} \alpha_j \lambda_j$.

The equilibrium point $z = 0$ is called *non-resonant* if the following condition holds:

$$\langle \alpha, \lambda \rangle - \lambda_i \neq 0 \quad \text{for all} \quad \alpha \neq \boldsymbol{e}_i \qquad (i = 1, \dots, m), \tag{2.2}$$

where $\boldsymbol{e}_i$ denotes the unit vector whose i-th component is 1 and others are all zero. In this case, the normal form reduces to the linear vector field

$$\dot{z}_i = \lambda_i z_i \qquad (i = 1, \dots, m).$$

More generally, the normal form X can be written as

$$X(z) = \sum_{i=1}^{m} \Big(\sum_{\langle \alpha, \lambda \rangle = \lambda_i} c_{i\alpha} z^\alpha \Big) \frac{\partial}{\partial z_i}\,.$$

As mentioned in the introduction, there exists a formal transformation $\varphi\colon z \mapsto z + O(|z|^2)$ which takes X into normal form. When does there exist a holomorphic (convergent) normalizing transformation φ ? It is a difficult question when the non-resonance condition is violated. For example, Hamiltonian case deals with $2n$-dimensional vector fields and their linear parts have eigenvalues in pairs $\pm\lambda_1, \dots, \pm\lambda_n$ and hence the above non-resonance condition is always violated. The normalization for Hamiltonian case is called *Birkhoff normalization.* Siegel [7, 8] showed that the existence of a convergent Birkhoff normalization depends sensitively on higher order terms of the Hamiltonian. In view of this fact, it would be natural to pursue more geometric approach to the problem of convergent normalization.

Let us give more precise description of Birkhoff normal form. The linear part of a Hamiltonian vector field near an equilibrium point has eigenvalues in pairs $\pm\lambda_1, \dots, \pm\lambda_n$, and is in diagonal form when the corresponding Hamiltonian has the form

$$H(z) = \sum_{i=1}^{n} \lambda_i x_i y_i + O(|z|^3), \qquad z = (x, y) \in \mathbb{C}^n \times \mathbb{C}^n\,.$$

The corresponding Hamiltonian vector field X_H is given by

$$\dot{x}_i = H_{y_i} = \lambda_i x_i + \cdots\,, \quad \dot{y}_i = -H_{x_i} = -\lambda_i y_i + \cdots \qquad (i = 1, \dots, n).$$

The Hamiltonian $H = H(z)$ is said to be in *Birkhoff normal form* if it commutes with its quadratic part, namely

$$\{H, H^0\} = 0, \qquad H^0 = \sum_{i=1}^{n} \lambda_i x_i y_i\,.$$

Here $\{\cdot, \cdot\}$ denotes the Poisson bracket between two functions. We note that $[X_G, X_H] = X_{\{G,H\}}$ for any functions G and H. Assume that

$$\langle \alpha, \lambda \rangle \neq 0 \quad \text{for all} \quad \alpha \in \mathbb{Z}^n \setminus \{0\},$$

where $\lambda = (\lambda_1, \dots, \lambda_n)$. This is the non-resonance condition in Hamiltonian case. It is well known that, under this non-resonance condition, there exists a formal symplectic transformation $\varphi\colon z \mapsto z + O(|z|^2)$ which takes the Hamiltonian H into Birkhoff normal form. In this case, the Birkhoff normal form $H \circ \varphi$ becomes a function of n variables $\omega_i = x_i y_i$ only. Therefore, if this transformation φ is convergent, transformed system is written as

$$\dot{x}_i = h_{\omega_i} x_i, \quad \dot{y}_i = -h_{\omega_i} y_i \qquad (i = 1, \dots, n),$$

which can be solved explicitly because ω_i and hence h_{ω_i} are constant along solutions. Here ω_i are Poisson commuting integrals of the transformed system and therefore the existence of a convergent Birkhoff transformation φ implies integrability of the original Hamiltonian system. Moreover, the converse holds true. Namely, there exists a holomorphic Birkhoff transformation φ near a non-resonant equilibrium point if the system X_H admits n holomorphic functionally independent integrals. Here n functions are called functionally independent if their derivatives are linearly independent on an open dense subset of a neighborhood of the origin. This was proved in general context in [3] by removing additional assumptions by Rüssmann [6] and Vey [12]. It is not assumed here that those integrals of X_H are Poisson commuting, however, they are necessarily so since any integral of X_H turns out to be a function of $\omega_1, \dots, \omega_n$.

This result has been generalized in several directions. In particular, the non-resonance condition was relaxed by Nguyen T. Zung [14] and its result was extended to general vector fields [13]. In order to state the results for general vector fields, we introduce the following definition.

Definition 2.2. An m-dimensional holomorphic vector field X is said to be *holomorphically Liouville integrable* if there exists an integer k such that $1 \le k \le m$ and the following hold:

[C.1] There exist k holomorphic vector fields $X_1 (= X), X_2, \dots, X_k$ such that $X_1, \dots, X_k$ are commuting with each other and are linearly independent on an open dense subset of a neighborhood of the origin.

[C.2] There exist $m - k$ functionally independent holomorphic integrals $G_1, \dots, G_{m-k}$ for those vector fields $X_1, \dots, X_k$.

This is a natural generalization of Liouville integrability of a Hamiltonian vector field (see [1]). If the vector fields $X_1, \dots, X_k$ are complete on a connected component of a regular level set defined by $G_i(z) = \text{const.}$, their flows give rise to a transitive $\mathbb{R}^k$-action on it and, in particular, a compact component of a regular level set turns out to be a k-dimensional torus on which the flows of $X_1, \dots, X_k$ are conditionally periodic. Therefore it would be natural to expect that, if the given system is Liouville integrable,

there exist a holomorphic normalizing transformation such that the transformed system can be solved explicitly. For this question, Zung proved the following result without any additional conditions.

Theorem 2.3 ([13]). *If X is a holomorphically Liouville integrable vector field, then there exists a holomorphic transformation φ which takes X into normal form.*

The Liouville integrability for a Hamiltonian system X_H corresponds to the case with $m = 2n$, $k = n$ and $X_i = X_{G_i}$ in Definition 2.2. In this case, the above φ can be taken as a symplectic transformation such that $H \circ \varphi$ is in Birkhoff normal form ([14]).

These are beautiful general results in the sense that there is no need to assume any restrictions on resonances. However, the definition of (Poincaré-Dulac or Birkhoff) normal forms means only that the vector field in normal form commutes with its linear part. We note that, in three or more degrees of freedom cases, Birkhoff normal forms in resonance cases may admit chaotic behaviors of solutions (c.f. [2]). From this viewpoint, it is not clear what restrictions are imposed on the normal forms in resonance cases by the Liouville integrability. In particular, it is not clear at all whether the normal form can be solved explicitly. Our result stated in the next section will answer this question by considering a restricted class of resonances.

§ 3. Statement of the result

We consider a restricted class of vector fields, that is, a $2n$-dimensional holomorphic vector field X near an equilibrium point $z = 0$ such that its linear part has eigenvalues in pairs $\lambda_{n+j} = -\lambda_j$ $(j = 1, \dots, n)$. Hamiltonian or reversible vector fields satisfy this condition for any equilibrium point. For general vector fields, we say that the equilibrium point $z = 0$ is of *center-type* if this condition holds. This is named after the fact that, if X is real analytic and λ_j are all purely imaginary, the origin $z = 0$ is an elliptic equilibrium point which is called a *center* especially for the case $n = 1$.

We assume that the linear part L is in diagonal form and hence

$$L = \sum_{j=1}^{n} \lambda_j \left(x_j \frac{\partial}{\partial x_j} - y_j \frac{\partial}{\partial y_j} \right). \tag{3.1}$$

We introduce the set $\mathcal{R}$ called *resonance lattice* as follows:

$$\mathcal{R} = \{k \in \mathbb{Z}^n \mid \langle k, \lambda \rangle = 0\},$$

where $k = (k_1, \dots, k_n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$. Since k runs over the ring $\mathbb{Z}^n$, instead of $\mathbb{Z}_+^m$, $\mathcal{R}$ is a discrete subgroup of $\mathbb{Z}^n$.

Definition 3.1. The equilibrium point $z = 0$ is said to be *of resonance degree* q if $\dim_{\mathbb{Z}} \mathcal{R} = q$.

We note that $1 \le q \le n-1$. Let $\rho^{(1)}, \dots, \rho^{(q)} \in \mathcal{R} \subset \mathbb{Z}^n$ be generators of the q-dimensional resonance lattice $\mathcal{R}$. Then, there exist $n-q$ linearly independent vectors $\rho^{(q+1)}, \dots, \rho^{(n)} \in \mathbb{Z}^n$ such that

$$\langle \rho^{(i)}, \rho^{(j)} \rangle = 0 \qquad \text{for} \;\; i = 1, \dots, q, \;\; j = q+1, \dots, n\,.$$

Let $\widehat{\mathcal{R}} = \mathrm{span}_{\mathbb{C}}(\rho^{(1)}, \dots, \rho^{(q)})$, namely we denote by $\widehat{\mathcal{R}}$ the complex vector space spanned by $\rho^{(1)}, \dots, \rho^{(q)}$. Then $\rho^{(q+1)}, \dots, \rho^{(n)} \in \mathbb{Z}^n$ constitute the basis of its orthogonal compliment $\widehat{\mathcal{R}}^{\perp}$ with respect to the Hermitian inner product of $\mathbb{C}^n$.

We define the following monomials in the coordinates $x_1, \dots, x_n, y_1, \dots, y_n$:

$$\begin{cases} \omega_k = x_k y_k \qquad (k = 1, \dots, n), \\ \omega_{n+j} = x^{\rho_+^{(j)}} y^{\rho_-^{(j)}}, \quad \rho^{(j)} = \rho_+^{(j)} - \rho_-^{(j)} \qquad (j = 1, \dots, q). \end{cases}$$

Here we used the multi-index notation and divided the vector $\rho^{(j)}$ with entries $\rho_k^{(j)}$ $(k = 1, \dots, n)$ into the positive part $\rho_+^{(j)}$ and the negative part $\rho_-^{(j)}$. Namely, $\rho_+^{(j)}$ and $\rho_-^{(j)}$ are vectors such that one of the pair of components ${\rho_+}_k^{(j)}$, ${\rho_-}_k^{(j)}$ is zero and the other is a nonnegative integer.

In the statement of the result below, we consider more general vector fields which do not necessarily begin with linear parts. In such cases, we also denote by X^0 the lowest order part of a vector field X and write X in expansions of vector-valued homogeneous polynomials as in (2.1). Our result is stated as follows:

Theorem 3.2. *Let X be a $2n$-dimensional holomorphic vector field near a center-type equilibrium point of resonance degree q. Assume that*

[A.1] *There exist $n-q$ holomorphic vector fields $X_1 (= X), X_2, \dots, X_{n-q}$ such that*

(1) $[X, X_j] = 0 \qquad (j = 1, \dots, n-q)$;

(2) *The lowest order parts $X_1^0, \dots, X_{n-q}^0$ are linearly independent on an open dense subset of the common domain of definition.*

[A.2] *There exist $n+q$ functionally independent and holomorphic integrals $G_1, \dots, G_{n+q}$ for $X_1, \dots, X_{n-q}$.*

Then there exists a holomorphic transformation φ such that

$$\varphi^* X_i = \sum_{j=q+1}^{n} P_{ij}(\omega) Z_j \quad (i = 1, \dots, n-q), \quad Z_j = \sum_{k=1}^{n} \rho_k^{(j)} \left(x_k \frac{\partial}{\partial x_k} - y_k \frac{\partial}{\partial y_k} \right), \tag{3.2}$$

where P_{ij} are convergent power series in $x_1, \dots, x_n, y_1, \dots, y_n$ and also can be considered as Laurent series in $\omega_1, \dots, \omega_{n+q}$. Moreover, $\omega_1, \dots, \omega_{n+q}$ are integrals of $\varphi^* X_1, \dots, \varphi^* X_{n-q}$, and any integral of those vector fields can be written as functions (Laurent series) of $n+q$ variables $\omega_1, \dots, \omega_{n+q}$.

Remarks. (i) The transformed system $\varphi^* X_i$ can be written as

$$\dot{x}_k = \Big(\sum_{j=q+1}^{n} P_{ij}(\omega) \rho_k^{(j)} \Big) x_k, \quad \dot{y}_k = -\Big(\sum_{j=q+1}^{n} P_{ij}(\omega) \rho_k^{(j)} \Big) y_k \qquad (k = 1, \dots, n).$$

This can be solved explicitly because $\omega_1, \dots, \omega_{n+q}$ turn out to be integrals of $\varphi^* X_i$.

(ii) In the assumption (1) of [A.1], the commuting relations among $X_2, \dots, X_n$ are not assumed. However, they follow from the special form (3.2) of those vector fields $X_1, \dots, X_{n-q}$. In fact, we have

$$[\varphi^* X_i, \varphi^* X_k] = \sum_{j=q+1}^{n} P_{ij}(\omega) [Z_j, \varphi^* X_k] + \sum_{j=q+1}^{n} \big(\varphi^* X_k\big) P_{ij}(\omega) Z_j = 0$$

since $P_{ij}(\omega)$ is an integral of $\varphi^* X_k$ (the assertion of the theorem) and a vector field in normal form commutes with $Z_{q+1}, \dots, Z_n$. In this sense, the assumption of this theorem implies that the vector field X is holomorphically Liouville integrable.

This theorem is closely related to the results by Stolovitch [9, 10] which are formulated using algebraic terminology (see also [11]). Actually the conclusion of Theorem 3.2 follows from the result of [10] provided that there exists a formal transformation φ satisfying (3.2). The arguments of the next section (§4) will show that the existence of a formal solution of (3.2) follows from the assumptions of Theorem 3.2, and the arguments of §5 give an alternative proof of Stolovitch's theorem in the framework of a special class of vector fields formulated above.

§ 4. Structure of simultaneous normalization

For the proof of Theorem 3.2, it will be crucial to use the structure of simultaneous normalization of vector fields $X_1, \dots, X_{n-q}$ as well as their integrals $G_1, \dots, G_{n+q}$. In this section, we discuss its formal aspects. In what follows, we consider a function to be a formal or convergent power series in $2n$ variables x_k, y_l and a vector field to be a vector whose components are such power series.

First, we give the definition of L-normal form for a vector field X and a function G, where L is a linear vector field of the form (3.1) which is generally different from the lowest order part of X.

Definition 4.1. (i) A vector field X is said to be in *L-normal form* if the identity $[X, L] = 0$ holds. It is also said to be in *L-normal form up to order $s + d - 1$* with $s = \deg X^0$ if we have

$$[X, L] = O(|z|^{s+d}).$$

(ii) A function G is in *L-normal form* if the identity $LG = 0$ holds. It is also said to be in *L-normal form up to order $t + d - 1$* ($t = \deg G^0$) if we have

$$LG = O(|z|^{t+d})).$$

The structure of simultaneous normalization is described as follows.

Lemma 4.2. *Assume that a vector field X with $X^0 = L$ is in L-normal form up to order $d(= s + d - 1)$. Then the following holds:*

(i) *If Y is another vector field satisfying $[X, Y] = 0$, then Y is in L-normal form up to order* $\deg Y^0 + d - 1$.

(ii) *If G is an integral of X, then G is in L-normal form up to order* $\deg G^0 + d - 1$.

This fact is well known and will play the key role in the proof of Theorem 3.2. To proceed, we need the following characterization of L-normal forms as power series.

Proposition 4.3. (i) *A function $G(z)$ is in L-normal form if and only if it is written as*

$$G(z) = \sum_{\alpha-\beta\in\mathcal{R}} c_{\alpha\beta} x^\alpha y^\beta.$$

In this case, $G(z)$ can be considered as a function of $n+q$ variables $\omega_1, \dots, \omega_{n+q}$, namely Laurent series in $\omega_1, \dots, \omega_{n+q}$.

(ii) *A $2n$-dimensional vector field X with $X(0) = 0$ is in L-normal form if and only if it can be written as*

$$\dot{x}_i = X_i(z) = \sum_{\alpha-\beta-\boldsymbol{e}_i\in\mathcal{R}} c_{i,\alpha\beta} x^\alpha y^\beta, \qquad \dot{y}_i = X_{i+n}(z) = \sum_{\alpha-\beta+\boldsymbol{e}_i\in\mathcal{R}} c_{i+n,\alpha\beta} x^\alpha y^\beta,$$

where $c_{i,\alpha\beta}, c_{i+n,\alpha\beta} \in \mathbb{C}$. In this case, X can be also written as

$$\dot{x}_i = p_i(\omega) x_i, \quad \dot{y}_i = q_i(\omega) y_i \qquad (i = 1, \dots, n), \tag{4.1}$$

where $p_i(\omega)$ and $q_i(\omega)$ are Laurent series in x_k, y_l $(k, l = 1, \dots, n)$ admitting only simple poles x_i^{-1} and y_i^{-1} respectively such that they can be considered as functions of $n + q$ variables $\omega_1, \dots, \omega_{n+q}$ and are written as Laurent series in those variables.

This proposition can be proved by direct calculation. Series $p_i(\omega)$ and $q_i(\omega)$ in the representation (4.1) can be written as

$$p_i(\omega) = \sum_{\alpha-\beta\in\mathcal{R},\, \alpha\in\mathbb{Z}^n_{+,i},\, \beta\in\mathbb{Z}^n_+} d_{i,\alpha\beta} x^\alpha y^\beta, \qquad q_i(\omega) = \sum_{\alpha-\beta\in\mathcal{R},\, \alpha\in\mathbb{Z}^n_+,\, \beta\in\mathbb{Z}^n_{+,i}} d_{i+n,\alpha\beta} x^\alpha y^\beta,$$

where $d_{i,\alpha\beta}, d_{i+n,\alpha\beta} \in \mathbb{C}$, $\mathbb{Z}^n_+$ is the set of n-dimensional vectors consisting of nonnegative integers and

$$\mathbb{Z}_{+,i} = \{\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n \,|\, \alpha_i \geq -1,\ \alpha_j \geq 0 \ \ (j \neq i)\}.$$

Now we consider formal L-normal forms of vector fields and functions. It is well known that there exists a formal transformation $\varphi = id + Y$ which takes $X = X_1$ with $X^0 = L$ into L-normal form, where Y is a vector whose components are formal power series in z.

Suppose that X_1 is already in L-normal form. Then, by Lemma 4.2, additional vector fields $X_2, \dots, X_{n-q}$, as well as those integrals $G_1, \dots, G_{n+q}$ are also in L-normal form. Therefore X_i are written in the form

$$\dot{x}_j = p_{ij}(\omega) x_j, \quad \dot{y}_j = q_{ij}(\omega) y_j \qquad (j = 1, \dots, n).$$

Also, $G_1, \dots, G_{n+q}$ turn out to be functions of $\omega_1, \dots, \omega_{n+q}$ and their functional independence implies

$$\det \left(\frac{\partial G_i}{\partial \omega_j} \right)_{i,j=1,\dots,n+q} \not\equiv 0. \tag{4.2}$$

Hence the condition $X_i G_j = 0$ $(i = 1, \dots, n-q, j = 1, \dots, n+q)$ is reduced to

$$X_i \omega_j = 0 \qquad (i = 1, \dots, n-q\,;\ j = 1, \dots, n+q).$$

By using n relations $X_i\omega_1 = \cdots = X_i\omega_n = 0$, we have

$$p_{ij} + q_{ij} = 0 \qquad (i = 1, \dots, n-q,\ j = 1, \dots, n).$$

Since p_{ij} and q_{ij} admit only simple poles of the form x_j^{-1} and y_j^{-1} respectively, these identities imply that there is no pole in the Laurent expansions p_{ij} and q_{ij}. Moreover, by the relations $X_i\omega_{n+1} = \cdots = X_i\omega_{n+q} = 0$, we see that

$$p_i(\omega) := \begin{pmatrix} p_{i1}(\omega) \\ \vdots \\ p_{in}(\omega) \end{pmatrix} \in \widehat{\mathcal{R}}^\perp = \mathrm{span}_{\mathbb{C}} \Big(\rho^{(q+1)} \cdots \rho^{(n)} \Big).$$

Therefore, those vector fields X_i are in L-normal form and can be written as

$$X_i = \sum_{j=q+1}^{n} P_{ij}(\omega) Z_j, \qquad Z_j = \sum_{k=1}^{n} \rho_k^{(j)} \Big(x_k \frac{\partial}{\partial x_k} - y_k \frac{\partial}{\partial y_k} \Big),$$

where $P_{ij}(\omega)$ are formal power series in x_k, y_l.

§ 5. Sketch of the convergence proof

The main theorem (Theorem 3.2) can be proved by using an iteration method. The simultaneous normalization structure plays a crucial role also in obtaining good estimates for the iteration of mappings. One iteration step is described by the following

Lemma 5.1. *Let X be a holomorphic vector field with $X^0 = L$ and assume that it is in L-normal form up to order d. Then there exists a polynomial vector field*

$$Y = Y^{d+1} + Y^{d+2} + \cdots + Y^{2d} \quad \textit{with} \quad P_N Y = 0$$

such that $\varphi := id + Y$ takes X into L-normal form up to order $2d$. Here Y^{d+j} denotes the vector whose components are homogeneous polynomials of degree $d+j$ and $P_N Y$ denotes the normal form part of Y.

Let $\mathcal{V}$ be the algeba of all vector fields which have power series expansions at the origin. We define the linear map $\operatorname{ad} L \colon \mathcal{V} \to \mathcal{V}$ by

$$\operatorname{ad} L \colon \mathcal{V} \ni X \mapsto [X, L] \in \mathcal{V}$$

so that the condition $X \in \operatorname{Ker} \operatorname{ad} L$ is equivalent to that X is in L-normal form. Since $\mathcal{V} = \operatorname{Ker} \operatorname{ad} L \oplus \operatorname{Im} \operatorname{ad} L$, we can introduce the projection operators

$$P_N \colon \mathcal{V} \to \operatorname{Ker} \operatorname{ad} L, \qquad P_R \colon \mathcal{V} \to \operatorname{Im} \operatorname{ad} L,$$

where we have $P_R = id - P_N$.

By Lemma 4.2, the transformation $\varphi = id + Y$ takes $n-q$ vector fields X_i as well as $n+q$ functions G_i simultaneously into L-normal form up to some finite order, respectively. This implies that the Y satisfies $2n$ systems of equations simultaneously.

To be precise, let s_i and t_i be the degrees of the lowest order parts X_i^0 and G_i^0, respectively, and suppose that X_i and G_i are in L-normal form up to order $s_i + d - 1$ and $t_i + d - 1$, respectively. Then they can be written as follows:

$$\begin{aligned} X_i(z) &= X_{i_N}(z) + \widehat{X}_i(z), \quad \widehat{X}_i(z) = O(|z|^{s_i+d}), \\ G_i(z) &= G_{i_N}(z) + \widehat{G}_i(z), \quad \widehat{G}_i(z) = O(|z|^{t_i+d}), \end{aligned}$$

where X_{i_N} and G_{i_N} are vector fields and functions in L-normal form. Let $\varphi = id + Y$ be a transformation given in Lemma 5.1. Then X_i are taken into

$$\varphi^* X_i = X_{i_N}(z) + \widehat{X}_i(z) - [X_{i_N}(z), Y(z)] + O(|z|^{s_i+2d}) \qquad (i = 1, \dots, n-q)$$

Therefore, $\varphi^* X_i$ is in L-normal form up to order $s_i + 2d - 1$ if and only if

$$P_R \widehat{X}_i(z) - [X_{i_N}(z), Y(z)] = O(|z|^{s_i+2d}) \tag{5.1}$$

and hence

$$\varphi^* X_i(z) = X'_{i_N}(z) + \widehat{X}'_i(z)\,; \quad X'_{i_N} = X_{i_N} + P_N^{2d-1}\widehat{X}_i, \quad \widehat{X}'_i = O(|z|^{s_i+2d}),$$

where $P_N^{2d-1}\widehat{X}_i = P_N(\widehat{X}_i^d + \cdots + \widehat{X}_i^{2d-1})$.

Recall that the normal form part X_{i_N} can be written in the form

$$X_{i_N} = \sum_{j=q+1}^{n} P_{ij}(\omega) Z_j$$

Here we note that $P_{ij}(\omega)$ are polynomials in $z = (x, y)$ which can be written as Laurent polynomials in $\omega_1, \dots, \omega_{n+q}$. Also we note that

$$[P_{ij}(\omega) Z_j, Y] = P_{ij}(\omega)[Z_j, Y] + \Big(Y P_{ij}(\omega)\Big) Z_j$$

with

$$Y P_{ij}(\omega) = \sum_{k=1}^{n+q} \frac{\partial P_{ij}}{\partial \omega_k} Y \omega_k \quad (i = 1, \dots, n-q;\ j = q+1, \dots, n).$$

Then equation (5.1) can be written as

$$\sum_{j=q+1}^{n} P_{ij}(\omega)[Z_j, Y] = P_R \widehat{X}_i - \sum_{j=q+1}^{n} \left(\sum_{k=1}^{n+q} \frac{\partial P_{ij}}{\partial \omega_k} Y \omega_k \right) Z_j + O(|z|^{s_i+2d}) \tag{5.2}$$
$$(i = 1, \dots, n-q)$$

Let us introduce the $(n-q)$-dimensional vector function $P_i(\omega) = \big(P_{i\,q+1}(\omega), \dots, P_{i\,n}(\omega)\big)$ and write it in the form

$$P_i(\omega) = P_i^0(\omega) + P_i^1(\omega) + \cdots + P_i^{d-1}(\omega), \qquad P_i^0(\omega) \not\equiv 0,$$

where $P_i^l(\omega) = \big(P_{i\,q+1}^l(\omega), \dots, P_{i\,q+2}^l(\omega), \dots, P_{i\,n}^l(\omega)\big)$ is the vector of homogeneous polynomials in $z = (x, y)$ of degree $s_i + l - 1$ such that each $P_{ij}^l(\omega)$ can be considered as a Laurent polynomial of $\omega_1, \dots, \omega_{n+q}$. One can prove that

$$X_1^0, \dots, X_{n-q}^0 \text{ are independent} \Longleftrightarrow \det\big(P_{ij}^0(\omega)\big)_{\substack{i=1,\dots,n-q \\ j=q+1,\dots,n}} \not\equiv 0.$$

Next we consider the functions $\varphi^* G_i$. Suppose that G_i are in L-normal form up to order $t_i + d - 1$, namely

$$G_i(z) = G_{i_N}(z) + \widehat{G}_i(z), \quad \widehat{G}_i(z) = O(|z|^{t_i+d}) \qquad (i = 1, \dots, n+q).$$

Then $\varphi^* G_i$ can be written as

$$\varphi^* G_i(z) = G_{i_N}(z) + \widehat{G}_i(z) + \langle \nabla G_{i_N}(z), Y(z) \rangle + O(|z|^{t_i+2d}).$$

Therefore, $\varphi^* G_i$ is in L-normal form up to order $t_i + 2d - 1$ if and only if

$$P_R \widehat{G}_i + \langle \nabla G_{i_N}, Y \rangle = O(|z|^{t_i+2d}),$$

which can be written as

$$\sum_{j=1}^{n+q} \frac{\partial G_{i_N}}{\partial \omega_j} Y \omega_j = -P_R \widehat{G}_i(z) + O(|z|^{t_i+2d}) \qquad (i = 1, \dots, n+q). \tag{5.3}$$

We have thus proved that the polynomial vector field Y given in Lemma 5.1 satisfies equations (5.2) and (5.3). We make use of this fact to get good estimates of Y. For this purpose, we compare homogeneous parts of (5.2) and (5.3). The comparison of terms of order $s_i + l - 1$ in both sides of (5.2) gives

$$\sum_{j=q+1}^{n} P_{ij}^0(\omega)[Z_j, Y^l] = F_i^{l-1}(z) \qquad (i = 1, \dots, n-q), \tag{5.4}$$

where

$$F_i^{l-1}(z) = P_R \widehat{X}_i^{l-1} - \sum_{\nu=0}^{l-d-1} \sum_{j=q+1}^{n} \sum_{k=1}^{n+q} \left(\frac{\partial P_{ij}^\nu}{\partial \omega_k} Y^{l-\nu} \omega_k \right) Z_j - \sum_{\nu=1}^{l-d-1} \sum_{j=q+1}^{n} P_{ij}^\nu(\omega)[Z_j, Y^{l-\nu}].$$

Also, the comparison of terms of order $t_i + l - 1$ of (5.3) gives

$$\sum_{j=1}^{n+q} \frac{\partial G_i^0}{\partial \omega_j} Y^l \omega_j = K_i^{l-1}(z), \tag{5.5}$$

where

$$K_i^{l-1}(z) = -P_R \widehat{G}_i^{l-1} - \sum_{\nu=1}^{l-d-1} \sum_{j=1}^{n+q} \frac{\partial G_i^\nu}{\partial \omega_j} Y^{l-\nu} \omega_j.$$

In the above, the partial derivatives $\partial P_{ij}^\nu / \partial \omega_k$, $\partial G_i^0 / \partial \omega_j$, $\partial G_i^\nu / \partial \omega_j$ may have poles as functions of $z = (x, y)$. It would be an obstruction to deriving estimates of $[Z_j, Y^l]$ and $Y^l \omega_j$ from (5.4) and (5.5). To eliminate it, we multiply the both sides of (5.4) and (5.5) by

$$M(z) := \prod_{j=1}^{n+q} \omega_j.$$

It makes easy to derive the desired estimates since $\omega_k\big(\partial P^\nu_{ij}/\partial\omega_k\big)$, $\omega_j\big(\partial G^0_i/\partial\omega_j\big)$ and $\omega_j\big(\partial G^\nu_i/\partial\omega_j\big)$ are polynomials in $z=(x,y)$.

In what follows, we always consider the system of equations (5.4) and (5.5) multiplied by $M(z)$. We first assume that $Y^{d+1}\omega_j,\dots,Y^{l-1}\omega_j$ are known and consider (5.5) as a system of linear equations for $Y^l\omega_j$, then we can solve it under the condition

$$g(z):=\det(g_{ij}(z))_{i,j=1,\dots,n+q}\not\equiv 0\,;\qquad g_{ij}(z)=M(z)\frac{\partial G^0_i}{\partial\omega_j}\,. \tag{5.6}$$

Next, substituting those solutions $Y\omega_j$ into (5.4), we consider (5.4) recursively as linear equations for $[Z_j,Y^l]$ and can solve it under the condition

$$p(z):=\det\big(p_{ij}(z)\big)_{\substack{i=1,\dots,n-q\\ j=q+1,\dots,n}}\not\equiv 0\,;\qquad p_{ij}(z)=M(z)P^0_{ij}(\omega). \tag{5.7}$$

These conditions (5.6) and (5.7) hold under the assumption of Theorem 3.2. In particular, condition (5.6) does not follow directly from the assumption of functional independence of $G_1,\dots,G_{n+q}$. However, one can use Ziglin's lemma (see [3, Appendix]) to construct polynomials of $G_1,\dots,G_{n+q}$ so that the lowest order parts of the new $n+q$ integrals are functionally independent.

We use the Cramer's formula to solve the systems of linear equations (5.4) and (5.5). For example, (5.5) can be solved as follows:

$$Y^l\omega_i=\frac{h^l_i(z)}{g(z)}\qquad (i=1,\dots,n+q;\ l=d+1,\dots,2d) \tag{5.8}$$

with

$$h^l_i(z)=\det\begin{pmatrix} g_{11}(z) & \cdots & M(z)K^l_1(z) & \cdots & g_{1\,n+q}(z)\\ \vdots & \ddots & \vdots & \ddots & \vdots\\ g_{n+q\,1}(z) & \cdots & M(z)K^l_{n+q}(z) & \cdots & g_{n+q\,n+q}(z)\end{pmatrix}.$$

We note that the polynomial $h^l_i(z)$ in the above is actually divisible by $g(z)$. In the same way, one can get a similar formula for $[Z_j,Y^l]$.

By using formula (5.8), one can get a good estimate of $Y\omega_j$. To state it, for some positive constants $\delta_1,\dots,\delta_n$ chosen appropriately, let us consider the polydisk

$$\Omega_r:=\{(x,y)\in\mathbb{C}^{2n}\mid |x_i|<\delta_i r,\quad |y_i|<\delta_i r\quad (i=1,\dots,n)\}$$

and introduce the following norms for holomorphic functions on $\overline{\Omega}_r$:

$$\|f\|_{r,m}=\frac{\|f\|_r}{r^m},\quad \|f\|_r:=\max_{z\in\overline{\Omega}_r}|f(z)|.$$

Here the $\|f\|_{r,m}$ is bounded for functions whose power series vanish up to order m. Then, since $Y^l\omega_j$ are holomorphic functions of z, the maximum value $\|Y^l\omega_j\|_r$ is already attained on the set

$$\Delta_r:=\{(x,y)\in\mathbb{C}^{2n}\mid |x_i|=|y_i|=\delta_i r\quad (i=1,\dots,n)\},$$

which is a part of the boundary of Ω_r. This implies that $\|Y^l\omega_j\|_r$ can be estimated as follows:

$$\|Y^l\omega_i\|_r \le \frac{\|h_i^l\|_r}{\min_{z\in\Delta_r}|g(z)|}\,.$$

By using this idea, one can get the estimates of $Y\omega_j$ and $[Z_j, Y^l]$. To state them, for any power series f we denote by $\widetilde{f}$ the following majorant series of f:

$$\widetilde{f} := \sum |c_{\alpha\beta}|x^\alpha y^\beta \quad \text{for} \quad f = \sum c_{\alpha\beta}x^\alpha y^\beta.$$

We introduce the following quantities:

$$|\partial\widehat{g}|_r := \sum_{i,j=1}^{n+q} \left\| M(z)\frac{\partial\big(\widetilde{G}_{i_N} - \widetilde{G}_i^0\big)}{\partial\omega_j}\right\|_{r,m_{ij}},$$

$$|\widehat{G}|_r := \sum_{i=1}^{n+q} \|M(z)\big(\widetilde{G}_i - \widetilde{G}_{i_N}\big)\|_{r,t_i-2},$$

$$|\widehat{P}|_r := \sum_{i=1}^{n-q}\sum_{j=q+1}^{n}\sum_{k=1}^{n+q}\left\| M(z)\frac{\partial\widetilde{P_{ij}}}{\partial\omega_k}\right\|_{r,s_i-2}, \quad |\widehat{X}|_r := \sum_{i=1}^{n-q}\|\widetilde{X_i}\|_{r,s_i-1}.$$

Then we obtain

Lemma 5.2. (i) *If $c_1|\partial\widehat{g}|_r < \frac{1}{2}$ ($c_1 > 0$ constant), then*

$$\|Y\omega_j\|_r \le 4c_1|\widehat{G}|_r \qquad (j = 1, \dots, n+q).$$

(ii) *There exist positive constants c_2, c_3 independent of r such that Y satisfies the estimate*

$$\big\|\big[Z_j, Y\big]\big\|_r \le c_2\big(|\widehat{X}|_r + |\widehat{G}|_r\big) \qquad (j = q+1, \dots, n)$$

provided that $c_3|\widehat{P}|_r < \frac{1}{2}$.

By using the second estimates, we can derive the estimate of the form

$$\|\widetilde{Y}\|_r \le c_4\big(|\widehat{X}|_r + |\widehat{G}|_r\big).$$

This is the crucial step in the proof of the main theorem. We can show uniform convergence of the iteration of normalizing transformations in the so-called rapidly convergent iteration scheme. Its details will be published elsewhere [5] .

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