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Exact WKB analysis, cluster algebras and Fock-Goncharov coordinates

By

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Abstract

This is a short summary of the paper [25] which investigates a cluster algebraic structure in exact WKB analysis of second order linear ordinary differential equations. We also compare the result [25] to the work [18] of Gaiotto, Moore and Neitzke.

§ 1. Introduction

In [25] the author and T. Nakanishi establish the relationship between *exact WKB analysis* of Schrödinger equations and *cluster algebras*. The one of the purpose of this article is to give a short summary of the result of [25].

Exact WKB analysis is a method to study the WKB (Wentzel-Kramers-Brillouin) solutions of the Schrödinger equation using the Borel resummation. It is initiated by Voros ([32]), and developed by many people (see [11, 27] for example). On the other hand, cluster algebras were introduced by Fomin and Zelevinsky ([14]) to study the coordinate rings of certain algebraic varieties. Recently, it turns out that cluster algebras are related with several branches of mathematics and physics; for example, representation theories of quivers, hyperbolic geometry, integrable systems, Donaldson-Thomas invariants and their wall-crossing, supersymmetric field theory, and so on. One of the main result of [25] was to embed exact WKB analysis in the above list of relating topics to cluster algebras.

The dictionary of these two theories are summarized in Table 1. There are two kinds of *Voros symbols* (for paths β and cycles γ), and they play roles of two kinds of

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Exact WKB analysis	Cluster algebras
Stokes graph and its mutation	B-matrix and its mutation
Voros symbols e^{W_β}	cluster x-variables
$\exp(\eta \oint_\gamma \sqrt{Q(z)}dz)$	coefficients (cluster y-variables)
Voros symbols e^{V_γ}	cluster $\hat{y}$-variables
Stokes phenomenon	mutation of cluster variables
identity of Stokes automorphisms (e.g., $\mathfrak{S}_{\gamma_1}\mathfrak{S}_{\gamma_2}\mathfrak{S}_{\gamma_1}^{-1}\mathfrak{S}_{\gamma_1+\gamma_2}^{-1}\mathfrak{S}_{\gamma_2}^{-1} = 1$)	periodicity of cluster algebra (e.g., $\mu_1^{(+)}\mu_2^{(+)}\mu_1^{(-)}\mu_2^{(-)}\mu_1^{(-)} = (1 \leftrightarrow 2)$)

Table 1. Dictionary: Exact WKB analysis and cluster algebras

(x- and $\hat{y}$-) *cluster variables*. In particular, the mutation of these cluster variables are realized as the *Stokes phenomenon* for the Voros symbols (Theorem 3.1).

On the other hand, a relation between (usual) WKB analysis and cluster algebras has been found in the work [18] of Gaiotto, Moore and Neitzke, through the study of *wall-crossing* of *BPS states* in four-dimensional field theory. They constructed cluster $\hat{y}$-variables in terms of solutions of a certain differential equation (associated with the *Hitchin equation* [22]), as the *Fock-Goncharov coordinates* of the moduli space of flat $SL(2,\mathbb{C})$-connections. The second purpose of this article is to compare the result of [25] to [18].

Consequently, we show that our Voros symbols for cycles, corresponding to cluster $\hat{y}$-variables, have the same expression as the Fock-Goncharov coordinates (Theorem 4.1). Namely, the (Borel sum of) Voros symbols are expressed as a cross-ratio of Wronskians of (Borel sum of) WKB solutions. This gives another proof of a formula of the Voros symbols for closed cycles in Theorem 3.1; the cluster mutation is nothing but the *Plücker relation* for the Wronskians. Note that this fact was pointed in [18]. From this result, we can expect that our Voros symbols are obtained as *conformal limit* of the Fock-Goncharov coordinates of [18] (see [16]).

This article is organized as follows. In Section 2 we recall some notions in exact WKB analysis. The main theorem of [25] is explained in Section 3. Finally, we will give the Fock-Goncharov type description of Voros symbols in Section 4.

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§ 2. Preliminary for exact WKB analysis

Here we briefly recall some notions in exact WKB analysis which is relevant to this paper. See [27] for more details.

§ 2.1. WKB solutions and Stokes graphs

We consider a second order linear ordinary differential equation (*Schrödinger equation*) of the form

$$\left(\frac{d^2}{dz^2} - \eta^2 Q(z)\right)\psi(z,\eta) = 0. \tag{2.1}$$

Here $\eta > 0$ is a large parameter, which plays a role of the inverse of Planck constant $\hbar$. In this paper we assume that $Q(z)$ (called *potential*) is a polynomial in z. (See Remark 3.2 for cases that $Q(z)$ is meromorphic.) We also assume the following throughout this paper:

$$\text{(A)} : Q(z) \text{ has only } \textit{simple} \text{ zeros.}$$

Zeros of $Q(z)$ are called *turning points* of the equation (2.1).

The *WKB solutions* of (2.1) are formal series solutions of the form

$$\psi_{\pm}(z,\eta) := \frac{1}{\sqrt{S_{\rm odd}(z,\eta)}}\exp\left(\pm\int_{z_0}^{z} S_{\rm odd}(z',\eta)dz'\right). \tag{2.2}$$

Here z_0 is a generic point (which determines the normalization of (2.2)), and

$$S_{\rm odd}(z,\eta) = \sum_{m=0}^{\infty}\eta^{1-2m}S_{2m-1}(z) = \eta S_{-1}(z) + \eta^{-1}S_1(z) + \eta^{-3}S_3(z) + \cdots \tag{2.3}$$

is a formal (Laurent) series of η^{-1} which is defined as the odd-order part of a formal solution of the Riccati equation $S^2 + dS/dz = \eta^2 Q(z)$ (see [27, Section 2]). The coefficients of $S_{\rm odd}(z,\eta)$ are determined recursively, and become (meromorphic) functions on the Riemann surface $\mathcal{R}$ of $\sqrt{Q(z)}$. $\mathcal{R}$ is a double cover of $\mathbb{P}^1$ branching at zeros of $Q(z)$, and at ∞ if $Q(z)$ is an odd degree polynomial. In particular, the first few terms are given by

$$S_{-1}(z) = \sqrt{Q(z)}, \quad S_1(z) = \frac{1}{32 Q(z)^{5/2}}\left(4Q(z)\frac{d^2Q(z)}{dz^2} - 5\left(\frac{dQ(z)}{dz}\right)^2\right), \ldots \tag{2.4}$$

Although WKB solutions are divergent in general, they are *Borel summable* under certain assumptions. Here the Borel summability indicates a well-definedness of *Borel* *sums.* The Borel sums of the WKB solutions give analytic solutions of the Schrödinger equation (2.1), and their *asymptotic expansions* for $\eta \to +\infty$ recover the original WKB solutions. See [10] or [27, Section 1] for Borel resummation method.

The Borel summability can be read off from the Stokes graph.

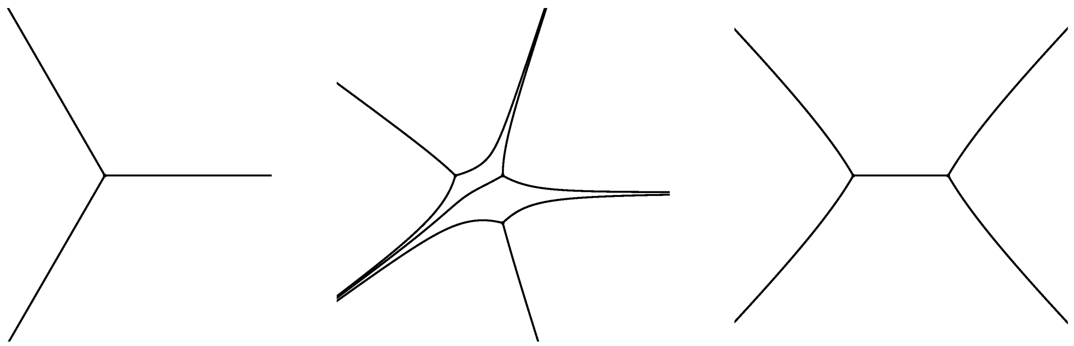
Definition 2.1 ([27, Definition 2.6])**.**

- A *Stokes curve* is a trajectory of the quadratic differential $Q(z)dz^2$ passing through a turning point. (A trajectory is a leaf of the foliation defined by $\mathrm{Im} \int^z \sqrt{Q(z')}dz' =$ constant.)
- The *Stokes graph* of (2.1) is the graph on $\mathbb{P}^1$ whose vertices are turning points and ∞, and whose edges are Stokes curves.
- A *Stokes segment* (or a *saddle trajectory*) is a Stokes curve connecting turning points.
- The Stokes graph is said to be *saddle-free* if there are no Stokes segment.

See Figure 1 for examples of Stokes graphs. Under the assumption (A), three Stokes curves emanate from each turning point. Figure 1 (c) depicts an example of a Stokes graph having a Stokes segment. It is known that, if the Stokes graph is saddle-free, then *Stokes regions* (faces of the Stokes graph) are one of

(i) *rectangular type* whose boundary contains two turning points, or

(ii) *digon type* (which appears near ∞) whose boundary has only one turning point ($d+2$ digon type Stokes regions appear near ∞, where d is the degree of polynomial $Q(z)$).



(a): $Q(z) = z$ (b): $Q(z) = z(z+1)(z+i)$ (c): $Q(z) = 1 - z^2$

Figure 1. Examples of Stokes graphs.

Theorem 2.2 ([28])**.** *If the Stokes graph is saddle-free, then the WKB solutions are Borel summable (as formal series of η^{-1}) when z lies on each Stokes region.*

The relationship between Borel resummed WKB solutions defined in adjacent Stokes regions are known as *Voros' connection formula* ([32, 27]). It will be used in Section 4.

§ 2.2. Voros symbols

Next we introduce an important notion, called Voros symbols. They are formal series defined by integrals of $S_{\rm odd}(z,\eta)dz$ along a homology class $\gamma \in H_1(\mathcal{R};\mathbb{Z})$ (we call it a *cycle*), or a relative homology class $\beta \in H_1(\mathcal{R},P;\mathbb{Z})$ (we call it a *path*). Here P is the inverse image of ∞ by the projection $\mathcal{R} \to \mathbb{P}^1$. Define

$$V_\gamma(\eta) := \oint_\gamma S_{\rm odd}(z,\eta)dz, \quad W_\beta(\eta) := \int_\beta \left(S_{\rm odd}(z,\eta) - \eta\sqrt{Q(z)}\right) dz \tag{2.5}$$

for a cycle γ and a path β. We can easily check that the formal series $W_\beta(\eta)$ is well-defined; that is, each coefficient of the formal series-valued 1-form $(S_{\rm odd}(z,\eta) - \eta\sqrt{Q(z)})dz$ is integrable at end-points of β.

Definition 2.3. The formal series $e^{V_\gamma(\eta)}$ and $e^{W_\beta(\eta)}$ are called *Voros symbols* for a cycle γ and a path β, respectively.

Note that $e^{V_\gamma(\eta)}$ is a formal series of η^{-1} with an exponential factor of the form

$$e^{V_\gamma(\eta)} = \exp\left(\eta \oint_\gamma \sqrt{Q(z)}dz\right)\left(1 + O(\eta^{-1})\right), \tag{2.6}$$

while $e^{W_\beta(\eta)}$ is a usual formal power series. Voros symbols are also divergent in general, and their Borel summability can also be read off from the topology of Stokes graph.

Theorem 2.4 ([28]). *The Voros symbol $e^{V_\gamma(\eta)}$ (resp., $e^{W_\beta(\eta)}$) for a cycle γ (resp., a path β), is Borel summable if γ (resp., β) never intersects with Stokes segments. In particular, the Voros symbols for any cycles and any paths are Borel summable if the Stokes graph is saddle-free.*

Remark 2.5. For the cases that $Q(z)$ is a rational function, the monodromy or connection matrices of (2.1) for the the Borel resummed WKB solutions are described by the Borel sums of Voros symbols (see [27, Section 3]).

§ 3. Cluster algebraic structure in exact WKB analysis

A *cluster algebra*, introduced in [14], is defined in terms of *seeds* and their *mutations* (see [14] for a precise definition of cluster algebras). In this section we review the result of [25]. That is, we associate a seed to the Schrödinger equation (2.1), and show that a deformation of the potential $Q(z)$ (by an S^1-action) causes a mutation for the seed. We will see that, cluster variables are realized by the (Borel resummed) Voros symbols, and the mutation is regarded as the Stokes phenomenon (for $\eta \to +\infty$) for the Voros symbols.

§ 3.1. Seeds in exact WKB analysis

Let us consider the Schrödinger equation (2.1) whose Stokes graph is saddle-free. In this subsection we define a seed from the Schrödinger equation. Here a seed is a triplet $(B, \mathbf{x}, \mathbf{y})$ consists of *skew-symmetric matrix* B of size $n \geq 1$, and two kinds of n-tuple of variables; called *cluster x-variables* and *cluster y-variables*, respectively. The integer n is the number of *rectangular* Stokes regions in the Stokes graph.

First, for each rectangular Stokes region D_i $(i = 1, \ldots, n)$, associate a path β_i (called the i-th *simple path*) and a cycle γ_i (called the i-th *simple cycle*) as indicated in Figure 2. That is, after taking an appropriate branch cut to determine the branch of $\sqrt{Q(z)}$, define them as follows:

- β_i is represented by a trajectory (regarded as a path on the Riemann surface $\mathcal{R}$) of the quadratic differential $Q(z)dz^2$ through a point in D_i, whose orientation is given so that $\mathrm{Re} \int^z \sqrt{Q(z)}dz$ increases along its positive direction.
- γ_i is represented by a closed cycle which encircles two turning points on the boundary of the Stokes region D_i, whose orientation is given by $\mathrm{Im} \oint_{\gamma_i} \sqrt{Q(z)}dz < 0$.

The orientations depend on the choice of the branch, but we can adopt any of them. This ambiguity doesn't matter when we define the Voros symbols for β_i and γ_i.

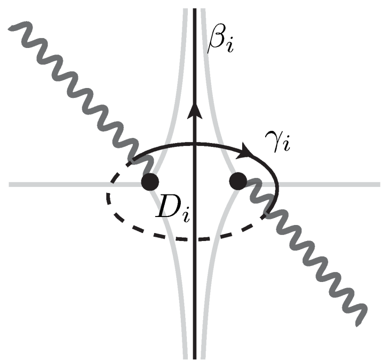


Figure 2. The path β_i and the cycle γ_i. The wiggly lines designate branch cuts. The solid (resp., dotted) part represents a part of path on the first (resp., the second) sheet of the Riemann surface $\mathcal{R}$.

As is shown in [18, 7], under the saddle-free assumption, the simple paths β_i's together with a similar paths in digon type Stokes regions near infinity (to be precise,

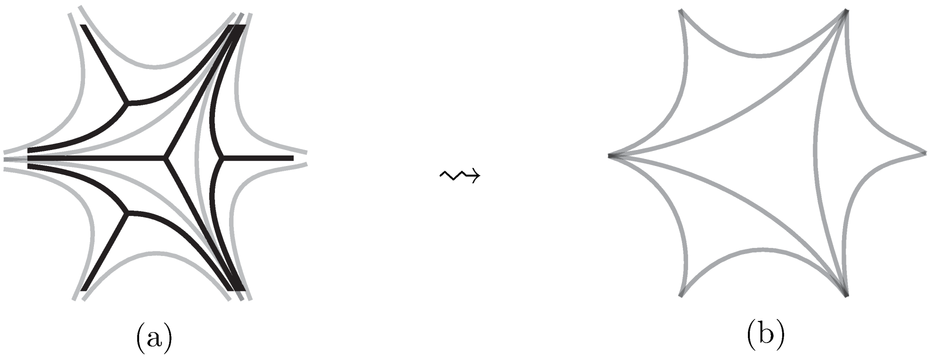


Figure 3. An example of a quartic polynomial $Q(z)$. (a): Thick lines depicts Stokes curves, while thin lines are paths. (b): The triangulation associated with $Q(z)$.

their images by projection $\mathcal{R} \to \mathbb{P}^1$) give a *triangulation* of a surface with boundary. The simple paths β_i's give diagonals of the triangulation, while the paths in digon-type Stokes regions give boundaries of the surface. For example, when $Q(z)$ is a quartic polynomial, we have a triangulation of a hexagon (see Figure 3 (b)).

Then, the skew symmetric matrix $B = (b_{ij})_{i,j=1}^n$ in our seed is defined from the triangulation as in [15]. Namely, we define

$$b_{ij} = \sum_{\Delta} b_{ij}^{\Delta} \tag{3.1}$$

where the summation is taken over all triangles in the triangulation, and

$$b_{ij}^{\Delta} = \begin{cases} 1 & \text{if } \beta_i \text{ and } \beta_j \text{ are different sides of } \Delta \text{, and the direction} \\ & \text{from } \beta_i \text{ to } \beta_j \text{ is counter-clockwise,} \\ -1 & \text{if } \beta_i \text{ and } \beta_j \text{ are different sides of } \Delta \text{, and the direction} \\ & \text{from } \beta_i \text{ to } \beta_j \text{ is clockwise,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Moreover, the cluster variables $\mathbf{x} = (x_i)_{i=1}^n$ and $\mathbf{y} = (y_i)_{i=1}^n$ in our seed are given by

$$x_i := \mathcal{S}[e^{W_{\beta_i}}], \quad y_i := \exp\left(\eta \oint_{\gamma_i} \sqrt{Q(z)}dz\right), \tag{3.3}$$

where $\mathcal{S}[e^{W_{\beta_i}}]$ is the Borel sum of the Voros symbols for β_i. Then, we can show that the Borel sum $\mathcal{S}[e^{V_{\gamma_i}}]$ of the Voros symbols for γ_i is given by

$$\mathcal{S}[e^{V_{\gamma_i}}] = y_i \prod_{j=1}^{n} x_j^{b_{ji}}. \tag{3.4}$$

This relation follows from a relation between simple cycles and simple paths (see [25, Proposition 6.27]). The right-hand side of the above equality is usually denoted by $\hat{y}_i$

(and called *cluster $\hat{y}$-variables*) in the theory of cluster algebras. Hence we set $\hat{y}_i := \mathcal{S}[e^{V_{\gamma_i}}]$ and $\hat{\mathbf{y}} := (\hat{y}_i)_{i=1}^n$.

§ 3.2. Mutation in exact WKB analysis

Mutation of cluster variables is a certain birational transformation. To see the mutation in the exact WKB analysis, we need to analyze the situation that a Stokes segment exists. For the purpose, let us consider the *S^1-family* of the potentials of the Schrödinger equation:

$$Q^{(\theta)}(z) := e^{2i\theta}Q(z) \quad (\theta \in \mathbb{R}). \tag{3.5}$$

Denote by G_θ the Stokes graph for $Q^{(\theta)}$ (G_0 is the original Stokes graph).

This S^1-action on the potential may cause a discontinuous change of Stokes graph in the following sense. For the sake of simplicity, suppose that the Stokes graph G_0 has a Stokes segment. Then, for a sufficiently small $\delta > 0$, the Stokes graphs $G_{\pm\delta}$ become saddle-free, and we can see that the topology of Stokes graphs $G_{\pm\delta}$ are different as in Figure 4. The discontinuous change of the Stokes graph (caused by the S^1-action) is called a *mutation of Stokes graphs.*

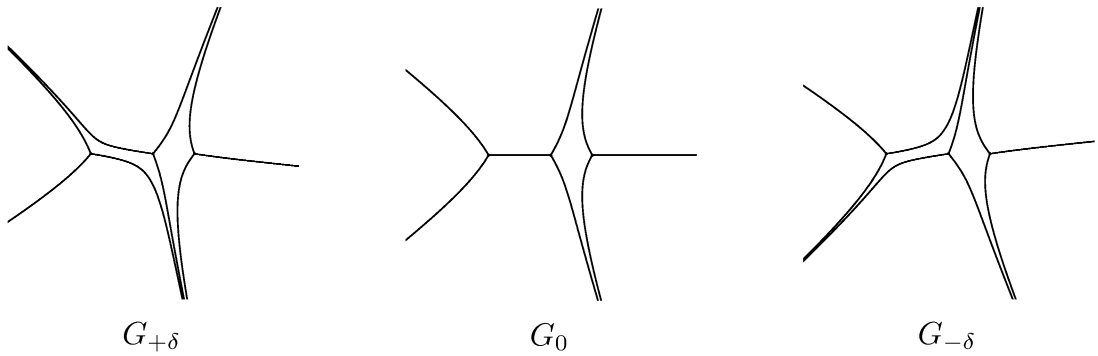


Figure 4. An example of a mutation of Stokes graphs.

It is known that such mutations of Stokes graphs cause a discontinuous change of Borel sums of WKB solutions and Voros symbols, and the formula describing the Stokes phenomenon is studied by [11, 3] etc. Such a discontinuous change of Borel sums are nothing but a *Stokes phenomenon* for $\eta \to +\infty$ because the Borel sums are asymptotically expanded to the original formal series. One of the main result of [25] is to find that the Stokes phenomenon for the Voros symbols realizes the mutations of cluster variables in the sense of [14].

To formulate the result of [25], fix a sign $\varepsilon \in \{+, -\}$ and set $G := G_{\varepsilon\delta}$ and $G' := G_{-\varepsilon\delta}$. Namely, we consider a "signed" mutation $\mu_k^{(\varepsilon)} : G \rightsquigarrow G'$ of Stokes graphs, where ε determines the direction of S^1-action, and $k \in \{1, \dots, n\}$ is the label of the rectangular

Stokes region which vanishes when the Stokes segment appears in G_0. Here we also assume that the Stokes segment appearing in G_0 is the *unique* Stokes segment.

As explained in the previous subsection, we can associate seeds $(B, \mathbf{x}, \mathbf{y})$, $\hat{\mathbf{y}}$ and $(B', \mathbf{x}', \mathbf{y}')$, $\hat{\mathbf{y}}'$ to G and G', respectively. These Borel sums depend on $\delta > 0$, and we can show that the limit $\delta \to +0$ of x_i, y_i, $\hat{y}_i$ and x_i', y_i', $\hat{y}_i'$ exist (see [25, (3.25) and (3.26)]). We use the same symbols (x_i etc.) for these limits. Then we have

Theorem 3.1 ([25, Theorem 7.5]). *The seeds defined above satisfy the following after taking the limit* $\delta \to +0$:

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+ & \text{otherwise.} \end{cases} \tag{3.6}$$

$$x'_i = \begin{cases} {x_k}^{-1} \left(\prod_{j=1}^{n} {x_j}^{[-\varepsilon b_{jk}]_+} \right) (1 + {\hat{y}_k}^{\varepsilon}) & i = k \\ x_i & i \neq k. \end{cases} \tag{3.7}$$

$$y'_i = \begin{cases} {y_k}^{-1} & i = k \\ y_i {y_k}^{[\varepsilon b_{ki}]_+} & i \neq k. \end{cases} \tag{3.8}$$

$$\hat{y}'_i = \begin{cases} {\hat{y}_k}^{-1} & i = k \\ \hat{y}_i {\hat{y}_k}^{[\varepsilon b_{ki}]_+} (1 + {\hat{y}_k}^{\varepsilon})^{-b_{ki}} & i \neq k. \end{cases} \tag{3.9}$$

Here $[a]_+ = \max(a, 0)$.

We can see that the mutation of Stokes graphs are consistent with the *flip* of the triangulation (i.e., replacement of diagonal to another one in a quadrilateral); it implies (3.6). The formulas for x_i and $\hat{y}_i$ follow from a result in [11] which describes the jump for Voros symbols caused by a Stokes phenomenon (relevant to the Stokes segment). The formulas in Theorem 3.1 are examples of mutation of cluster variables with coefficients in the *tropical semifield* (to be precise, ε should be chosen as the *tropical sign*; see [25] for details). Thus we have seen that the Borel sum of Voros symbols realize the cluster variables.

In our framework, the *pentagon identity for Stokes automorphisms*

$$\mathfrak{S}_{\gamma_1} \mathfrak{S}_{\gamma_2} \mathfrak{S}_{\gamma_1}^{-1} \mathfrak{S}_{\gamma_1+\gamma_2}^{-1} \mathfrak{S}_{\gamma_2}^{-1} = 1 \tag{3.10}$$

obtained in [11] follows from a *periodicity* of cluster algebras of the type A_2. Here $\mathfrak{S}_\gamma$ is the Stokes automorphism (associated to a cycle γ) which describes the Stokes phenomenon for Voros symbols relevant to the mutation of Stokes graph (see [11]). We can see that the cycles appearing in (3.10) is nothing but the *c-vectors*, and the signs on the Stokes automorphisms are the *tropical signs* (associated with a mutation period in the cluster algebra of the type A_2). See [30] for the terminologies. We can obtain similar identities of Stokes automorphisms from any period of mutation sequence in the corresponding cluster algebra (see [25]).

However, the above result is *not* enough to study the cluster algebras. A cluster algebra is defined by iterations of mutations, but our result concerns with a single mutation. We have to develop our framework so that we can analyze the whole structure of corresponding cluster algebras.

Remark 3.2. Theorem 3.1 is generalized to cases where $Q(z)$ is a meromorphic function (satisfying certain assumptions). Then, other kind of Stokes segments appear in meromorphic cases: If $Q(z)$ has a double pole, then a *loop type Stokes segment* may appear around the double pole. The formula describing the change of the WKB solutions and Voros symbols are studied in [1]. In this case the loop type Stokes segment doesn't cause a mutation of cluster variables, but a *local rescaling* of cluster x-variables. Local rescale is an operation which commutes with mutations of cluster variables (see [25, Theorem 7.10]). On the other hand, when $Q(z)$ has a *simple pole*, the mutation formula relevant to a Stokes segment connecting a turning point and simple pole is an example of a mutation in *generalized cluster algebras* introduced in [9]. See [26] for the simple pole case. It seems to be interesting to analyze situations that more than two Stokes segments interact in the mutation of the Stokes graph (one particular example, called *juggle*, is analyzed in [18]).

Remark 3.3. Quadratic differentials and their trajectories are also important in the study of *stability conditions* ([5]) on certain categories associated with a triangulation (or N-angulation) of surface with boundaries; see [6, 7, 24, 31]. In particular, the mutation of Stokes graphs are related to a kind of *wall-crossing* in the space of stability conditions.

§ 4. Voros symbols and Fock-Goncharov coordinates

Here we give some comments on a similarity of the result in [25] and that of Gaiotto, Moore and Neitzke [18].

§ 4.1. Voros symbol as a cross-ratio of Wronskians

Let us continue to consider the Schrödinger equation (2.1) satisfying the assumptions in Section 2, and suppose that the Stokes graph is saddle-free. Using the idea of [18], we will give an expression of Borel resummed Voros symbols $\mathcal{S}[e^{V_{\gamma_i}}]$ for the simple cycle γ_i in terms of the Borel resummed WKB solutions.

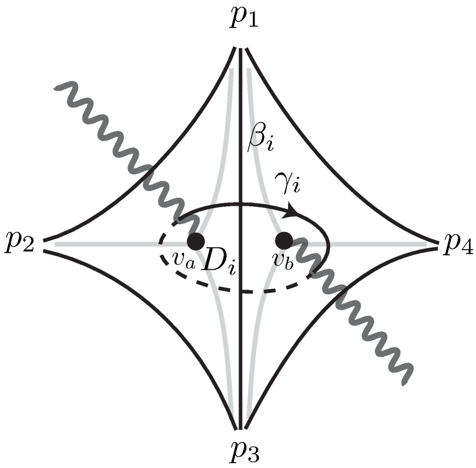


Figure 5. The quadrilateral Quad_i.

For any label $i \in \{1, \dots, n\}$, let Quad_i be the quadrilateral whose edges are given by edges of the triangulation defined in Section 3.1 and containing the simple path β_i as its diagonal. Let $p_1, \dots, p_4$ be the vertices of Quad_i; we set p_1 at one of vertices where β_i is attached, and set p_2, p_3 and p_4 in counter-clockwise order as indicated in Figure 5. Taking branch cuts as in Figure 5, we fix the branch of $\sqrt{Q(z)}$ on $(\mathrm{Quad}_i \setminus \{\text{branch cuts}\})$. For $m = 1, \dots, 4$, define a WKB solution $\psi_m(z, \eta)$ by

(4.1)

$$\psi_m(z,\eta) = \begin{cases} \psi_-(z,\eta) & \text{if } \mathrm{Re} \int_v^z \sqrt{Q(z)}dz > 0 \text{ on Stokes curves which flow into } p_m, \\ \psi_+(z,\eta) & \text{if } \mathrm{Re} \int_v^z \sqrt{Q(z)}dz < 0 \text{ on Stokes curves which flow into } p_m, \end{cases}$$

where v is a turning point where the Stokes curve in question emanates. If a WKB solution has an exponentially small phase on a Stokes curve, then the Borel sum of the WKB solution is well-defined even if z lies on the Stokes curve ([27, Theorem 2.23]). Therefore, the Borel sum of ψ_m is well-defined near p_m. We denote it by $\Psi_m(z, \eta)$. Then we have

Theorem 4.1. *The Borel resummed Voros symbol $e^{V_{\gamma_i}(\eta)}$ for the cycle γ_i has the following expression:*

$$\mathcal{S}[e^{V_{\gamma_i}}] = -\frac{\mathrm{Wr}[\Psi_1, \Psi_2]\ \mathrm{Wr}[\Psi_3, \Psi_4]}{\mathrm{Wr}[\Psi_2, \Psi_3]\ \mathrm{Wr}[\Psi_4, \Psi_1]}. \tag{4.2}$$

Here $\mathrm{Wr}[\Psi_k, \Psi_m]$ is the Wronskian of Borel resummed WKB solutions (evaluated at a point in z_* in Quad_i):

$$\mathrm{Wr}[\Psi_k, \Psi_m] = \Psi_k(z,\eta)\frac{d\Psi_m}{dz}(z,\eta) - \frac{d\Psi_k}{dz}(z,\eta)\Psi_m(z,\eta). \tag{4.3}$$

Note that the right hand-side of (4.2) is independent of z since the Wronskians are independent of z. We give a proof of Theorem 4.1 in the rest of this subsection. Our proof is based on the Voros' connection formula [32].

Proof. To specify the situation, we assume that the branch of $\sqrt{Q(z)}$ on the first sheet is chosen so that

$$\psi_1 = \psi_-, \quad \psi_2 = \psi_-, \quad \psi_3 = \psi_+, \quad \psi_4 = \psi_+. \tag{4.4}$$

Note that $\mathrm{Re} \int_v^z \sqrt{Q(z)}dz$ has different signs on adjacent Stokes curves emanating from a turning point v if no branch cut runs between the two Stokes curves. Then, the cycle γ_i has the orientation shown in Figure 5. Our proof proceeds when the branch is chosen differently.

First, we note that the right-hand side of (4.2) is independent of the normalization (i.e., the choice of the lower end-point in (2.2)) of each WKB solution ψ_m. Therefore, to evaluate (4.2), we can chose the following special normalizations:

$$\psi_1 = \psi_{-,v_a}, \quad \psi_2 = \psi_{-,v_a}, \quad \psi_3 = \psi_{+,v_b}, \quad \psi_4 = \psi_{+,v_b}. \tag{4.5}$$

Here v_a and v_b are turning points in Quad_i (specified in Figure 5), and $\psi_{\pm,v}$ is the WKB solution defined by choosing the lower end point z_0 in (2.2) at a turning point v (see [27, Section 2]). At the level of formal series, simple computations show that the Wronskians for these WKB solutions are given by

$$\mathrm{Wr}[\psi_{+,v_a}, \psi_{-,v_a}] = \mathrm{Wr}[\psi_{+,v_b}, \psi_{-,v_b}] = -2, \tag{4.6}$$

$$\mathrm{Wr}[\psi_{+,v_a}, \psi_{-,v_b}] = -2\exp\left(\frac{1}{2}V_{\gamma_i}\right), \tag{4.7}$$

$$\mathrm{Wr}[\psi_{+,v_b}, \psi_{-,v_a}] = -2\exp\left(-\frac{1}{2}V_{\gamma_i}\right). \tag{4.8}$$

For example,

$$\begin{aligned}
&\mathrm{Wr}[\psi_{+,v_a}, \psi_{-,v_b}] \\
&\quad = \left(-S_{\mathrm{odd}} - \frac{1}{2S_{\mathrm{odd}}}\frac{dS_{\mathrm{odd}}}{dz}\right)\psi_{+,v_a}\psi_{-,v_b} - \left(+S_{\mathrm{odd}} - \frac{1}{2S_{\mathrm{odd}}}\frac{dS_{\mathrm{odd}}}{dz}\right)\psi_{+,v_a}\psi_{-,v_b} \\
&\quad = -2\exp\left(\int_{v_a}^{v_b} S_{\mathrm{odd}}dz\right) = -2\exp\left(\frac{1}{2}\oint_{\gamma_i} S_{\mathrm{odd}}dz\right).
\end{aligned}$$

On the other hand, the *Voros' connection formula* relates the Borel sums of WKB solutions on Stokes curves if the lower end-points z_0 in (2.2) are chosen at a turning point ([32, Section 6]; see also [27, Theorem 2.23]). The resulting formula shows that Ψ_m's are evaluated on D_i as

$$\Psi_1 = \Psi^{D_i}_{-,v_a}, \quad \Psi_2 = \Psi^{D_i}_{-,v_a} + i\Psi^{D_i}_{+,v_a}, \quad \Psi_3 = \Psi^{D_i}_{+,v_b}, \quad \Psi_4 = \Psi^{D_i}_{+,v_b} + i\Psi^{D_i}_{-,v_b}. \tag{4.9}$$

Here $\Psi_{\pm,v}^{D_i}$ is the Borel sum of the WKB solution $\psi_{\pm,v}$ defined on the Stokes region D_i. Since the Borel resummation commutes with the operation such as sum, multiplication and differentiation with respect to z, the desired formula (4.2) follows from (4.6), (4.7), (4.8) and (4.9). This completes the proof of Theorem 4.1. □

Theorem 4.1 gives another proof of (3.9) in Theorem 3.1. That is, the transformation law (3.9) for the Borel resummed Voros symbols $\mathcal{S}[e^{V_{\gamma_i}}]$ are nothing but the *Plücker relation* for Wronskians:

(4.10)
$$\mathrm{Wr}[\Psi_{m_1},\Psi_{m_2}]\mathrm{Wr}[\Psi_{m_3},\Psi_{m_4}] = \mathrm{Wr}[\Psi_{m_1},\Psi_{m_3}]\mathrm{Wr}[\Psi_{m_2},\Psi_{m_4}]+\mathrm{Wr}[\Psi_{m_1},\Psi_{m_4}]\mathrm{Wr}[\Psi_{m_3},\Psi_{m_2}].$$

Here Ψ_{m_i}'s are any solutions of the Schrödinger equation (2.1).

This idea (i.e., cluster $\hat{y}$-variable = cross-ratio of Wronskians) is due to [18]. In fact, the above expression (4.2) of cluster $\hat{y}$-variables is the same as the *Fock-Goncharov coordinate* (c.f., [13]), as we will see in the next subsection.

§ 4.2. Fock-Goncharov coordinates

Here we briefly explain a part of the work [18] of Gaiotto, Moore and Neitzke.

Based on some physical motivations, [18] constructs cluster $\hat{y}$-variables in terms of solutions of a differential equation (or flat sections) using a combinatorics of Stokes graphs. The flat $SL(2,\mathbb{C})$-connection they discussed contains parameters $\zeta \in \mathbb{C}^{\times}$ and $R > 0$, and takes the form

$$\mathcal{A} := \frac{R}{\zeta}\varphi + D + R\zeta\bar{\varphi}, \tag{4.11}$$

where $D := D_z dz + D_{\bar{z}} d\bar{z}$ and $\varphi := \varphi_z dz$ are traceless 2×2 matrix-valued meromorphic 1-form on $\mathbb{P}^1$ (and satisfy *Hitchin equations* [22]), and $\bar{\ }$ denotes the Hermitian conjugate. Therefore, unlike the previous sections, the differential equation for flat sections of $\mathcal{A}$ contains an *anti-holomorphic* part.

The Stokes graph (in the WKB analysis for $\zeta \to 0$) of the flat connection $\mathcal{A}$ is defined by the quadratic differential $\det\varphi$ as well as Definition 2.1. Assume that the quadratic differential $\det\varphi$ has only simple zeros, and the Stokes graph is saddle-free. Then we can consider the triangulated surface as in Section 3.1. Let $\mathrm{Quad}_i, p_1, \ldots, p_4$ are the same as in the Section 4.1. Then, the cluster $\hat{y}$-variable constructed in [18] is given by

$$\mathcal{X}_{\gamma_i} := -\frac{(s_1\wedge s_2)(s_3\wedge s_4)}{(s_2\wedge s_3)(s_4\wedge s_1)}. \tag{4.12}$$

Here s_m is a flat section (i.e., $(d+\mathcal{A})s_m = 0$) specified by its behavior when $z \to p_m$ by the similar rule as (4.1), and all s_m in (4.12) are evaluated at a point z_* in Quad_i. Since $\mathcal{A}$ is traceless, $\mathcal{X}_{\gamma_i}$ doesn't depend on z_*.

This is called the *Fock-Goncharov coordinate* of the moduli space of flat $SL(2,\mathbb{C})$-connections (c.f., [13]). The Fock-Goncharov coordinates depend on the topology of the Stokes graph, and it is shown in [18, Section 7.6.1] that, under the mutation of the Stokes graph, they transform as cluster $\hat{y}$-variables.

The Fock-Goncharov coordinates contain an anti-holomorphic part; it depends on the complex conjugate of parameters contained in φ. Thus, our Voros symbols $e^{V_{\gamma_i}}$ of the Schrödinger equation are *holomorphic analogue* of the Fock-Goncharov coordinates $\mathcal{X}_{\gamma_i}$ of [18]. It is observed in [16] that a Schrödinger equation of the form (2.1) appears as a limit ($R, \zeta \to 0$, with fixed $R/\zeta = \eta$) of the flat connection $\mathcal{A}$. The limit is called *conformal limit*. We expect that, if the conformal limit of the Fock-Goncharov coordinates exists, they give our Voros symbols.

The Fock-Goncharov coordinates of [18] play important roles in their analysis of the *wall-crossing formulas* for *BPS spectrum* of four-dimensional field theory (c.f., [29]), that is the main focus of [18]. The wall-crossing formulas can be recognized as identities of (compositions of) cluster transformations acting on the Fock-Goncharov coordinates (the equality (3.10) is an example of the wall-crossing formula). This is expected to be an *isomonodromic property* for the Stokes factors of the Fock-Goncharov coordinates (c.f., [8, 12, 17]). It seems to be interesting to analyze the isomondromic property and wall-crossing phenomenon from the view point of the exact WKB analysis.

Remark 4.2. In the exact WKB analysis for *higher order* differential equations, the configuration of Stokes graph becomes more complicated; we need to consider *virtual turning points* and *new Stokes curves* (see [2, 4, 23]). Interestingly, the same notion of Stokes graphs for higher order equations also appears in the work of Gaiotto, Moore and Neitzke [20, 21] in the study of BPS spectrum for four-dimensional field theories specified by higher rank group. They call the graph *spectral network*. Moreover, the *2d-4d wall-crossing* discussed in [19, 20] seems to be closely related to the Stokes phenomena occurring to WKB solutions. It also seems to be interesting to relate these works and the exact WKB analysis.

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