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RELATIONS AMONG SPLITTINGS OF COHOMOLOGIES OF p-GROUPS WITH RANK 2

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1. Introduction

Let P be a p-group and BP its classifying space. We study the stable splitting and splitting of cohomology

$$(*)\quad BP \cong X_1 \vee ... \vee X_i,$$

$$(**)\quad H^*(P) \cong H^*(X_1) \oplus ... \oplus H^*(X_i) \quad (for\ * > 0)$$

where X_i are irreducible spaces in the stable homotopy category.

From the answer of the Segal conjecture by Carlsson, the splittings $(*)$ are given by only using modular representation theory by Benson-Feshbach [Be-Fe] and Martino-Priddy [Ma-Pr]. In fact, their theorems say that such a decomposition is decided only by structures of simple modules of the mod(p) double Burnside algebra $A(P,P)$. These theorems do not use splittings of cohomology $(**)$.

In particular, Dietz and Dietz-Priddy [Di], [Di-Pr] gave the stable splitting $(*)$ for groups P with $rank_p(P) = 2$ for $p \geq 5$. However it was not used splittings $(**)$ of the cohomology $H^*(P)$, and the cohomologies $H^*(X_i)$ were not given there. In [Hi-Ya1],[Hi-Ya2], we give the cohomology $H^*(X_i)$ (and hence $(**)$) for $P = p_+^{1+2}$ the extraspecial p group of order p^3 and exponent p. Their cohomology $H^*(X_i)$ are very complicated but have rich structures, in fact p_+^{1+2} is a p-Sylow subgroup of many interesting groups, e.g., $GL_3(\mathbb{F}_p)$ and many simple groups e.g. J_4 for $p = 3$.

In this paper, we give the decomposition of $H^*(P) = H^*(P;\mathbb{Z})/(p,\sqrt{0})$ for other $rank_pP = 2$ groups for odd primes p. In fact the double Burnside algebra $A(P,P)$ acts on $\sqrt{0} \subset H^*(P;\mathbb{Z})$.

In most cases, $H^*(X_i)$ are seemed not to have so rich structure as p_+^{1+2}, since $H^*(P) \subset H^*(p_+^{1+2})$ as graded modules, and they are not

p-Sylow subgroups of so interesting groups. However, we hope that from our computations, it becomes more clear that the relations among splittings of $H^*(P)$ of groups P with $rank_p(P) = 2$. In particular, we note that the irreducible components of Bp_+^{1+2} are most *fine* in those of $rank_p = 2$ groups, namely, the cohomology $H^*(X_i(P))$ can be written as a sum of submodules of $H^*(X_k(p_+^{1+2}))$ (Theorem 7.2, Theorem 7.5, Corollary 7.6).

Theorem 1.1. *For $p \geq 5$, let P be a non-abelian p-group of $rank_p P = 2$, which is not a metacyclic group. For each primitive idempotent $e \in A(P,P)$, there is an idempotent $f \in A(p_+^{1+2}, p_+^{1+2})$ such that $eH^*(P) \cong fH^*(p_+^{1+2})$. Namely, for each irreducible component $X_i(P)$ of BP, we can take some index set $J(i,P)$ such that*

$$H^*(X_i(P)) \cong \bigoplus_{j \in J(i,P)} H^*(X_j(p_+^{1+2})).$$

Remark. Note that $X_i(P) \not\cong \bigvee_{j \in J(i,P)} X_j(p_+^{1+2})$ in the stable homotopy category, because $X_i(P)$ is irreducible.

This paper is planed as follows. In §2 we recall the relation between $A(P,P)$ and the stable splitting. In §3, we note $Out(P)$-actions. In §4 – §6, we give the decomposition of $H^*(P)$ for metacyclic groups, $C(r)$ groups (such that $C(3) = p_+^{1+2}$), $G(r,e)$ groups respectively. In §7, we study the relation of splittings among groups studied in §4 – §6.

2. The double Burnside algebra and stable splitting

Let us fix an odd prime p and $k = \mathbb{F}_p$. For a finite groups G, let $A_{\mathbb{Z}}(G,G)$ be the double Burnside group defined by the Grothendieck group generated by (G,G)-bisets with free right G-actions. Each element Φ in $A_{\mathbb{Z}}(G,G)$ is generated by elements $[Q,\phi] = (G \times_{(Q,\phi)} G)$ for some subgroup $Q \leq G$ and a homomorphism $\phi : Q \to G$. In this paper, we use the notation

$$[Q,\phi] = \Phi : G \geq Q \xrightarrow{\phi} G.$$

By the composition, the group $A_{\mathbb{Z}}(G,G)$ becomes a ring, and call it the (integral) double Burnside algebra.

For each $\Phi = [Q,\phi] \in A_{\mathbb{Z}}(G,G)$, we can define a Φ-action on $H^*(G;k)$ by

$$\Phi(x) = [Q,\phi] \cdot x = Tr_Q^G \phi^*(x) \quad \textit{for } x \in H^*(G;k).$$

In particular, for a finite group G, we have an $A_{\mathbb{Z}}(G,G)$-module structure on $H^*(G;k)$ and $H^*(G;\mathbb{Z})$.

Recall Quillen's theorem [Qu] such that the restriction map $H^*(G;k) \to \lim_V H^*(V;k)$ is an F-isomorphism (i.e. the kernel and cokernel are nilpotent) where V ranges elementary abelian p-subgroups of G. Using this theorem, it is easily see ([Hi-Ya1])

Lemma 2.1. *Let $\sqrt{0}$ be the nilpotent ideal in $H^*(G;k)$ (or $H^*(G;\mathbb{Z})/p$). Then $\sqrt{0}$ itself is an $A_{\mathbb{Z}}(G,G)$-module.*

In this paper we consider the cohomology modulo nilpotent elements. We simply write

$$H^*(G) = H^*(G;\mathbb{Z})/(p,\sqrt{0}).$$

By the preceding lemma, $H^*(G)$ has the $A_{\mathbb{Z}}(G,G)$-module structures.

Let a ring R act on $H^*(G)$ (e.g., $R = A_{\mathbb{Z}}(G,G)$, $k[Out(G)]$). Suppose that there is an R-filtration $F_1 \subset ... \subset F_n \cong H^*(P)$ such that

$$grH^*(G) = \bigoplus F_{i+1}/F_i \cong \bigoplus m_j M_j \quad for \ * > 0$$

with simple R-modules M_j. Then we write $H^*(G) \leftrightarrow \bigoplus m_j M_j$.

Throughout this paper, we assume that degree $* > 0$ so that $H^*(X \vee X') \cong H^*(X) \oplus H^*(X')$. In this paper, $H^*(G) \cong A$ for an graded ring A means an graded module isomorphism otherwise stated, while (in most cases) it is induced from the ring isomorphism $grH^*(G) \cong A$ for some filtrations of $H^*(G)$.

Let $BG = BG_p$ be the p-completion of the classifying space of G. Recall that $\{BG, BG\}_p$ is the (p-completed) group generated by stable homotopy self maps. It is well known from the Segal conjecture (Carlsson's theorem [Ca], [Ma-Pr]) that this group is isomorphic to the double Burnside algebra $A_{\mathbb{Z}}(G,G)^\wedge$ completed by the augmentation ideal. Since the transfer is represented as a stable homotopy map Tr, an element $\Phi = [Q,\phi] \in A(G,G)$ is represented as a sum of maps $\Phi \in \{BG, BG\}_p$

$$\Phi : BG \stackrel{Tr}{\to} BQ \stackrel{B\phi}{\to} BG.$$

Of course, for $x \in H^*(G)$, we have $Tr^*(B\phi)^*(x) = Tr_Q^G \phi^*(x)$.

Let us write $A(G,G) = A_{\mathbb{Z}}(G,G) \otimes k$. Hereafter we consider in the case $G = P$ for a p-group P. Given a primitive idempotents decomposition of the unity of $A(P,P)$

$$1 = e_1 + ... + e_n,$$

we have an indecomposable stable splitting

$$BP \cong X_1 \vee ... \vee X_n \quad with \ e_i BP = X_i.$$

In this paper, an isomorphism $X \cong Y$ for spaces means that it is a stable homotopy equivalence. Recall that $M_i = A(P,P)e_i/(rad(A(P,P)e_i)$ is a simple $A(P,P)$-module where $rad(-)$ is the Jacobson radical. By Wedderburn's theorem, the above decomposition is also written as

$$BP \cong \bigvee_j (\bigvee_k X_{jk}) = \bigvee_j m_j X_{j1} \quad where\ m_j = dim(M_j)$$

where $A(P,P)e_{jk}/rad(A(P,P)e_{jk}) \cong M_j$ for all k, and $dim(M_j)$ is the dimension of M_j over the field $End(M_j)$ which is isomorphic to k in cases considered in this paper. Therefore the stable splitting of BP is completely determined by the idempotent decomposition of the unity in the double Burnside algebra $A(P,P)$.

Here X_i is only defined in the stable homotopy category. (So strictly, the cohomology ring $H^*(X_i)$ is not defined.) However we can define $H^*(X_i)$ as a graded submodule of the cohomology ring $H^*(P)$ by

$$H^*(X_i) = e_i \cdot H^*(P) \quad (= e_i^* H^*(P)\ stably).$$

From Benson-Feshbach [Be-Fe] and Martino-Priddy [Ma-Pr], it is known that each simple $A(P,P)$-module is written as

$$S(P,Q,V) \quad for\ Q \leq P,\ and\ V : simple\ k[Out(Q)] - module.$$

Then the main theorem of stable splitting of BP is stated as follow.

Theorem 2.2. *(Benson-Feshbach* [Be-Fe], *Martino-Priddy* [Ma-Pr]*) There are indecomposable stable spaces $X_{S(P,Q,V)}$ for $S(P,Q,V) \neq 0$ such that*

$$BP \cong \bigvee_{Q \leq P} (dimS(P,Q,V))X_{S(P,Q,V)}.$$

3. $Out(P)$-MODULES

Let R be a subring of $A(P,P)$. For a simple R-module S_R, we can define the idempotent e_{S_R} and the stable space $Y_{S_R} = e_{S_R}BP$ which decomposes BP, while it is (in general) not irreducible. In particular, we take the group algebra $k[Out(P)]$ of the outer automorphism group $Out(P)$ as the ring R.

Lemma 3.1. *For $Out(P)$-simple modules R_i with $dim(R_i) = n_i$, we have*

$$BP = n_1 Y_1 \vee ... \vee n_s Y_s \quad where\ Y_i = e_{R_i} BP$$

for idempotents e_{R_i} in $k[Out(P)]$. Then each Y_i decomposes

$$Y_i = m_{i,1} X_{S_{i1}} \vee ... \vee m_{i,t} X_{S_{it}} \quad for\ X_{S_{ij}} = e_{S_{ij}} BP$$

where $e_{S_{ij}}$ are idempotents in $A(P,P)$ with $dim(S_{ij}) = n_i m_{i,j}$.

An irreducible summands $X_{S(P,Q,V)}$ are called dominant summands if $Q = P$ ([Ni], [Ma-Pr]). Let $X_{S=S(P,Q,V)}$ be a non-dominant summand for a proper subgroup Q. Then it is known ([Ni],[Ma-Pr]) that the corresponding idempotent $e_S \in A(P,P)$ is generated by elements $P > Q \to P$ and $P \to Q \to P$. Hence when there is no non-trivial map $P \to Q$, we see $H^*(X_S) \cong e_S H^*(BP) \subset Tr_Q^P H^*(Q)$.

Corollary 3.2. *Let V be a simple $Out(P)$-module. Then we have decomposition*

$$Y_V \cong X_{S(P,P,V)} \vee \bigvee_{Q \neq P} X_{S(P,Q,W)}.$$

For a simple $Out(P)$-module V, define a stable summand $Y(V)$ by

$$e_V = \sum_{V_i \cong V} e_i, \quad Y(V) = \bigvee_{V_{jk} \cong V} Y_{jk} = e_V BP.$$

Lemma 3.3. *Given a simple $Out(P)$-module V, we have*

$$H^*(Y(V)) \leftrightarrow \bigoplus_{i=1} V[k_i], \qquad 0 \leq k_1 \leq ... \leq k_s \leq ...$$

where $[k_s]$ is the operation ascending degree k_s.

In this paper, we compute the decomposition of $H^*(P)$ as follows. We first study cohomologies of non-dominant summands (i.e., compute the decomposition of proper subgroups $Q \subset P$). Next compute $H^*(Y_V)$ for a simple $Out(P)$-module V by using above Lemma 3.3. Then we compute $H^*(X_{S(P,P,V)})$ from Corollary 3.2 by considering non-dominant summands mainly using the transfer map. Thus we get the decomposition from Theorem 2.2.

4. Metacyclic groups for $p \geq 3$

For $p \geq 5$, groups P with $rank_p P = 2$ are classified by Blackburn (see Thomas [Th], Dietz-Priddy [Di-Pr]). They are metacyclic groups, groups $C(r)$ and $G(r', e)$. In this section, we consider metacyclic p groups P for $p \geq 3$

$$0 \to \mathbb{Z}/p^m \to P \to \mathbb{Z}/p^n \to 0. \qquad (4.1)$$

These groups are represented as

$$P = \langle a, b | a^{p^m} = 1, a^{p^{m'}} = b^{p^n}, [a,b] = a^{rp^\ell} \rangle \quad r \neq 0 \ mod(p). \quad (4.2)$$

It is known by ([Hu],[Th]) that $H^{even}(P;\mathbb{Z})$ is multiplicatively generated by Chern classes of complex representations. Let us write

$$\begin{cases} y = c_1(\rho), & \rho : P \to P/\langle a\rangle \to \mathbb{C}^* \\ v = c_{p^{m-\ell}}(\eta), & \eta = Ind_H^P(\xi), \quad \xi : H = \langle a, b^{p^{m-\ell}}\rangle \to \langle a\rangle \to \mathbb{C}^* \end{cases}$$

where ρ, ξ are nonzero linear representations.

By using Quillen's theorem and the fact that P has just one conjugacy class of maximal abelian p-subgroups, we can prove

Theorem 4.1. *(Theorem 5.45 in* [Ya]*) For any metacyclic p-group P with $p \geq 3$, we have a ring isomorphism*

$$H^*(P) \cong k[y, v], \quad |v| = 2p^{m-\ell}.$$

For a non split metacyclic groups, it is proved that BP itself is irreducible [Di]. Hence we consider a split metacyclic group, it is written as

$$P = M(\ell, m, n) = \langle a, b | a^{p^m} = b^{p^n} = 1, [a, b] = a^{p^\ell}\rangle \qquad (4..3)$$

for $m > \ell \geq max(m - n, 1)$. The outer automorphism is the semidirect product

$$Out(P) \cong (p - group) : \mathbb{Z}/(p-1).$$

The p-group acts trivially on $H^*(P)$, and $j \in \mathbb{Z}/(p-1)$ acts as $a \mapsto a^j$ on P, and it acts on $H^*(P)$ as $j^* : v \mapsto jv$.

There are $p - 1$ simple $\mathbb{Z}/(p-1)$-modules $S_i \cong k\{v^i\}$. We consider the decomposition by idempotens for $Out(P)$. Let us write $Y_i = e_{S_i}BP$ and

$$H^*(Y(S_i)) \cong (dim(S_i))H^*(Y_i) \subset H^*(P)$$

(in the notation Y_i from Lemma 2.1). Hence we have the decomposition for $Out(P)$-idempotents

$$H^*(Y_i) \cong k[y, V]\{v^i\}, \quad V = v^{p-1}.$$

We assume $P \neq M(1, 2, 1)$. Then $Tr_H^P(x) = 0$ for $x \in H^*(H)$ for each proper subgroup H of P. By [Di], we have splitting

$$(*) \quad BP \cong \bigvee_{i=0}^{p=2} X_i \vee \bigvee_{i=0}^{p-2} \bar{L}(1, i).$$

Here we write $X_i = e_{S(P,P,S_i)}BP$ identifying S_i as the $A(P, P)$ simple module (but not the simple $Out(P)$-module).

The summand $\bar{L}(1, i)$ is defined as follows. Recall that $H^*(\langle b\rangle) \cong k[y]$. We get $B\langle b\rangle \cong \bigvee_{i=0}^{p-2} \bar{L}(1, i)$, with $H^*(\bar{L}(1, i)) \cong k[Y]\{y^i\}$. Let $\Phi \in A(P, P)$ be defined by the map $\Phi : P \geq P \to \langle b\rangle \subset P$ which induces the isomorphisms $\Phi \cdot H^*(P) \cong k[y] \subset H^*(Y_0)$. This shows $X_{S(P,\langle b\rangle,S'_i)} \cong \tilde{L}(1,i)$, and we have

$$(**) \quad Y_i \cong \begin{cases} X_i & i \neq 0 \\ X_0 \vee \bigvee_{j=0}^{p-2} \bar{L}(1,j) & i = 0. \end{cases}$$

Theorem 4.2. *Let $P = M(\ell,m,n)$ with $(\ell,m,n) \neq (1,2,1)$. Then we have*

$$H^*(X_i) \cong \begin{cases} k[y,V]\{v^i\} & i \neq 0 \\ k[y,V]\{V\} & i = 0. \end{cases}$$

Proof. For $i \neq 0$, we have $H^*(Y_i) \cong H^*(X_i)$. For $i = 0$, we see

$$H^*(X_0) \cong H^*(Y_0) \ominus H^*(\vee_{j=0}^{p-2} L(1,j)) \cong k[y,V] \ominus k[y] \cong k[y,V]\{V\}$$

where $A \ominus B \cong C$ means $A \cong B \oplus C$. □

For the case $(\ell,m,n) = (1,2,1)$, see [Hi-Ya3]. In this case we see $Tr \neq 0$ in $A(P,P)$.

5. $C(r)$ GROUPS FOR $p \geq 3$

The group $C(r)$, $r \geq 3$ is the p-group of order p^r such that

$$C(r) = \langle a,b,c | a^p = b^p = c^{p^{r-2}} = 1,\ [a,b] = c^{p^{r-3}} \rangle$$

for $r \geq 3$. (In partiular, $C(3) = p_+^{1+2}$.) Hence we have a central extension

$$0 \to \mathbb{Z}/p^{r-2} \to C(r) \to \mathbb{Z}/p \times \mathbb{Z}/p \to 0.$$

For each $r \geq 3$, the cohomology $H^*(C(r))$ is isomorphic to $H^*(C(3))$. Denote $C(3) = p_+^{1+2}$ by E. The cohomology of E is well known ([Lw],[Le])

$$H^*(E) \cong (k[y_1,y_2]/(y_1^p y_2 - y_1 y_2^p) \oplus k\{C\}) \otimes k[v]. \qquad (5.1)$$

Here y_1 (resp. y_2) is the first Chern class $c_1(e_1)$ (resp. $c_1(e_2)$) for the nonzero linear representation $e_1 : E \to \langle a\rangle \to \mathbb{C}^*$ (resp. $e_2 : E \to \langle b\rangle \to \mathbb{C}^*$). The elements C and v are also represented by Chern classes

$$c_i(Ind_A^E(e)) = \begin{cases} v & for\ i = p \\ C & for\ i = p-1 \end{cases}$$

where $e : A \to \langle c\rangle \to \mathbb{C}^*$ is a non zero linear representation, for any maximal elementary abelian subgroup A. Hence $|y_i| = 2, |C| = 2(p-1), |v| = 2p$. It is known $Cy_i = y_i^p$, $C^2 = Y_1 + Y_2 - Y_1Y_1$ where $Y_i = y_i^{p-1}$ and $V = v^{p-1}$.

From the formula (4.1), we get the another expression of $H^*(E)$ (Proposition 9 in [Gr-Le])

$$H^*(E) \cong k[C,v]\{y_1^i y_2^j \mid 0 \le i,j \le p-1,\ (i,j) \ne (p-1,p-1)\}. \qquad (5.2)$$

For $j = (p-1)+i$ with $0 \le i \le p-2$. Define $T(A)^i$ by

$$H^j(E) \supset T(A)^i, \quad T(A)^i = k\{y_1^{p-1}y_2^i,\ y_1^{p-2}y_2^{i+1},\ \ldots,\ y_1^i y_2^{p-1}\}.$$

Then we can identify $T(A)^i$ as an $Out(E)$-module such that $T(A)^i \cong S(A)^{p-1-i} \otimes det^i[2i]$. In fact, from (5.2), we also have

Theorem 5.1. *(Theorem 4.4 in* [Hi-Ya1]*) Let us write $\mathbb{CA} = H^*(E)^{Out(E)} \cong k[C,V]$. Then there is a decomposition of $Out(E)$-module such that*

$$H^*(E) \leftrightarrow \mathbb{CA} \otimes (\bigoplus_{q=0}^{p-2}\bigoplus_{i=0}^{p-2}(S(A)^i \otimes v^q \oplus T(A)^i \otimes v^q))$$

where $S(A)^i \otimes v^q \cong S(A)^i \otimes det^q$ and $T(A)^i \otimes v^q \cong S(A)^{p-1-i} \otimes det^{i+q}[2i]$.

(I) $P = C(r)$ for $r > 3$.

By Dietz and Priddy, the stable splitting is known. The splitting is given as

$$BP \cong \bigvee_{i,q}(i+1)X_{i,q} \vee \bigvee_q (q+1)L(1,q) \vee pL(1,p-1)$$

where $0 \le i \le p-1$, $0 \le q \le p-2$ and $L(1,p-1) = L(1,0)$. Transfers from proper subgroups are always zero when $r > 3$.

Theorem 5.2. *Let $P = C(r)$ and $r \ge 4$. Then*

$$(i+1)H^*(X_{i,q}) \cong \begin{cases} (i+1)H^*(Y_{i,q}) & if\ q \ne 0 \\ \mathbb{CA} \otimes (S(A)^i\}\{V\} \oplus T^{p-1-i}(A)v^i) & q = 0, i \ne p-1 \\ \mathbb{CA} \otimes S(A)^{p-1}\{V\} & q = 0,\ i = p-1. \end{cases}$$

(II) $C(3) = p_+^{1+2}$.

In this case, the decomposition of cohomology is given in [Hi-Ya1] but it is quite complicated. By Dietz-Priddy, the splitting is given as

$$BP \cong \bigvee_{i,q}(i+1)X_{i,q} \vee \bigvee_q ((p+1)L(2,q) \vee (q+1)L(1,q)) \vee pL(1,p-1)$$

where $0 \le i \le p-1$ and $0 \le q \le p-2$. The different places from $r \ge 4$ are the existence of

$$L(2,q) = X_S \quad for\ S = S(P,A,S(A)^{p-1} \otimes det^q),$$

which are induced from the transfer (see §9 in [Hi-Ya1] for details). For the cohomology $H^*(X_{i,q})$ see also [Hi-Ya1].

6. $G(r,e)$ FOR $p \geq 5$

Throughout this section, we assume $p \geq 5$. The group $G = G(r,e), r \geq 4$ (and e is 1 or a quadratic non residue modulo p) is defined as

$$\langle a,b,c | a^p = b^p = c^{p^{r-2}} = [b,c] = 1, [a,b^{-1}] = c^{ep^{r-3}}, [a,c] = b\rangle.$$

The subgroup $\langle a,b,c^p\rangle$ is isomorphic to $C(r-1)$. Hence we have the extension

$$1 \to C(r-1) \to G(r,e) \to \mathbb{Z}/p \to 0.$$

Of course $E = C(3) \subset C(r-1) \subset G(r,e)$.

By [Ya], we have an isomorphism

$$H^*(G(r,e)) \cong H^*(E)^{\langle c\rangle}.$$

The invariant ring $H^*(C(3))^{\langle c\rangle}$ is multiplicatively generated by

$$y_1,\ C,\ v,\ y_2^i w \quad \text{where } w = y_2^p - y_1^{p-1}y_2, \quad 0 \leq i \leq p-3$$

since $c^* : y_2 \mapsto y_2 + y_1$ and $C^2 = Y_1^2 + y_2^{p-2}w$. Hence we have

Lemma 6.1. *We have isomorphisms*

$$(1)\quad H^*(G(r,e)) \cong (k[y_1] \oplus k[y_2]\{w\} \oplus k\{C\}) \otimes k[v]$$

$$\cong \mathbb{CA} \otimes \bigoplus_{q=0}^{p-2} (k\{1, y_1, ..., y_1^{p-1}\}\{v^q\} \oplus k\{1, y_2, ..., y_2^{p-3}\}\{wv^q\}).$$

Corollary 6.2. *We have additively* $H^*(G(r,e)) \cong \bigoplus_{i,q} H^*(Y_{i,q}(E))$.

The outer automorphism is $Out(P) \cong (p-group) : (\mathbb{Z}/2 \times \mathbb{Z}/(p-1))$ (see [Di-Pr] for details). Here the action $i \in \mathbb{Z}/2$ induces $i : a \mapsto a^{-1}$ and $k \in \mathbb{Z}/(p-1)$ induces $k : c \mapsto c^k$. Hence all simple $\mathbb{Z}/2 \times \mathbb{Z}/(p-1)$-modules are represented as $k\{v^i\}$ and $k\{y_1v^i\}$ for $0 \leq i \leq p-2$. Using this and Lemma 6.1, we get

Lemma 6.3. *Let $P = G(r,e)$ with $r \geq 4$. For $Out(P)$-module decomposition component $Y_{i,q}(P)$ of BP, we have additively*

$$H^*(Y_{i,q}(P)) \cong \begin{cases} \bigoplus_{j=even} H^*(Y_{j,q}(E)) & if\ i = 0 \\ \bigoplus_{j=odd} H^*(Y_{j,q}(E)) & if\ i = 1 \end{cases}$$

where $0 \leq i \leq 1$, $0 \leq j \leq p-1$ *and* $0 \leq q \leq p-2$.

(I) $G(r,e)$ for $r > 4$.

The stable splitting is given by Dietz-Priddy [Di-Pr]

$$BG(r,e) \cong \bigvee_{i,q} X_{i,q}(G(r,e)) \vee \bigvee_q X_{p-1,q}(C(r-1)) \vee \bigvee_q L(1,q)$$

where $i \in \mathbb{Z}/2$ and $0 \leq q \leq p-2$.

Theorem 6.4. *For $r > 4$, we have*

$$H^*(X_{i,q}(G(r,e)) \cong \begin{cases} H^*((\bigvee_{j=ev}^{p-3} X_{j,0}(C(r-1))) \vee L(1,0)) & if\ i = q = 0 \\ H^*(\bigvee_{j=ev}^{p-3} X_{j,q}(C(r-1))) & if\ i = 0, q \neq 0 \\ H^*(\bigvee_{j=odd}^{p-2} X_{j,q}(C(r-1)) & if\ i = 1 \end{cases}.$$

(II) $G(4,e)$

In this case cohomology is the same as (I). However the stable splitting is not same as (I) and it is also given by Dietz and Priddy [Di-Pr]

$$BG(r,e) \cong \bigvee_{i,q} X_{i,q}(G(r,e)) \vee \bigvee_{q} (X_{p-1,q}(C(r-1)) \vee L(2,q) \vee L(1,q))$$

where $i \in \mathbb{Z}/2$ and $0 \leq q \leq p-2$. The problems are only to see that these $H^*(L(2,q))$ go to what $H^*(Y_{i,q'})$. For details see [Hi-Ya3].

7. Relations among BP with $rank_p P = 2$.

The following lemma is immediate from preceding sections.

Lemma 7.1. *Let $P = C(r)$ (or $G(r+1,e)$) for $r \geq 3$. Then for $0 \leq q \leq p-2$, non-dominant summands are $L(1,q)$, $L(2,q)$ (and $X_{p-1,q}(C(r))$ for $P = G(r+1,e)$).*

For stable homotopy spaces X, X', let us write $X \cong_H X'$ when $H^*(X) \cong H^*(X')$ as graded modules. Theorem 1.1 in the introduction is a immediate consequence of the above lemma and the following theorem about dominant summands, which follows, for example, from Theorem 6.4 when $G = G(r,e)$, $r > 4$.

Theorem 7.2. *Let $P = C(r)$ (or $G(r+1,e)$) for $r \geq 3$. Given $0 \leq i \leq p-1$ (or $i = 0$ or 1) and $0 \leq q \leq p-2$, there are $0 \leq a_j, b_k, c \leq 1$ such that we have the isomorphism*

$$X_{i,q}(P) \cong_H \bigvee_{j=0}^{p-1} a_j X_{j,q}(E) \vee \bigvee_{k=0}^{p-2} b_k L(2,k) \vee cL(1,0)$$

In particular, $c = 1$ if and only if $i = q = 0$ and $P = G(r+1,e)$.

Next, we study split metacyclic groups. For stable spaces $X = X_{i,j}(C(r))$ or $X = Y_{i,j}(C(r))$, let SX be the virtual object defined by (strictly the module $H^*(SX)$ is defined)

$$H^*(SX) = H^*(X) \ \cap \ \mathbb{CA} \otimes (\bigoplus_{q=0}^{p-2} k\{1, y_1, ..., y_1^{p-2}\}\{v^q\})$$

where we identify it as the submodule of $\mathbb{CA}\otimes(\bigoplus_q S(A)^*\{v^q\}) \subset H^*(E)$ in Theorem 4.1. Then we see

$$H^*(S(BE)) \cong \mathbb{CA} \otimes (\bigoplus_q (k\{1, y_1, ..., y_1^{p-2}\}\{v^q\}) \cong k[y_1, v]$$

identifying $C = Y = y_1^{p-1}$ as graded modules.

Recall that for the split metacyclic group $M = M(\ell, m, n)$, we have $H^*(M) \cong k[y, v]$ with $|v| = 2p^{m-\ell}$ from Theorem 4.1. In particular, when $m - \ell = 1$, we see $H^*(M) \cong H^*(S(BE))$. The results in §5 imply the following theorem

Theorem 7.3. *Let $M = M(m-1, m, n)$. Then we have*

$$H^*(X_q(M)) \cong \begin{cases} \bigoplus_{j=0}^{p-2} H^*(SX_{j,q}(C(r))), & \text{for } r > 3, \quad \text{if } (m,n) \neq (2,1) \\ \bigoplus_{j=0}^{p-2} H^*(SX_{j,q}(E)) & \text{if } (m,n) = (2,1). \end{cases}$$

At last in this section, we consider the cases $m - \ell > 1$. From the results in §5, it is almost immediate

Proposition 7.4. *Let $m - \ell > 1$. Then we have*

$$H^*(X_i(M(\ell, m, n)) \cong H^*(X_i(M(m-1, m, n)) \cap k[y, v^{p^{m-\ell-1}}].$$

From these results, we get

Theorem 7.5. *For $p \geq 5$, let P be a non-abelian p-group of $rank_p P = 2$. Then there is a submodule $HS(P) \subset H^*(E)$ such that for each primitive idempotent e in $A(P, P)$, there is an idempotent $f \in A(E, E)$ such that $eH^*(P) \cong HS(P) \cap fH^*(E)$. When P is not metacyclic, we can take $HS(P) = H^*(E)$.*

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