数理解析研究所講究録
第2061巻 2018年 120-125

On the low dimensional cohomology groups of the IA-automorphism group of the free group of rank three

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Abstract

In this announcement we consider the structure of the rational cohomology groups of the IA-automorphism group IA_3 of the free group of rank three by using combinatorial group theory and representation theory. In particular, we detect a non-trivial irreducible component in the second cohomology group of IA_3, which does not contained in the image of the cup product map of the first cohomology groups. We also show that the image of the triple cup product map of the first cohomology groups in the third cohomology group is trivial. As a corollary, we obtain that the fourth term of the lower central series of IA_3 has finite index in that of the Andreadakis-Johnson filtration.

1 Introduction

Let F_n be a free group of rank $n \geq 2$ with basis $x_1, \ldots, x_n$, and $\mathrm{Aut}\,F_n$ the automorphism group of F_n. As far as we know, the first contribution to the study of the (co)homology groups of $\mathrm{Aut}\,F_n$ was given by Nielsen [36] in 1924, who showed $H_1(\mathrm{Aut}\,F_n, \mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ for $n \geq 2$ by using a presentation for $\mathrm{Aut}\,F_n$. Now we have a broad range of results for the (co)homology groups of $\mathrm{Aut}\,F_n$ due to many authors. In 1984, Gersten [16] showed $H_2(\mathrm{Aut}\,F_n, \mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ for $n \geq 5$ In 1980s, by introducing the Outer space, Culler and Vogtmann [10] made a breakthrough in the computation of homology groups of the outer automorphism groups $\mathrm{Out}\,F_n$ of free groups F_n. To put it briefly, the Outer space is an analogue of the Teichmüller space on which the mapping class of a surface naturally acts. By using the geometry of the Outer space, Hatcher and Vogtmann [17] computed $H_4(\mathrm{Aut}\,F_4, \mathbf{Q}) = \mathbf{Q}$, and On the other hand, by using sophisticated homotopy theory, Galatius [14] showed that the stable integral homology groups of $\mathrm{Aut}\,F_n$ are isomorphic to those of the symmetric group $\mathfrak{S}_n$ of degree n. In particular, the stable rational homology groups $H_q(\mathrm{Aut}\,F_n, \mathbf{Q})$ are trivial for $n \geq 2q+1$. This result

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is a quite contrast to the case of the mapping class groups of surfaces. Intuitively, we can see this from the fact that the free group has no geometric extra structure like surface groups.

With respect to unstable cohomology groups, $\mathrm{Aut}\,F_n$ behave in much different and mysterious way. The unstable cohomology groups of the (outer) automorphism groups of free groups has also been studied by many authors. For unstable case, the Outer space is a powerful tool for computation of the cohomology groups. For example, in 1993, Brady [6] computed the integral cohomology groups of $\mathrm{Out}\,F_3$. Other than Hatcher and Vogtmann's results for the rational cohomology as mentioned above, Gerlitz showed $H_7(\mathrm{Aut}\,F_5, \mathbf{Q}) = \mathbf{Q}$ in 2002, and Ohashi [38] computed $H_8(\mathrm{Aut}\,F_6, \mathbf{Q}) = \mathbf{Q}$. On the other hand, in 1999, Morita [33] constructed a series of unstable homology classes of $\mathrm{Out}\,F_n$ with Kontsevich's results [22] and [23]. (See also [34].) These homology classes are called the Morita classes. It is known that the first and the second one are non-trivial, and hence are generators of $H_4(\mathrm{Aut}\,F_4, \mathbf{Q})$ and $H_8(\mathrm{Aut}\,F_6, \mathbf{Q})$ respectively. (See [34] and [9] respectively.) Today, the non-triviality of the higher Morita classes is under intense study by many authors.

Let H be the abelianization of F_n, and IA_n the kernel of the natural homomorphism $\mathrm{Aut}\,F_n \to \mathrm{Aut}\,H$ induced from the abelianization homomorphism $F_n \to H$. The group IA_n is called the IA-automorphism group of F_n. By the spectral sequence of the group extension of IA_n by $\mathrm{Aut}\,H$, the cohomology groups of IA_n are closely related to those of $\mathrm{Aut}\,F_n$. However, the structure of the cohomology groups of IA_n is far from well-understood in contrast to those of $\mathrm{Aut}\,F_n$. To our best knowledge, in the (co)homology groups of IA_n, completely determined and explicitly written down one is only the first integral homology group $H_1(\mathrm{IA}_n, \mathbf{Z})$, which is obtained by Cohen-Pakianathan [7, 8], Farb [13] and Kawazumi [21] independently. Krstić and McCool [24] showed that IA_3 is not finitely presentable. This shows that the second homology group $H_2(\mathrm{IA}_3, \mathbf{Z})$ is not finitely generated. This fact also follows by a recent work of Bestvina, Bux and Margalit [4]. By using the Outer space, they showed that the quotient group of IA_n by the inner automorphism group $\mathrm{Inn}\,F_n$ has a $2n-4$-dimensional Eilenberg-Maclane space, and that $H_{2n-4}(\mathrm{IA}_n/\mathrm{Inn}\,F_n, \mathbf{Z})$ is not finitely generated. For $n \geq 4$, it is not known whether IA_n is finitely presentable or not. Namely, at the present stage, even $H_2(\mathrm{IA}_n, \mathbf{Z})$ is not determined explicitly. Pettet [39] determined the image of the rational cup product of the first cohomologies in $H^2(\mathrm{IA}_n, \mathbf{Q})$, and gave its irreducible GL-decomposition. Furthermore, recently Day and Putman [11] obtained an explicit finite set of generators for $H_2(\mathrm{IA}_n, \mathbf{Z})$ as a $\mathrm{GL}(n, \mathbf{Z})$-module.

In this announcement, we mainly study the second rational cohomology group $H^2(\mathrm{IA}_n, \mathbf{Q})$ for the case where $n = 3$. In particular, we detect a new $\mathrm{GL}(3, \mathbf{Q})$-irreducible component of $H^2(\mathrm{IA}_3, \mathbf{Q})$ by using combinatorial group theory and representation theory. By Pettet [39], the $\mathrm{GL}(3, \mathbf{Q})$-irreducible decomposition of the image of the cup product $\cup_{\mathbf{Q}} : \Lambda^2 H^1(\mathrm{IA}_3, \mathbf{Q}) \to H^2(\mathrm{IA}_3, \mathbf{Q})$. We obtain the following.

Theorem 1. *The quotient module* $H^2(\mathrm{IA}_3, \mathbf{Q})/\mathrm{Im}(\cup_{\mathbf{Q}})$ *contains the* $\mathrm{GL}(3, \mathbf{Q})$*-irreducible representation* $D^{-3} \otimes_{\mathbf{Q}} [5, 1]$.

where D is the one dimensional representation coming from the determinant, and $[\lambda]$

is the irreducible polynomial representation associated to the Young diagram λ.

In order to show Theorem 1, we use our previous results about the Andreadakis-Johnson filtration $\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$ and the Johnson homomorphisms of $\mathrm{Aut}\, F_3$. Historically, the Andreadakis-Johnson filtration was originally introduced by Andreadakis [1] in the 1960s. In 1980s, Johnson used this filtration to study the group structure of the mapping class groups of surfaces. Andreadakis conjectured that the filtration $\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$ coincides with the lower central series $\mathrm{IA}_n = \mathcal{A}'_n(1) \supset \mathcal{A}'_n(2) \supset \cdots$. Andreadakis showed that this conjecture is true for $n = 2$ and any $k \geq 2$, and $n = 3$ and $k \leq 3$. Bachmuth [2] showed $\mathcal{A}'_n(2) = \mathcal{A}_n(2)$ for any $n \geq 2$. This result is also induced from the fact that the first Johnson homomorphism is the abelianization of IA_n by independent works Cohen-Pakianathan [7, 8], Farb [13] and Kawazumi [21]. Pettet [39] showed that $\mathcal{A}'_n(3)$ has at most finite index in $\mathcal{A}_n(3)$ for any $n \geq 4$. Bartholdi [3] showed that the "rational" version of the Andreadakis conjecture is not true for $n = 3$. From our computation in the proof of Theorem 1, as a corollary, we also obtain the following.

Corollary 1. $\mathcal{A}_3(4)/\mathcal{A}'_3(4)$ *is finite.*

We remark that this fact is also obtained by Bartholdi's computation.

Finally, we consider the third rational cohomology group $H^3(\mathrm{IA}_3, \mathbf{Q})$. The results by work of Bestvina, Bux and Margalit [4] as mentioned above, we see that $H^3(\mathrm{IA}_3, \mathbf{Q})$ is infinitely generated. The following theorem shows that non-trivial elements in $H^3(\mathrm{IA}_3, \mathbf{Q})$ cannot be detected by the triple cup product of the first cohomology group of IA_3.

Theorem 2. *The image of the triple cup product*

$$\cup^3_{\mathrm{IA}_3} : \Lambda^3 H^1(\mathrm{IA}_3, \mathbf{Q}) \to H^3(\mathrm{IA}_3, \mathbf{Q})$$

is trivial.

We remark that the arguments and techniques which we use in this paper are applicable to study the cohomology groups of IA_n for general $n \geq 4$. However, the amount of calculation and the complexity vastly increase with the increasing n. In the present paper, we give the first combinatorial group theoretic approach to the study of the low dimensional cohomology groups of the IA-automorphism groups of free groups.

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