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AN ALTERNATIVE CONSTRUCTION OF THE SU(2) CHERN-SIMONS PERTURBATION THEORY

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1. Introduction

In this note, we give an alternative construction of the 2-loop term of the Chern–Simons perturbation theory that is an invariant of a closed 3–manifold with a local system. This construction is deeply inspired by R. Bott and A. S. Cattaneo in [2], however our construction is more flexible than the original one. By using our construction, we gave a Morse theoretic description of the 2-loop term of the $SU(2)$ Chern–Simons perturbation theory in [5].

The Chern–Simons perturbation theory established by S. Axelrod and I. M. Singer in [1] and M. Kontsevich in [3] gives a topological invariant of a closed oriented 3-manifold with an acyclic local system. Let M be a closed oriented 3–manifold and let E be a local system on M. In the Chern–Simons perturbation theory, a propagator plays an important role. A propagator is a closed 2-form (with a twisted coefficient given by E) on the 2-point configuration space of M satisfying some conditions near the diagonal of $M \times M$. In this note, we mitigate these conditions.

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2. 2-loop term of the SU(2)-Chern–Simons perturbation theory

In this section, we give a construction of 2-loop term of the Chern–Simons perturbation theory. This construction is deeply inspired by Bott and Cattaneo in [2].

Let M be a closed oriented 3-manifold, and let $\rho : \pi_1(M) \to SU(2)$ be an irreducible representation. The composition of ρ and the adjoint representation $SU(2) \to \mathrm{Aut}(\mathfrak{su}(2))$ is an orthonormal representation of $\pi_1(M)$. We denote by $E = E_\rho$ the local system corresponding to this orthonormal representation. We assume that E is acyclic, namely

$$H^i(M; E) = 0,$$

for any i. For $x \in M$, we denote by E_x the object of E corresponding to $x \in M$.

For a submanifold B of a manifold A, we denote by $B\ell(A, B)$ the manifold obtained by a real blowing up of A along B, namely $B\ell(A,B) = (A \setminus B) \cup S\nu_B$. Here ν_B is the normal bundle of B and $S\nu_B$ is a unit sphere bundle of ν_B. Let $C_2(M) = B\ell(M^2, \Delta)$, where $\Delta = \{(x,x) \mid x \in M\} \subset M^2$ is the diagonal. $C_2(M)$ is a compactification of the configuration space $M^2 \backslash \Delta$. We denote by $q : C_2(M) \to M^2$ the blow down map. Since the normal bundle ν_Δ is isomorphic to the tangent bundle TM, we have $\partial C_2(M) = q^*(\Delta) \cong STM \cong M \times S^2$. Let $T : C_2(M) \to C_2(M)$ be the involution induced by the involution

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$T_0 : M^2 \to M^2, (x, y) \mapsto (y, x)$. The involution T also acts on $\partial C_2(M) \cong STM$. We denote by $H^*_+(\partial C_2(M)), H^*_-(\partial C_2(M))$ the $+1$, -1 eigen space under the involution T_*, respectively.

Let $p_i : M^2 \to M$ be the projection, for $i = 1, 2$. Then $p_1^* E \otimes p_2^* E$ is a local system on M^2 and $F = q^*(p_1^* E \otimes p_2^* E)$ is a local system on $C_2(M)$.

The following lemma says how $SU(2)$ is special.

Lemma 2.1. $(E \otimes E)^- = 0$. *In particular* $H^2_-(\Delta; E \otimes E) = 0$.

Proof. The Lie bracket $[\cdot, \cdot]$ induces a local system morphism $b : E \otimes E \to E, f(x \otimes y) = [x, y]$. Since $\mathfrak{su}_2$ is semi-simple, $b : E_x \otimes E_x \to E_x$ is subjective for each $(x, x) \in \Delta$. Then we have $\dim(\ker b) = 6$. Since $b(T(x \otimes y)) = -b(x \otimes y)$, we have $(E_x \otimes E_x)^+ \subset \ker b$. On the other hand $\dim((E_x \otimes E_x)^+) = 6$. Then $(E \otimes E)^- \cong E$. Since $T_0|_\Delta = \mathrm{id}$, $H^2_-(\Delta; E \otimes E) = H^2(\Delta; (E \otimes E)^-)$. Therefore $H^2(\Delta; (E \otimes E)^-) = H^2(\Delta; E) = 0$. □

Let $\mathbb{R}$ be the trivial local system. We define a local system morphism $c : \mathbb{R} \to E \otimes E$ as follows: For any $x \in M$, take an orthonormal basis $\{e^x_1, e^x_2, e^x_3\}$ of $E_x \cong \mathfrak{su}_2$. Then $\sum_{i=1,2,3} e^x_i \otimes e^x_i \in E_x \otimes E_x$ is independent of the choice of the basis. So we set $c(1) = \sum_{i=1,2,3} e^x_i \otimes e^x_i$.

Let $p : \partial C_2(M) \cong STM \cong M \times S^2 \to S^2$. Let $\omega_{S^2} \in \Omega^2(S^2; \mathbb{R})$ be a closed 2-form satisfying $\int_{S^2} \omega_{S^2} = 1$. Then $[p^*\omega_{S^2} - T^*p^*\omega_{S^2}]$ generates [1] $H^2_-(\partial C_2(M); \mathbb{R}) \cong H^2(S^2; \mathbb{R})$.

Definition 2.2. A closed 2-form $\omega \in \Omega^2(C_2(M); F)$ is said to be a *propagator* if there is a closed 2-form $\omega_\mathbb{R} \in \Omega^2(\partial C_2(M); \mathbb{R})$ and there is a closed 2-form $\eta \in \Omega^2(\Delta; E \otimes E)$ satisfy the following conditions:

- $[\omega_\mathbb{R}] = \frac{1}{2}[p^*\omega_{S^2} - T^*p^*\omega_{S^2}]$,
- $T^*\omega_\mathbb{R} = -\omega_\mathbb{R}, T^*\eta = -\eta$, (of course, $T^*\omega = -\omega$)
- $\omega|_{\partial C_2(M)} = c_*\omega_\mathbb{R} + q^*\eta$.

Lemma 2.3. *For any M and E, there is a propagator.*

Proof. Thanks to Lemma 2.1, in the following cohomology long exact sequence of the pair $(C_2(M), \partial C_2(M))$, $H^3_-(C_2(M), \partial C_2(M); F) \cong H^3_-(M^2, \Delta; p_1^*E \otimes p_2^*E) \cong H^2_-(\Delta; E \otimes E) = 0$.

$$\ldots \to H^2_-(C_2(M); F) \to H^2_-(\partial C_2(M); F) \to H^3_-(C_2(M), \partial C_2(M); F) \to \ldots.$$

Then for any $\omega_\mathbb{R}$ and η, there is a propagator. □

We define a local system morphism $\mathrm{Tr} : E \otimes E \otimes E \to \mathbb{R}$ by $\mathrm{Tr}(x \otimes y \otimes z) = \langle [x, y], z \rangle$. Here $[\cdot, \cdot]$ is the Lie bracket and $\langle \cdot, \cdot \rangle$ is the Killing form.

Let ω be a propagator that $\omega|_{\partial C_2(M)} = c_*\omega_\mathbb{R} + q^*\eta$.

Definition 2.4.

$$Z_\Theta(\omega) = \int_{C_2(M)} \mathrm{Tr}\omega^3 \in \mathbb{R},$$

$$Z_{O-O}(\omega) = \int_{C_2(M)} \mathrm{Tr}(\omega \wedge \eta_1 \wedge \eta_2) \in \mathbb{R},$$

$$Z(\omega) = Z_\Theta(\omega) - 3Z_{O-O}(\omega).$$

Here $\eta_1 = q^*p_1^*\eta$ and $\eta_2 = q^*p_2^*\eta$.

[1] It is easy to check that the cohomology class $[p^*\omega_{S^2} - T^*p^*\omega_{S^2}]$ is independent from the choice of isomorphism $STM \cong M \times S^2$.

Let X be a closed oriented 4-manifold such that the Euler characteristic of X is zero and $\partial X = M$. Take a sub $\mathbb{R}^3$ bundle T^vX of TX satisfying $T^vX|_M = TM$. Let ST^vX be the unit sphere bundle of T^vX. Let F_X be the tangent bundle along the fiber of the sphere bundle $ST^vX \to X$. Take a closed 2-form $\omega_X \in \Omega^2(ST^vX;\mathbb{R})$ such that:

- $[\omega_X] = \frac{1}{2}e(F_X)$ where $e(F_X)$ is the Euler class of F_X,
- $\omega_X|_{STM} = \omega_{\mathbb{R}}$ under the identification $STM \cong \partial C_2(M)$.

Lemma 2.5 (Proposition 5.3 in [4]). $\int_{ST^vX}\omega_X^3 - \frac{3}{4}\mathrm{Sing}X$ *is independent of the choice of* X *and* ω_X.

Definition 2.6. $I(\omega) = \int_{ST^vX}\omega_X^3 - \frac{3}{4}\mathrm{Sing}X$.

Theorem 2.7. $Z(M;E) = Z(\omega) - 6I(\omega) \in \mathbb{R}$ *is an invariant of* (M,E).

Remark 2.8. This invariant or similar invariants were given by Kontsevich in [3], Axelrod, Singer in [1], Bott and Cattaneo in [2]. They used more limited propagators, in particular the restriction of propagators to $\partial C_2(M)$ were written by using a framing of M. They used a signature defect of the framing instead of our $I(\omega)$.

3. Proofs

Let ω, ω' be propagators such that $\omega|_{\partial C_2(M)} = c_*\omega_{\mathbb{R}} + q^*\eta$ and $\omega'|_{\partial C_2(M)} = c_*\omega'_{\mathbb{R}} + q^*\eta'$.

Lemma 3.1. $[\eta] = [\eta'] = 0, [\omega_{\mathbb{R}}] = [\omega'_{\mathbb{R}}]$.

Proof. This is a direct consequence of the condition about propagators and Lemma 2.1. □

Lemma 3.2. *If* $\omega|_{\partial C_2(M)} = \omega'|_{\partial C_2(M)}$, *then* $Z(\omega) = Z(\omega')$.

Proof. $Z_\Theta(\omega) - Z_\Theta(\omega') = \int_{C_2(M)}\mathrm{Tr}((\omega-\omega')(\omega^2+\omega\omega'+(\omega')^2) = (*)$. Since $\omega|_{\partial C_2(M)} = \omega'|_{\partial C_2(M)}$, $[\omega - \omega'] \in H^2(C_2(M);F)$. Thanks to Lemma 2.1, in the following cohomology long exact sequence of the pair $(C_2(M), \partial C_2(M))$, we have $H^2_-(C_2(M),\partial C_2(M);F) \cong H^2_-(M^2,\Delta;p_1^*E\otimes p_2^*E) = H^1_-(\Delta;E\otimes E) = 0$.

$$\ldots \to H^2_-(C_2(M),\partial C_2(M);F) \to H^2_-(C_2(M);F) \to H^2_-(\partial C_2(M);F) \to \ldots.$$

This implies that $[\omega - \omega'] = 0 \in H^2(C_2(M);F)$. Therefore $(*) = 0$.

It is easy to see that $Z_{O-O}(\omega) = \int_{C_2(M)}\mathrm{Tr}(\omega\eta_1\eta_2) = \int_{C_2(M)}\mathrm{Tr}(\omega'\eta_1\eta_2)$. □

Proposition 3.3. *If* $\omega_{\mathbb{R}} = \omega'_{\mathbb{R}}$, *then* $Z(\omega) = Z(\omega')$.

Proof. There is a cochain $\xi \in \Omega^1_-(\Delta; E \otimes E)$ such that $d\xi = \eta'$. Thanks to Lemma 3.2, it is enough to show that the case $\omega' = \omega + d\xi$ and $\eta = 0$.

$$\begin{aligned} Z_\Theta(\omega') - Z_\Theta(\omega) &= \int_{C_2(M)} \mathrm{Tr}((\omega + d\xi)^3 - \omega^3) \\ &= \int_{C_2(M)} \mathrm{Tr}(3\omega^2 d\xi + 3\omega(d\xi)^2 + (d\xi)^3) \\ &= \int_{\partial C_2(M)} \mathrm{Tr}(3(c_*\omega_{\mathbb{R}})^2\xi + 3\xi c_*\omega_{\mathbb{R}} d\xi + \xi(d\xi)^2) \\ &= \int_{\partial C_2(M)} \mathrm{Tr}(3\xi c_*\omega_{\mathbb{R}} d\xi) \\ &= 3\int_\Delta \mathrm{Tr}(\xi d\xi). \end{aligned}$$

On the other hand,

$$\begin{aligned} Z_{O-O}(\omega') - Z_{O-O}(\omega) &= \int_{C_2(M)} \mathrm{Tr}((\omega + d\xi)\eta_1\eta_2) \\ &= \int_{C_2(M)} \mathrm{Tr}(\omega q^* p_1^* d\xi \eta_2 + d\xi\eta_1\eta_2) \\ &= \int_{\partial C_2(M)} \mathrm{Tr}(c_*\omega_{\mathbb{R}}\xi\eta + \xi\eta^2) \\ &= \int_\Delta \mathrm{Tr}(\xi d\xi). \end{aligned}$$

Therefore $Z_\Theta(\omega) - 3Z_{O-O}(\omega) = Z_\theta(\omega') - 3Z_{O-O}(\omega)$. □

Proof of Theorem 2.7. It is enough to show that the case of $\eta = \eta' = 0$. Thanks to Lemma 3.1, there is a 1-form $\nu \in \Omega^1(\partial C_2(M); \mathbb{R})$ such that $\omega'_{\mathbb{R}} - \omega_{\mathbb{R}} = d\nu$. Thanks to Lemma 3.2, we can assume that $\omega' = \omega + c_* d\nu$.

$$\begin{aligned} Z_\Theta(\omega') - Z_\Theta(\omega) &= \int_{C_2(M)} \mathrm{Tr}(c_* d\nu(\omega^2 + \omega\omega' + (\omega'^2))) \\ &= \int_{\partial C_2(M)} \mathrm{Tr}(c_*(1) \otimes c_*(1) \otimes c_*(1))\nu(\omega_{\mathbb{R}}^2 + \omega_{\mathbb{R}}\omega'_{\mathbb{R}} + (\omega'_{\mathbb{R}})^2) \\ &= 6\int_{\partial C_2(M)} \nu(\omega_{\mathbb{R}}^2 + \omega_{\mathbb{R}}\omega'_{\mathbb{R}} + (\omega'_{\mathbb{R}})^2). \\ Z_{O-O}(\omega') - Z_{O-O}(\omega) &= 0 - 0 = 0. \end{aligned}$$

We can take $\omega'_X = \omega_X + d\nu$. Then we have

$$\begin{aligned} I(\omega') - I(\omega) &= \int_{ST^vX} d\nu(\omega_X^2 + \omega_X\omega'_X + (\omega'_X)^2) \\ &= \int_{STM} \nu(\omega_{\mathbb{R}}^2 + \omega_{\mathbb{R}}\omega'_{\mathbb{R}} + (\omega'_{\mathbb{R}})^2). \end{aligned}$$

Therefore $Z(\omega) - 6I(\omega) = Z(\omega') - 6I(\omega')$. □

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