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TOPOLOGICAL ENTROPY AND TOPOLOGICAL STRUCTURES OF CONTINUA

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1. Introduction

During the last thirty years or so, many interesting connections between dynamical systems and continuum theory have been studied by many authors (see [1,2,6,7,9-15,17,19,22-25,27,28]). We are interested in the following fact that chaotic topological dynamics should imply existence of complicated topological structures of underlying spaces. In many cases, such continua (=compact connected metric spaces) are indecomposable continua which are central subjects of continuum theory in topology. We know that many indecomposable continua often appear as chaotic attractors of dynamical systems. Also, in many cases, the composants of such indecomposable continua are strongly related to stable or unstable (connected) sets of the dynamics. For instance, in the theory of dynamical systems and continuum theory, the Knaster continuum (= Smale's horse shoe), the pseudo-arc, solenoids and Wada's lakes (= Plykin attractors) etc., are well-known as such indecomposable continua.

In [3], by use of ergodic theory method, Blanchard, Glasner, Kolyada and Maass proved that if a map $f : X \to X$ of a compact metric space X has positive topological entropy, then there is an uncountable δ-scrambled subset of X for some $\delta > 0$ and hence the dynamics (X, f) is Li-Yorke chaotic. In [18], Kerr and Li developed local entropy theory and gave a new proof of this theorem. Moreover, they proved that X contains a Cantor set Z which yields more chaotic behaviors (see [18, Theorem 3.18]). In [2], Barge and Diamond showed that for piecewise monotone surjections of graphs, the conditions of having positive entropy, containing a horse shoe and the inverse limit space containing an indecomposable subcontinuum are all equivalent. In [24], Mouron proved that if X is an arc-like continuum which admits a homeomorphism f with positive topological entropy, then X contains an indecomposable subcontinuum. In [6], as an extension of the Mouron's theorem, we proved that if G is any graph and a homeomorphism f on a G-like continuum X has positive topological entropy, then X contains an indecomposable subcontinuum. Moreover, if G is a tree, there is a pair of two distinct points x and y of X such that the pair (x, y) is an IE-pair of f and the irreducible continuum between x and y in X is an indecomposable subcontinuum.

In this note, for any graph G we define a new notion of "free tracing property by free chains" on G-like continua and by use of this notion, we prove that a positive topological entropy homeomorphism on a G-like continuum admits a Cantor set Z such that every tuple of finite points in Z is an IE-tuple of f and Z has the free tracing property by free chains. Also, we prove that the Cantor set Z is related to both the chaotic behaviors of Kerr and Li [18] in dynamical systems and composants of indecomposable continua in topology. Our main result is Theorem 3.3 whose proof is also a new proof of [6]. Also, we study dynamical properties of continuum-wise expansive homeomorphisms. In this case, we obtain more precise results concerning continuum-wise stable sets of chaotic continua and IE-tuples.

2. Definitions and notations

In this note, we assume that all spaces are separable metric spaces and all maps are continuous. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}$ the set of integers.

Let X be a compact metric space and $\mathcal{U}, \mathcal{V}$ be two covers of X. Put

$$\mathcal{U} \vee \mathcal{V} = \{U \cap V | \ U \in \mathcal{U}, V \in \mathcal{V}\}.$$

The quantity $N(\mathcal{U})$ denotes minimal cardinality of subcovers of $\mathcal{U}$. Let $f : X \to X$ be a map and let $\mathcal{U}$ be an open cover of X. Put

$$h(f,\mathcal{U}) = \lim_{n\to\infty} \frac{\log N(\mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \ldots \vee f^{-n+1}(\mathcal{U}))}{n}.$$

The *topological entropy of* f, denoted by $h(f)$, is the supremum of $h(f,\mathcal{U})$ for all open covers $\mathcal{U}$ of X. The reader may refer to [3,4,5,6,8,18,22-25,27,28] for important facts concerning topological entropy. Positive topological entropy of map is one of generally accepted definitions of chaos.

We say that a set $I \subseteq \mathbb{N}$ has *positive density* if

$$\liminf_{n\to\infty} \frac{|I \cap \{1,2,...,n\}|}{n} > 0.$$

Let X be a compact metric space and $f : X \to X$ a map. Let $\mathcal{A}$ be a collection of subsets of X. We say that $\mathcal{A}$ has an *independence set with positive density* if there exists a set $I \subset \mathbb{N}$ with positive density such that for all finite sets $J \subseteq I$, and for all $(Y_j) \in \prod_{j\in J} \mathcal{A}$, we have that

$$\bigcap_{j\in J} f^{-j}(Y_j) \neq \emptyset.$$

We observe a simple but important and and useful fact that if I is an independence set with positive density for $\mathcal{A}$ then for all $k \in \mathbb{Z}$, $k + I$ is an independence set with positive density for $\mathcal{A}$. For convenience, we may assume that I satisfies the condition (kl); for all $(Y_j) \in \prod_{j\in J} \mathcal{A}$ and any $Y_0 \in \mathcal{A}$

$$(kl) \ Y_0 \cap \bigcap_{j\in J} f^{-j}(Y_j) \neq \emptyset.$$

We now recall the definition of IE-tuple. Let $(x_1, \ldots, x_n)$ be a sequence of points in X. We say that $(x_1, \ldots, x_n)$ is an *IE-tuple for* f if whenever $A_1, \ldots, A_n$ are open sets containing $x_1, \ldots, x_n$, respectively, we have that the collection $\mathcal{A} = \{A_1, \ldots, A_n\}$ has an independence set with positive density. In the case that $n = 2$, we use the term IE-pair. We use IE_k to denote the set of all IE-tuples of length k.

Let $f : X \to X$ be a map of a compact metric space X with metric d and let $\delta > 0$. A subset S of X is a *δ-scrambled set* of f if $|S| \geq 2$ and for any $x, y \in S$ with $x \neq y$, then one has

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n\to\infty} d(f^n(x), f^n(y)) \geq \delta.$$

We say that $f : X \to X$ is *Li-Yorke chaotic* if there is an uncountable subset S of X such that for any $x, y \in S$ with $x \neq y$, then one has

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

Also, f has *sensitive dependence on initial conditions* if there is a positive number $c > 0$ such that for any $x \in X$ and any neighborhood U of x, one can find $y \in U$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) \geq c$.

Let X_i $(i \in \mathbb{N})$ be a sequence of compact metric spaces and let $f_{i,i+1} : X_{i+1} \to X_i$ be a map for each $i \in \mathbb{N}$. The *inverse limit* of the inverse sequence $\{X_i, f_{i,i+1}\}_{i=1}^{\infty}$ is the space

$$\varprojlim\{X_i, f_{i,i+1}\} = \{(x_i)_{i=1}^{\infty} \mid x_i = f_{i,i+1}(x_{i+1}) \text{ for each } i \in \mathbb{N}\} \subset \prod_{i=1}^{\infty} X_i$$

which has the topology inherited as a subspace of the product space $\prod_{i=1}^{\infty} X_i$.

If $f : X \to X$ is a map, then we use $\varprojlim(X, f)$ *to denote the inverse limit of* X *with* f *as the bonding maps*, i.e.,

$$\varprojlim(X, f) = \left\{(x_i)_{i=1}^{\infty} \in X^{\mathbb{N}} \mid f(x_{i+1}) = x_i \ (i \in \mathbb{N})\right\}.$$

Let $\sigma_f : \varprojlim(X, f) \to \varprojlim(X, f)$ be the *shift homeomorphism* defined by

$$\sigma_f(x_1, x_2, x_3, ...,) = (x_2, x_3, ...,).$$

A *continuum* is a compact connected metric space. We say that a continuum is *nondegenerate* if it has more than one point. A continuum is *indecomposable* (see [19,20,23,26]) if it is nondegenerate and it is not the union of two proper subcontinua. For any continuum H, the set $c(p)$ of all points of the continuum H, which can be joined with the point p by a proper subcontinuum of H, is said to be the *composant* of the point $p \in H$ (see [20, p.208]). Note that for an indecomposable continuum H, the following are equivalent;

(1) the two points p, q belong to same composant of H;
(2) $c(p) \cap c(q) \neq \emptyset$;
(3) $c(p) = c(q)$.

So, we know that if H is an indecomposable continuum, the family

$$\{c(p) \mid p \in H\}$$

of all composants of H is a family of uncountable mutually disjoint sets $c(p)$ which are connected and dense F_σ-sets in H (see [20, p.212, Theorem 6]). Note that a (nondegenerate) continuum X is indecomposable if and only if there are three distinct points of X such that any subcontinuum of X containing any two points of the three points coincides with X, i.e., X is irreducible between any two points of the three points.

Let H be an indecomposable continuum. We say that a subset Z of H is *vertically embedded* to the composants of H if no two of points of Z belong to the same composant of H, i.e., if x, y are any distinct points of Z and E is any subcontinuum of H containing x and y, then $E = H$.

A map g from X onto G is an *ϵ-map* $(\epsilon > 0)$ if for every $y \in G$, the diameter of $g^{-1}(y)$ is less than ϵ. A continuum X is *G-like* if for every $\epsilon > 0$ there is an ϵ-map from X onto G. For any finite polyhedron G, X is G-like if and only if X is homeomorphic to an inverse limit of an inverse sequence of G. Arc-like continua are those which are G-like for $G = [0, 1]$. Our focus in this article is on G-like continua where G is a graph (= connected 1-dimensional compact polyhedron). A graph G is a *tree* if G contains no simple closed curve. A continuum X is *tree-like* if for any $\epsilon > 0$ there exist a tree G_ϵ and an ϵ-map from X onto G_ϵ. In this case, G_ϵ depends on ϵ. If $\mathcal{G}$ is a collection of subsets of X, then the *nerve* $N(\mathcal{G})$ of $\mathcal{G}$ is the polyhedron whose vertices are elements of $\mathcal{G}$ and there is a simplex $< g_1, g_2, ..., g_k >$ with distinct vertices $g_1, g_2, ..., g_k$ if

$$\bigcap_i g_i \neq \emptyset.$$

In this paper, we consider the only case that nerves are graphs.

If $\{C_1, \ldots, C_n\}$ is a subcollection of $\mathcal{G}$ we call it a *chain* if $C_i \cap C_{i+1} \neq \emptyset$ for $1 \leq i < n$ and $\overline{C_i} \cap \overline{C_j} \neq \emptyset$ implies that $|i - j| \leq 1$. We say that $\{C_1, \ldots, C_n\}$ is a *free chain in* $\mathcal{G}$ if it is a chain and, moreover, for all $1 < i < n$ we have that $C \in \mathcal{G}$ with $\overline{C} \cap \overline{C_i} \neq \emptyset$ implies that $C = C_i$, $C = C_{i-1}$ or $C = C_{i+1}$. By the *mesh* of a finite collection $\mathcal{G}$ of sets, we means the largest of diameters of elements of $\mathcal{G}$. Note that for a graph G, a continuum X is a G-like if and only if for any $\epsilon > 0$, there is a finite open cover $\mathcal{G}$ of X such that $N(\mathcal{G}) = G$ (which means that $N(\mathcal{G})$ and G are homeomorphic) and the mesh of $\mathcal{G}$ is less than ϵ. The Knaster continuum (= Smale's horse shoe) and the pseudo-arc are arc-like continua, solenoids are circle-like continua and Plykin attractors are $(S_1 \vee S_2 \vee \cdots \vee S_m)$-like continua, where $S_1 \vee S_2 \vee \cdots \vee S_m$ $(m \geq 3)$ denotes the one point union of m circles S_i. Such spaces are typical indecomposable continua. The reader may refer to [20] and [26] for standard facts concerning continuum theory.

Let X be a continuum and $m \in \mathbb{N}$. Suppose that A_i $(1 \leq i \leq m)$ are m (nonempty) open sets in X and x_i $(1 \leq i \leq m)$ are m distinct points of X. We identify the order $A_1 \to A_2 \to \cdots \to A_m$ and the converse order $A_m \to A_{m-1} \to \cdots \to A_1$. Then we consider the equivalence class

$$[A_1 \to A_2 \to \cdots \to A_m] = \{A_1 \to A_2 \to \cdots \to A_m; A_m \to A_{m-1} \to \cdots \to A_1\}.$$

Suppose that $\mathcal{G}$ is a finite open cover of X. We say that a chain $\{C_1, \cdots, C_n\} \subseteq \mathcal{G}$ *follows from the pattern* $[A_1 \to A_2 \to \cdots \to A_m]$ if there exist

$$1 \leq k_1 < k_2 < \cdots < k_m \leq n \text{ or } 1 \leq k_m < k_{m-1} < \cdots < k_1 \leq n$$

such that $C_{k_i} \subset A_i$ for each $i = 1, 2, ..., m$. In this case, more precisely we say that the chain $[C_{k_1} \to C_{k_2} \to \cdots \to C_{k_m}]$ follows from the pattern $[A_1 \to A_2 \to \cdots \to A_m]$. Similarly, we say that a chain $\{C_1, \ldots, C_n\} \subseteq \mathcal{G}$ *follows from the pattern* $[x_1 \to x_2 \to \cdots \to x_m]$ if there exist

$$1 \leq k_1 < k_2 < \cdots < k_m \leq n \text{ or } 1 \leq k_m < k_{m-1} < \cdots < k_1 \leq n$$

such that $x_i \in C_{k_i}$ for each $i = 1, 2, ..., m$, where

$$[x_1 \to x_2 \to \cdots \to x_m] = \{x_1 \to x_2 \to \cdots \to x_m;\ x_m \to x_{m-1} \to \cdots \to x_1\}.$$

More precisely, we say that the chain $[C_{k_1} \to C_{k_2} \to \cdots \to C_{k_m}]$ follows from the pattern $[x_1 \to x_2 \to \cdots \to x_m]$.

Let Z be a subset of a G-like continuum X. We say that Z has *the free tracing property by (resp. free) chains* if for any $\epsilon > 0$, any $m \in \mathbb{N}$ and any order $x_1 \to x_2 \to \cdots \to x_m$ of any m distinct points x_i $(i = 1, 2, ..., m)$ of Z, there is an open cover $\mathcal{U}$ of X such that the mesh of $\mathcal{U}$ is less than ϵ, the nerve $N(\mathcal{U})$ of $\mathcal{U}$ is G and there is a (resp. free) chain in $\mathcal{U}$ which follows from the pattern $[x_1 \to x_2 \to \cdots \to x_m]$.

Example 1. (1) Let $X = [0,1]$ be the unit interval and D a subset of X. If $|D| \geq 3$, D does not have the free tracing property by chains.
(2) Let $X = S^1$ be the unit circle and D a subset of X. If $|D| \leq 3$, then D has the free tracing property by free chains. If $|D| \geq 4$, then D does not have the free tracing property by chains.

For the case that X is a tree-like, we obtain the following proposition.

Proposition 2.1. *Let X be a tree-like continuum and let D be a subset of X with $|D| \geq 3$. Then the following are equivalent.*

- *(1) For any order $x_1 \to x_2 \to x_3$ of three distinct points x_i $(i = 1, 2, 3)$ of D and any $\epsilon > 0$, there is an open cover $\mathcal{U}$ of X such that the mesh of $\mathcal{U}$ is less than ϵ, the nerve $N(\mathcal{U})$ of $\mathcal{U}$ is a tree and there is a chain in $\mathcal{U}$ which follows from the pattern $[x_1 \to x_2 \to x_3]$.*
- *(2) D has the free tracing property by chains; for any $\epsilon > 0$, any $m \in \mathbb{N}$ and any order $x_1 \to x_2 \to \cdots \to x_m$ of any m distinct points x_i $(i = 1, 2, ..., m)$ of D, there is an open cover $\mathcal{U}$ of X such that the mesh of $\mathcal{U}$ is less than ϵ, the nerve $N(\mathcal{U})$ of $\mathcal{U}$ is a tree and there is a chain in $\mathcal{U}$ which follows from the pattern $[x_1 \to x_2 \to \cdots \to x_m]$.*
- *(3) The minimal continuum H in X containing D is indecomposable and no two of points of D belong to the same composant of H, i.e., D is vertically embedded to the composants of H.*

3. Topological entropy on G-like continua and Cantor sets which have the free tracing property by free chains

In [3], by use of ergodic theory method, Blanchard, Glasner, Kolyada and Maass proved that if a map $f : X \to X$ of a compact metric space X has positive topological entropy, then there is an uncountable δ-scrambled set of f for some $\delta > 0$ and hence the dynamics (X, f) is Li-Yorke chaotic. In [8], Huang and Ye studied local entropy theory and they gave a characterization of positive topological entropy by use of entropy tuples. Moreover, in [18], by use of local entropy theory (IE-tuples), Kerr and Li proved the following more precise theorem.

Theorem 3.1. ([18, Theorem 3.18]) *Suppose that $f : X \to X$ is a positive topological entropy map on a compact metric space X, and $x_1, x_2, ..., x_m$ $(m \geq 2)$ are finite distinct points of X such that the tuple $(x_1, x_2, ..., x_m)$ is an IE-tuple of f. If A_i $(i = 1, 2, ..., m)$ is any neighborhood of x_i, then there are Cantor sets $Z_i \subset A_i$ such that the following conditions hold;*
(1) *every tuple of finite points in the Cantor set $Z = \cup_i Z_i$ is an IE-tuple;*
(2) *for all $k \in \mathbb{N}$, k distinct points $y_1, y_2, ..., y_k \in Z$ and any points $z_1, z_2, ..., z_k \in Z$, one has*

$$\liminf_{n\to\infty} \max\{d(f^n(y_i), z_i)| \ 1 \leq i \leq k\} = 0.$$

In particular, Z is a δ-scrambled set of f for some $\delta > 0$.

In [6], by use of local entropy theory (IE-tuples), we proved the following theorem.

Theorem 3.2. ([6]) *Suppose that G is any graph and $f : X \to X$ is a homeomorphism on a G-like continuum X with positive topological entropy. Then X contains an indecomposable subcontinuum. Moreover, if G is a tree, there is a pair of two distinct points x and y of X such that the pair (x, y) is an IE-pair of f and the irreducible continuum between x and y in X is an indecomposable subcontinuum.*

The next theorem is the main theorem in this note which is a structure theorem for positive topological entropy homeomorphisms on G-like continua. The theorem implies that for any graph G, a positive topological entropy homeomorphism on a G-like continuum X admits Cantor set Z which yields both some complicated structures in topology and the chaotic behaviors of Kerr and Li [18] in dynamical systems. Especially, the Cantor set Z has the free tracing property by free chains.

Theorem 3.3. *Let G be any graph, X a G-like continuum and $f : X \to X$ a homeomorphism on X with positive topological entropy. Suppose that $x_1, x_2, ..., x_m$ $(m \geq 2)$ are finite distinct points of X such that the tuple $(x_1, x_2, ..., x_m)$ is an IE-tuple of f and A_i $(i = 1, 2, ..., m)$ is any neighborhood of x_i. Then there are Cantor sets $Z_i \subset A_i$ $(i = 1, 2, ..., m)$ and an indecomposable subcontinuum H of X such that the following conditions hold;*
(1) H is the unique minimal subcontinuum in X containing $Z = \cup_{i=1}^m Z_i$ and the Cantor set Z is vertically embedded to the composants of H; i.e., if x, y are distinct points of Z, then the irreducible continuum $Ir(x, y; H)$ between x and y in H is H,
(2) Z has the free tracing property by free chains; for any $k \in \mathbb{N}$ and any order $x_1 \to x_2 \to \cdots \to x_k$ of k distinct points x_i $(i = 1, 2, ..., k)$ of Z and any $\epsilon > 0$, there is an open cover $\mathcal{U}$ of X such that the mesh of $\mathcal{U}$ is less than ϵ, the nerve $N(\mathcal{U})$ of $\mathcal{U}$ is G and there is a free chain in $\mathcal{U}$ which follows from the pattern $[x_1 \to x_2 \to \cdots \to x_k]$,
(3) every tuple of finite points in the Cantor set Z is an IE-tuple of f, and
(4) for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, ..., y_k \in Z$ and any points $z_1, z_2, ..., z_k \in Z$, the following condition holds

$$\liminf_{n \to \infty} \max\{d(f^n(y_i), z_i) | \ 1 \leq i \leq k\} = 0.$$

In particular, Z is a δ-scrambled set of f for some $\delta > 0$.

In the statement of Theorem 3.3, we need the condition that X is a G-like continuum for a graph G.

Example 2. Let $g : Z \to Z$ be a homeomorphism on a Cantor set Z which has positive topological entropy. Let $X = \text{Cone}(Z)$ be the cone of Z and let $f : X \to X$ be a homeomorphism which is the natural extension of g. Then $h(f) > 0$ and X is tree-like, but X is not G-like for any graph G. Note that X contains no indecomposable subcontinuum. Also, if D is a subset with $|D| \geq 3$, then D does not have the free tracing property by chains.

Example 3 (Boronski and Oprocha [29]). There is a map $f : I = [0, 1] \to I$ such that the shift homeomorphism

$$\sigma_f : \varprojlim(I, f) \to \varprojlim(I, f)$$

of f is Li-Yorke chaotic for some $\delta > 0$ and the inverse limit $\varprojlim(I, f)$ is Suslinean. In particular, $\varprojlim(I, f)$ contains no indecomposable continua. Of course, $h(f) = 0$.

We will freely use the following facts from the local entropy theory.

Proposition 3.4. ([18, Propositions, 3.8, 3.9]) *Let X be a compact metric space and let $f : X \to X$ be a map.*

(1) Let $(A_1, \dots, A_k)$ be a tuple of closed subsets of X which has an independent set of positive density. Then, there is an IE-tuple $(x_1, \dots, x_k)$ with $x_i \in A_i$ for $1 \leq i \leq k$.
(2) $h(f) > 0$ if and only if f has an IE-pair (x_1, x_2) with $x_1 \neq x_2$.
(3) IE_k is closed and $f \times \dots \times f$ invariant subset of X^k.
(4) If $(A_1, \dots, A_k)$ has an independence set with positive density and, for $1 \leq i \leq k$, $\mathcal{A}_i$ is a finite collection of sets such that $A_i \subseteq \cup \mathcal{A}_i$, then there is $A'_i \in \mathcal{A}_i$ such that $(A'_1, \dots, A'_k)$ has an independence set with positive density.

To prove Theorem 3.3, we need the following results.

Proposition 3.5. ([6, Proposition 3.1]) *Let $I \subseteq \mathbb{N}$ be a set with positive density and $n \in \mathbb{N}$. Then, there is a finite set $F \subseteq I$ with $|F| = n$ and a positive density set B such that $F + B \subseteq I$.*

Proposition 3.6. ([6, Proposition 3.2]) *Let X be a compact metric space and let $f : X \to X$ be a map. Let $\mathcal{A}$ be a collection which has an independence set with positive density and $n \in \mathbb{N}$. Then, there is a finite set F with $|F| = n$ such that*

$$\mathcal{A}_F = \{\bigcap_{i \in F} f^{-i}(Y_i) : Y_i \in \mathcal{A}\}$$

has an is an independence set with positive density.

Let $m \geq 2$ and let $\{1, 2, ..., m\}^n$ be the set of all functions from $\{1, 2, ..., n\}$ to $\{1, 2, ..., m\}$. For $\sigma \in \{1, 2, ..., m\}^n$ $(m \geq 2)$, we write $\sigma = (\sigma(1), \sigma(2), ...\sigma(n))$, where $\sigma(i) \in \{1, 2, ..., m\}$. Note that $|\{1, 2, ..., m\}^n| = m^n$.

Proposition 3.7. (cf. [6, Proposition 3.3]) *Let $m, n \in \mathbb{N}$, and $\sigma_1, \dots, \sigma_{[(m-1)n+1][(m-1)^n+1]}$ be any sequence of distinct elements of $\{1, ..., m\}^n$. Then there are $1 \leq i \leq n$ and*

$$1 \leq k_1 < k_2 < k_3 < \dots < k_m \leq [(m-1)n + 1][(m-1)^n + 1]$$

such that $\sigma_{k_j}(i) = j$ for $j = 1, ..., m$.

To check the chaotic behaviors of Kerr and Li ([18, Theorem 3.18]), we need the following lemma.

Lemma 3.8. *Let $f : X \to X$ be a map of a compact metric space X. Suppose that $(A_1, \dots, A_k)$ is a tuple of closed subsets of X which has an independent set of positive density. Then there is a tuple $(A'_1, \dots, A'_k)$ of closed subsets of X which has an independent set with positive density such that $A'_j \subset A_j$ $(j = 1, 2, ..., k)$, and if $h : \{1, 2, ..., k\} \to \{1, 2, ..., k\}$ is any function, then there is $n_h \in \mathbb{N}$ such that $f^{n_h}(A'_j) \subset A_{h(j)}$ for each $j = 1, 2, ..., k$.*

Proposition 3.9. *Let X be a G-like continuum for a graph G and let T be a Cantor set in X.*
(1) *Suppose that T has the free tracing property by chains. Then any minimal continuum H in X containing T is indecomposable and there is $s \in \mathbb{N}$ such that for any composant c of H, $|c \cap T| \leq s$. In particular, no infinite points of T belong to the same composant of H. Also, there is a subset Z of T such that Z is a Cantor*

set and Z is vertically embedded to the composants of H.
(2) Moreover, if T has the free tracing property by free chains, then there is the unique minimal continuum H in X containing T and T is itself vertically embedded to the composants of H.

The following lemma is the key lemma to prove the main theorem.

Lemma 3.10. *Let G be a graph and let f be a homeomorphism on a G-like continuum X with positive topological entropy. Suppose that $\mathcal{A}$ is a finite open collection of X which has an independence set of f with positive density, any distinct elements of $\mathcal{A}$ are disjoint, and $|\mathcal{A}| = m \geq 2$. Then for any $\epsilon > 0$ and any order $A_1 \to A_2 \to \cdots \to A_m$ of all elements of $\mathcal{A}$, there exists a finite open cover $\mathcal{V}$ of X satisfying the following conditions;*
(1) the mesh of $\mathcal{V}$ is less than ϵ,
(2) the nerve $N(\mathcal{V})$ of $\mathcal{V}$ is G,
(3) for each $A \in \mathcal{A}$ there is a shrink $s(A) \in \mathcal{V}$ with $s(A) \subset A$ such that

$$s(\mathcal{A}) = \{s(A) |\ A \in \mathcal{A}\}$$

has an independence set with positive density, and
(4) there is a free chain $[s(A_1) \to s(A_2) \to \cdots \to s(A_m)]$ from $s(A_1)$ to $s(A_m)$ in $\mathcal{V}$ which follows from the pattern $[A_1 \to A_2 \to \cdots \to A_m]$.

As a corollary of Theorem 3.3, we have the following results.

Corollary 3.11. *Let G be any graph. If $f : Y \to Y$ is a positive entropy map on a G-like continuum Y, then there exist an indecomposable subcontinuum H of $X = \varprojlim(Y, f)$ and a Cantor set Z in H satisfies the following conditions;*
(1) Z is vertically embedded to the composants of H,
(2) Z has the free tracing property by free chains,
(3) every tuple of finite points in the Cantor set Z is an IE-tuple of the shift map σ_f and
(4) for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, ..., y_k \in Z$ and any points $z_1, z_2, ..., z_k \in Z$, the following condition holds

$$\liminf_{n\to\infty} \max\{d(\sigma_f^n(y_i), z_i) |\ 1 \leq i \leq k\} = 0.$$

In particular, Z is a δ-scrambled set of σ_f for some $\delta > 0$.

For a special case, we have the following.

Corollary 3.12. *Let X be one of the Knaster continuum, solenoids or Plykin attractors. If f is any positive topological entropy homeomorphism on X, then there is a Cantor set Z in X such that the Cantor set Z satisfies the following conditions;*
(1) Z is vertically embedded to the composants of X,
(2) Z has the free tracing property by free chains,
(3) every tuple of finite points in the Cantor set Z is an IE-tuple of f, and
(4) for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, ..., y_k \in Z$ and any points $z_1, z_2, ..., z_k \in Z$, the following condition holds

$$\liminf_{n\to\infty} \max\{d(f^n(y_i), z_i) |\ 1 \leq i \leq k\} = 0.$$

In particular, Z is a δ-scrambled set of f for some $\delta > 0$.

An onto map $f : X \to Y$ of continua is *monotone* if for any $y \in Y$, $f^{-1}(y)$ is connected. In [16], we proved that if G is a graph and $f : X \to X$ is a monotone map on a G-like continuum X which has positive topological entropy, then X contains an indecomposable subcontinuum. Here we give the following more precise result.

Theorem 3.13. *Suppose that G is a graph and X is a G-like continuum. If $f : X \to X$ is a monotone map on X with positive topological entropy, then there exist an indecomposable subcontinuum H of X and a Cantor set Z in H such that the Cantor set Z satisfies the following conditions;*

(1) *Z is vertically embedded to the composants of H,*

(2) *every tuple of finite points in the Cantor set Z is an IE-tuple of f,*

(3) *for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, ..., y_k \in Z$ and any points $z_1, z_2, ..., z_k \in Z$, the following condition holds*

$$\liminf_{n\to\infty} \max\{d(f^n(y_i), z_i)|\ 1 \le i \le k\} = 0.$$

In particular, Z is a δ-scrambled set of f for some $\delta > 0$.

A continuum E is an *n-od* ($2 \le n < \infty$) if E contains a subcontinuum A such that the complement of A in E is the union n nonempty mutually separated sets, i.e.,

$$E - A = \bigcup\{E_i|\ i = 1, 2, ..., n\}$$

for some subsets E_i satisfying the condition:

$$\overline{E_i} \cap E_j = \emptyset\ (i \ne j).$$

For any continuum X, let

$$T(X) = \sup\{n \mid \text{there is an } n\text{-od in } X\}.$$

Note that if X is a G-like continuum for a graph G, then $T(X) < \infty$.

To prove Theorem 3.13, we need the following lemma.

Lemma 3.14. (cf. [16, Lemma 2.3]) *Let X and Y be continua with $T(X) < \infty$. Suppose that $f : X \to Y$ is an (onto) monotone map, H' is an indecomposable subcontinuum of X and Z' is a Cantor set which is vertically embedded to the composants of H'. If $H = f(H')$ is nondegenerate, then H is an indecomposable subcontinuum of Y and there is a subset Z of $f(Z')$ such that Z is a Cantor set and Z is vertically embedded to the composants of H.*

4. Chaotic continua of continuum-wise expansive homeomorphisms and IE-tuples

In this section, we study dynamical behaviors of continuum-wise expansive homeomorphisms related to IE-tuples and chaotic continua in topology. Any continuum-wise expansive homeomorphism f on a continuum X has positive topological entropy and hence f has IE-tuples (see Theorem 4.1 below). Also, X contains a chaotic continuum and chaotic continuum has uncountable mutually disjoint (unstable or) stable dense connected F_σ-sets (see Theorem 4.1). In this section, we study some precise results of IE-tuples related to (unstable) stable connected sets of chaotic continua and composants of indecomposable continua.

A homeomorphism $f : X \to X$ of a compact metric space X with metric d is called *expansive* ([5,13]) if there is $c > 0$ such that for any $x, y \in X$ and $x \ne y$, then there is an integer $n \in \mathbb{Z}$ such that

$$d(f^n(x), f^n(y)) > c.$$

A homeomorphism $f : X \to X$ of a compact metric space X is *continuum-wise expansive* (resp. *positively continuum-wise expansive*) [15] if there is $c > 0$ such that if A is a nondegenerate subcontinuum of X, then there is an integer $n \in \mathbb{Z}$ (resp. a positive integer $n \in \mathbb{N}$) such that

$$\operatorname{diam} f^n(A) > c,$$

where $\operatorname{diam} B = \sup\{d(x,y) | \, x, y \in B\}$ for a set B. Such a positive number c is called an *expansive constant* for f. Note that each expansive homeomorphism is continuum-wise expansive, but the converse assertion is not true. There are many continuum-wise expansive homeomorphisms which are not expansive (see [15]). These notions have been extensively studied in the area of topological dynamics, ergodic theory and continuum theory (see [5,10-15,27]).

The hyperspace 2^X of X is the set of all nonempty closed subsets of X with the Hausdorff metric d_H. Let

$$C(X) = \{A \in 2^X | \; A \text{ is connected}\}.$$

Note that 2^X and $C(X)$ are compact metric spaces (e.g., see [20] and [26]). For a homeomorphism $f : X \to X$ and for each closed subset H of X and $x \in H$, the *continuum-wise σ-stable sets* $V^\sigma(x; H)$ $(\sigma = s, u)$ of f are defined as follows:

$$V^s(x; H) = \{y \in H | \text{ there is } A \in C(H) \text{ such that } x, y \in A \text{ and } \lim_{n\to\infty} \operatorname{diam} f^n(A) = 0\},$$

$$V^u(x; H) = \{y \in H | \text{ there is } A \in C(H) \text{ such that } x, y \in A \text{ and } \lim_{n\to\infty} \operatorname{diam} f^{-n}(A) = 0\}.$$

Note that

$$V^s(x; H) \subset W^s(x) = \{y \in X | \lim_{n\to\infty} d(f^n(y), f^n(x)) = 0\},$$

$$V^u(x; H) \subset W^u(x) = \{y \in X | \lim_{n\to\infty} d(f^{-n}(y), f^{-n}(x)) = 0\}.$$

A subcontinuum H of X is called a *σ-chaotic continuum* (see [13]) of f (where $\sigma = s, u$) if

(1) for each $x \in H$, $V^\sigma(x; H)$ is dense in H, and
(2) there is $\tau > 0$ such that for each $x \in H$ and each neighborhood U of x in X, there is $y \in U \cap H$ such that

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) \geq \tau \text{ in case } \sigma = s, \text{ or}$$

$$\liminf_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) \geq \tau \text{ in case } \sigma = u.$$

We know that if $f : X \to X$ is a continuum-wise expansive homeomorphism, then $V^\sigma(z; H)$ is a connected F_σ-set containing z. If H is a σ-chaotic continuum of f, then the decomposition $\{V^\sigma(z; H) | \, z \in H\}$ of H is an uncountable family of mutually disjoint, dense connected F_σ-sets in H. Note that σ-chaotic continua of f have very similar structures of composants of indecomposable continua. In fact, for the case of 1-dimensional continua, σ-chaotic continua may be indecomposable (see [10]).

Example 4. Let $f : T^2 \to T^2$ be an Anosov diffeomorphism on the 2-dimensional torus T^2, say

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Then f is expansive and T^2 itself is a σ-chaotic continuum of f for $\sigma = u, s$. Note that T^2 contains no indecomposable σ-chaotic subcontinuum.

For continuum-wise expansive homeomorphisms, we have obtained the following results (see [11,13,15]).

Theorem 4.1. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism on a continuum X. Then the followings hold.*

(1) ([15, Theorem 4.1]) *f has positive topological entropy and hence there are IE-tuples.*

(2) ([13, Theorem 3.6 and Theorem 4.1]) *There is a σ-chaotic continuum H of f. Moreover, if H is a u-chaotic continuum (resp. s-chaotic continuum), then there exists a Cantor set Z in H satisfying the conditions;*
(i) no two of points of Z belong to the same $V^u(x; H)$ $(x \in X)$ (resp. $V^s(x; H)$ $(x \in X)$), i.e., Z is vertically embedded to $V^\sigma(x, H)$ $(x \in H)$,
(ii) Z is a δ-scrambled set of f^{-1} for some $\delta > 0$ (resp. f).

(3) ([11, Theorem 2.4]) *Moreover, if $f : X \to X$ is a positively continuum-wise expansive homeomorphism, then X contains a u-chaotic continuum H such that H is indecomposable and the set of composants of H coincides to $\{V^u(x; H) |\ x \in H\}$. Also, there exists a Cantor set Z in H satisfying the conditions;*
(i) Z is vertically embedded to the composants $V^u(x, H)$ $(x \in H)$,
(ii) Z is a δ-scrambled set of f^{-1} for some $\delta > 0$.

(4) (*d*) ([11, Corollary 2.7]) *Moreover, if G is any graph and X is a G-like continuum, then X contains a σ-chaotic continuum H such that H is indecomposable and the set of composants of H coincides to $\{V^\sigma(x; H) |\ x \in H\}$. Moreover if $\sigma = u$ (resp. s), then there exists a Cantor set Z in H satisfying the conditions;*
(i) Z is vertically embedded to $V^\sigma(x, H)$ $(x \in H)$,
(ii) Z is a δ-scrambled set of f^{-1} for some $\delta > 0$ (resp. f).

(5) ([11, Theorem 2.6]) *Moreover, if X is a continuum in the plane $\mathbb{R}^2$, then X contains a σ-chaotic continuum H of f such that H is indecomposable and the set of composants of H coincides to $\{V^\sigma(x, H) |\ x \in H\}$. Moreover if $\sigma = u$ (resp. s), then there exists a Cantor set Z in H satisfying the conditions;*
(i) Z is vertically embedded to $V^\sigma(x, H)$ $(x \in H)$,
(ii) Z is a δ-scrambled set of f^{-1} for some $\delta > 0$ (resp. f).

We consider the case that σ-chaotic continua are periodic. By combining Theorem 3.1 and Theorem 4.1, we have the following results.

Corollary 4.2. *Let $f : X \to X$ be a continuum-wise expansive homeomorphism on a continuum X. Suppose that X contains a periodic σ-chaotic continuum H of f. Then there exists a Cantor set Z in H such that if $\sigma = u$ (resp. $\sigma = s$), then the following conditions hold;*
(1) *Z is vertically embedded to $V^\sigma(x, H)$ $(x \in H)$,*
(2) *every tuple of finite points in the Cantor set Z is an IE-tuple of f^{-1} (resp. f), and*
(3) *for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, ..., y_k \in Z$ and any points $z_1, z_2, ..., z_k \in$*

Z, *the following condition holds*

$$\liminf_{n\to\infty} \max\{d(f^{-n}(y_i), z_i)|\ 1 \le i \le k\} = 0$$

$$(resp. \liminf_{n\to\infty} \max\{d(f^{n}(y_i), z_i)|\ 1 \le i \le k\} = 0).$$

Similarly, we have the following result.

Corollary 4.3. *Suppose that $f : X \to X$ is a positively continuum-wise expansive homeomorphism on a continuum X such that X has a periodic u-chaotic continuum H which is indecomposable and the set of composants of H coincides to $\{V^u(x; H)|\ x \in H\}$. Then there exists a Cantor set Z in H which is vertically embedded to the composants of H and satisfies the conditions;*
(1) *if x, y belong to the same composant of H, then $\lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0$,*
(2) *every tuple of finite points in the Cantor set Z is an IE-tuple of f^{-1}, and*
(3) *for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, ..., y_k \in Z$ and any points $z_1, z_2, ..., z_k \in Z$, the following condition holds*

$$\liminf_{n\to\infty} \max\{d(f^{-n}(y_i), z_i)|\ 1 \le i \le k\} = 0.$$

For special cases, we have the following.

Corollary 4.4. *Suppose that X is one of the Knaster continuum, Plykin attractors or solenoids. If $f : X \to X$ is a continuum-wise expansive homeomorphism on X, then f or f^{-1} is positively continuum-wise expansive. In particular, if f is positively continuum-wise expansive, then there exists a Cantor set Z in X such that the Cantor set Z is vertically embedded to the composants of X and satisfies the conditions;*
(1) *if x, y belong to the same composant of X, then $\lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0$,*
(2) *every tuple of finite points in the Cantor set Z is an IE-tuple of f^{-1},*
(3) *Z has the free tracing property by free chains, and*
(4) *for all $k \in \mathbb{N}$, any distinct k points $y_1, y_2, ..., y_k \in Z$ and any points $z_1, z_2, ..., z_k \in Z$, the following condition holds*

$$\liminf_{n\to\infty} \max\{d(f^{-n}(y_i), z_i)|\ 1 \le i \le k\} = 0.$$

For the case of the shift map $\sigma_f : \varprojlim(G, f) \to \varprojlim(G, f)$ of a map $f : G \to G$ on a graph G which has sensitive dependence on initial conditions, we can find a periodic indecomposable s-chaotic continuum in $\varprojlim(G, f)$. Hence we have the following corollary.

Corollary 4.5. *Suppose that $f : G \to G$ is a map on a graph G which has sensitive dependence on initial conditions and $\sigma_f : X = \varprojlim(G, f) \to X$ is the shift map of f. Then there exists an indecomposable s-chaotic continuum H in X such that $\sigma_f^n(H) = H$ for some $n \in \mathbb{N}$ and the set of composants of H coincide to $\{V^s(x; H)|x \in H\}$. Hence there is a Cantor set Z in H such that Z is vertically embedded to the composants of H and satisfies the conditions;*
(1) *if x, y belong to the same composant of H, then $\lim_{n\to\infty} d(\sigma_f^n(x), (\sigma_f^n(y)) = 0$,*
(2) *every tuple of finite points in the Cantor set Z is an IE-tuple of σ_f,*
(3) *Z has the free tracing property by free chains, and*

(4) *for all* $k \in \mathbb{N}$*, any distinct* k *points* $y_1, y_2, ..., y_k \in Z$ *and any points* $z_1, z_2, ..., z_k \in Z$*, the following condition holds*

$$\liminf_{n\to\infty} \max\{d(\sigma_f^n(y_i), z_i) |\ 1 \leq i \leq k\} = 0.$$

Example 5. Let $f : I = [0,1] \to I$ be the map defined by $f(t) = 4t(1-t)$ ($t \in I$). Note that f has sensitive dependence on initial conditions and $\varprojlim(I, f)$ is the Knaster continuum. Then $\varprojlim(I, f)$ is the s-chaotic continuum of the shift homeomorphism $\sigma_f : \varprojlim(I, f) \to \varprojlim(I, f)$ satisfying the conditions of Corollary 4.5, where $H = \varprojlim(I, f)$.

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