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SOME RECENT RESULTS ON EXTENSIONS AND QUASI-HOMOMORPHISMS OF TOPOLOGICAL ABELIAN GROUPS

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To the memory of Paweł Domański (1959-2016)

ABSTRACT. The present paper is an enlarged version of a talk given at the RIMS Meeting on Set Theoretic and Geometric Topology held in Kyoto University from June 12 to June 14, 2017. Its goal, as well of that of the talk it grew from, is to give a motivation for the theory of extensions of topological abelian groups, including some recently published results. It has no pretenses at completeness, but on the way to presenting the main theorems we touch on several topics (as those of quasi-homomorphisms, cross sections or three-space problems) that arise naturally in connection with this subject.

1. EXTENSIONS

1.1. **Algebraic theory.** (All groups in this paper are abelian.) There are several ways to look at group extensions, even from a purely algebraic setting and with no homological background to start with. One can naturally arrive to this concept e. g. from splitting problems, that is, those dealing with conditions on the group X, its subgroup H, or both, which guarantee that H is a direct summand of X. For many purposes it is accurate enough to say that an abelian group X is an extension of the abelian group G by the abelian group H if H can be embedded in X in such a way that the corresponding quotient group X/H is isomorphic to G. It soon becomes evident, though, that the right definition must feature not only these objects, but also the morphisms linking them together.

Definition 1.1. Let G and H be abelian groups. An *extension* of G by H is a short exact sequence of groups and homomorphisms

$$0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$$

where X is an abelian group and 0 denotes a one-element group. In other words, $\imath$ is injective, π is onto, and $\imath(H) = \mathrm{Ker}\pi$.

Definition 1.2. Let G and H be abelian groups. Let $E_j : 0 \to H \xrightarrow{\imath_j} X_j \xrightarrow{\pi_j} G \to 0$ $(j = 1, 2)$ be two extensions of G by H. We say that E_1 and E_2 are *equivalent* if there exists an isomorphism $T : X_1 \to X_2$ for which $T \circ \imath_1 = \imath_2$ and $\pi_2 \circ T = \pi_1$.

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$$\begin{tikzcd}[column sep=small]
 & & X_1 \arrow[dd, "T"] \arrow[dr, "\pi_1"] & & \\
0 \arrow[r] & H \arrow[ur, "\imath_1"] \arrow[dr, "\imath_2"'] & & G \arrow[r] & 0 \\
 & & X_2 \arrow[ur, "\pi_2"'] & &
\end{tikzcd}$$

It is an easy consequence of the Five Lemma that any group homomorphism $T : X_1 \to X_2$ making the above diagram commutative is actually an isomorphism.

Definition 1.3. An extension of abelian groups $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ *splits algebraically* if it is equivalent to the trivial extension $0 \to H \xrightarrow{\imath_H} H \times G \xrightarrow{\pi_G} G \to 0$. Here $\imath_H$ (resp. π_G) is the canonical inclusion into the product $H \times G$ (resp. the projection onto G), that is, $\imath_H(h) = (h, 0)$ and $\pi_G(h, g) = g$ for every $h \in H, g \in G$.

The following Proposition reduces the concept of a splitting extension to less technical conditions. The proof is not difficult.

Proposition 1.4. *Let $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ be an extension of abelian groups. The following conditions are equivalent:*

(i) *E splits algebraically.*
(ii) *There exists a homomorphism $P : X \to H$ with $P \circ \imath = id_H$.*
(iii) *There exists a homomorphism $S : G \to X$ with $\pi \circ S = id_G$.*

In what follows $\mathbb{T}$ will denote the quotient group $\mathbb{R}/\mathbb{Z}$. A natural example of a non-splitting extension of abelian groups is $0 \to \mathbb{Z} \xrightarrow{\imath} \mathbb{R} \xrightarrow{\pi} \mathbb{T} \to 0$ where $\imath$ is the inclusion and π is the corresponding quotient mapping.

It is quite remarkable (although of course it is hardly news for anyone who knows the basics of abelian group theory) that both the class of free abelian groups and that of divisible abelian groups can be characterized by their behaviour with respect to splitting extensions:

Theorem 1.5. (a) *Let H be an abelian group. Then H is divisible if and only if every extension of abelian groups of the form $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ splits algebraically.*
(b) *Let G be an abelian group. Then G is free (that is, $G \cong \mathbb{Z}^{(I)}$ for some index set I) if and only if every extension of abelian groups of the form $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ splits algebraically.*

1.2. **Extensions of topological abelian groups.** As we will see next, the topological-group counterparts of the notions and general principles just introduced come across as quite natural.

Definition 1.6. Let G and H be topological abelian groups. An *extension of topological abelian groups*, or briefly a *topological extension* of G by H is a short exact sequence

$$0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$$

where X is a topological abelian group and the arrows represent relatively open, continuous homomorphisms.

In other words, $\imath$ is an embedding, π is a quotient mapping, and $\imath(H) = \mathrm{Ker}\pi$.

With the notations of Definition 1.6, note that the embedded copy of H in X is necessarily a closed subgroup because it is the kernel of a continuous homomorphism.

Definition 1.7. Let G and H be topological abelian groups. Let $E_j : 0 \to H \xrightarrow{\imath_j} X_j \xrightarrow{\pi_j} G \to 0$ $(j = 1, 2)$ be two topological extensions of G by H. We say that E_1 and E_2 are *equivalent* if there exists a topological isomorphism $T : X_1 \to X_2$ for which $T \circ \imath_1 = \imath_2$ and $\pi_2 \circ T = \pi_1$.

With the notations of Definition 1.7, note that any continuous group homomorphism $T : X_1 \to X_2$ satisfying $T \circ \imath_1 = \imath_2$ and $\pi_2 \circ T = \pi_1$ is already a topological isomorphism. This follows from the corresponding, above discussed algebraic property, and Merzon's Lemma [11, Lemma 1].

Definition 1.8. An extension of abelian groups $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ *splits* if it is equivalent to the trivial extension $0 \to H \xrightarrow{\imath_H} H \times G \xrightarrow{\pi_G} G \to 0$, where $\imath_H$ and π_G are as in Definition 1.3 and $H \times G$ carries the product topology.

The following Proposition follows from Proposition 1.4 and elementary considerations concerning continuity:

Proposition 1.9. *Let $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ be an extension of topological abelian groups. The following conditions are equivalent:*

(i) *E splits.*
(ii) *There exists a continuous homomorphism $P : X \to H$ with $P \circ \imath = id_H$.*
(iii) *There exists a continuous homomorphism $S : G \to X$ with $\pi \circ S = id_G$.*

Item (ii) in Proposition 1.9 is a well known characterization of the fact that the subgroup $\imath(H)$ splits topologically from X, that is, there is a closed subgroup $Y \leq X$ such that $[(x, y) \in \imath(H) \times Y \mapsto x + y \in X]$ is a topological isomorphism (see Theorem 6.6 in [2]).

In what follows, if there is no risk of ambiguity, the term "extension" will be used to denote an extension of topological abelian groups.

2. Groups G and H for which $\mathrm{Ext}(G, H) = 0$)

We are interested in finding necessary and/or sufficient conditions on the topological abelian groups G and H for every extension of G by H to split. This would mean that there is a unique way (the trivial one) to embed H as a closed subgroup of another topological abelian group with the property that the corresponding quotient is topologically isomorphic to G.

Definition 2.1. Given two topological abelian groups G and H, we write $\mathrm{Ext}(G, H) = 0$ if every extension of the form $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ splits. If this is not the case we will write $\mathrm{Ext}(G, H) \neq 0$.

Behind these notations there lies the fact that one can define an operation between (classes of equivalences of) extensions, which gives rise to a group structure, and the situation where only trivial extensions are available corresponds to this group being trivial. This is a well known notion in homological algebra. We refer the interested reader to [4] for the definition and some general properties of the group Ext, which we are not going to examine here.

A first nontrivial example where only the trivial extension exists follows:

Proposition 2.2. *If G is a locally compact abelian group and H is either $\mathbb{R}$ or $\mathbb{T}$ then* $\mathrm{Ext}(G, H) = 0$.

Proof. Fix an extension $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ where $H = \mathbb{R}$ (resp. $H = \mathbb{T}$). Since both H and G are locally compact, so is X [12, Lemma 7.2.4]. Since $\imath(H)$ is topologically isomorphic to $\mathbb{R}$ (resp. to $\mathbb{T}$) it splits from the locally compact abelian group X [2, Theorem 6.16]. By Proposition 1.9, the extension splits. □

There are many examples of extensions $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ featuring locally compact abelian groups, which split algebraically but not topologically; see for instance [2, Examples 6.17 and 6.19] or Proposition 3.5 below.

Also, Proposition 2.2 is no longer true without the hypothesis of local compactness of G. For a counterexample we pick the following notable result, which was independently proved by N. J. Kalton, M. Ribe and J. W. Roberts in the late 70's:

Theorem 2.3. [15, 19, 20] $\mathrm{Ext}(\ell_1, \mathbb{R}) \neq 0$

(Note that by Theorem 1.5(a), any extension with $\mathbb{R}$ on the left end splits algebraically.) Here ℓ_1 stands for the topological group underlying the Banach space of all summable sequences of real numbers, with the norm $\|(x_n)\|_1 = \sum |x_n|$. Of course the original result pertains to the theory of metric linear spaces; actually it answered in the negative the by then long-standing three-space problem for local convexity. Thus its proofs can be boiled down to the construction of a short exact sequence $0 \to \mathbb{R} \to X \to \ell_1 \to 0$ of complete metric linear spaces and continuous, relatively open linear mappings such that the corresponding embedded copy of $\mathbb{R}$ is not complemented in X. It is easy to convince oneself that this same extension provides a proof for Theorem 2.3, which is a statement about topological abelian groups and continuous homomorphisms.

It might be interesting to compare Theorem 2.3 with the following result of F. Cabello:

Theorem 2.4. [9, Theorem 1(b)] $\mathrm{Ext}(G, H) = 0$ *whenever G is either $\mathbb{R}$ or $\mathbb{T}$ and H is (the additive topological group underlying) a Banach space.*

At this point one might wonder how the property $\mathrm{Ext}(G, H) = 0$ behaves with respect to the formation of subgroups, completions, products and other operations in the variety of topological abelian groups. A comprehensive account would probably cause us to lose the plot; we refer the reader to [4] for details. We do include next two results concerning quotients which we are going to need in what follows.

The following result was proved in [5] (Theorem 21) for $H = \mathbb{T}$ but the same proof works in the general case:

Theorem 2.5. *Let G and H be topological abelian groups and let $M \leq G$ be a closed subgroup of G.*

(a) *If $\mathrm{Ext}(G/M, H) = 0$ then every continuous homomorphism of M to H extends to a continuous homomorphism from G to H.*
(b) *If every continuous homomorphism of M to H extends to a continuous homomorphism from G to H and $\mathrm{Ext}(G, H) = 0$ then $\mathrm{Ext}(G/M, H) = 0$ as well.*

Item (a) of Theorem 2.5 is a source for examples of non-splitting extensions. The reader who is familiar with duality theory of topological abelian groups will probably want to consider its particularization for $H = \mathbb{T}$: *If M is a closed, not dually embedded subgroup of G then* $\mathrm{Ext}(G/M, \mathbb{T}) \neq 0$. (The definitions of basic duality concepts such that of a dually embedded subgroup, and some nontrivial sufficient conditions as well as examples of subgroups lacking this property can be found in [3]).

Theorem 2.5(b) gives a sufficient condition for the property $\mathrm{Ext}(\cdot, H) = 0$ to remain invariant under a quotient mapping. Results which go the other way around (from the quotient(s) to the group) are much harder to come by, and need much more restrictive assumptions. To make sense of the following theorem, note that a subgroup P of a topological abelian group G is said to be *admissible* if G/P admits a weaker metrizable group topology, and that we call a family of admissible subgroups *cofinal* if every admissible subgroup of G contains one of its members.

Theorem 2.6. [6, Theorem 3.5] *If G and H are topological abelian groups, H is metrizable and locally compact and* $\mathrm{Ext}(G/P, H) = 0$ *for any P in a cofinal family of admissible subgroups of G then* $\mathrm{Ext}(G, H) = 0$.

3. Cross sections

We have seen (Proposition 1.9) that a given extension can be shown to split by finding a continuous homomorphism that is a right inverse for its quotient mapping. However, in many cases a weaker version of this property will already have meaningful consequences. Thus it makes sense to introduce the following general notion:

Definition 3.1. Let $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ be an extension of topological abelian groups. A *cross section* for E is any mapping $\rho : G \to X$ which satisfies $\pi \circ \rho = id_G$. We will always assume that $\rho(0) = 0$.

For instance, we know that any extension admitting a cross section which is a homomorphism splits algebraically (Proposition 1.4). In general we are more interested in keeping continuity (which can be global, local or just at one point) even if we lose additivity. The following result is more or less known; its proof can be found e. g. in [5, Proposition 31]:

Proposition 3.2. *Let $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ be an extension of topological abelian groups. If X is metrizable then E admits a cross section which is continuous at zero.*

Note that metrizability is a three-space property, i. e. if both G and H in the extension $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ are metrizable, then so is X. This property is often invoked when (as it is the case in the problems we are dealing with here) our input is the groups at both ends of the extension rather than the one on the middle.

When trying to choose representatives of the classes making up a given quotient in a continuous fashion, is natural to turn to Michael's selection theorems. The following theorem, which is far from being exhaustive, contains three examples of applications of Michael's results and some of its known corollaries to our setting:

Theorem 3.3. *Let $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ be an extension of topological abelian groups.*

(i) [18, Theorem 2] *If X is metrizable and G is paracompact and zero-dimensional then $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ has a continuous cross section.*
(ii) [8, Proposition 7.1] *If $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ is an extension of complete metric linear spaces and H is locally convex, then E has a continuous cross section.*
(iii) [17, Corollary 1.3] *If X is metrizable, H is complete and G is zero-dimensional then $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ has a cross section which is continuous on a neighborhood of zero.*

Note that Theorem 3.3(ii) implies in particular that the nonsplitting extension witnessing $\mathrm{Ext}(\ell_1, \mathbb{R}) \neq 0$ (Theorem 2.3) admits a globally continuous cross section.

Finally we present two notable examples of nonsplitting extensions which split algebraically and admit locally or globally continuous cross sections. The first one is based on a construction by T. C. Stevens [21].

Proposition 3.4. [4, 7.2.6] *There is a nonsplitting extension of the form* $0 \to \mathbb{R} \xrightarrow{\imath} X \xrightarrow{\pi} (\mathbb{R}, \tau) \to 0$ *admitting a cross section which is continuous on a neighborhood of zero, where* τ *is a metrizable group topology on* $\mathbb{R}$ *weaker than the usual one.*

Proposition 3.5. [7, Proposition 18] *For every compact, connected abelian group* H *which is not topologically isomorphic to a product of copies of* $\mathbb{T}$ *there exists a compact, totally disconnected abelian group* G *and a non-splitting extension* $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ *which splits algebraically and has a globally continuous cross section.*

4. Extensions of free abelian topological groups by compact ones

Michael's "zero-dimensional" selection theorem is also an essential ingredient of the following result. Recall that for a completely regular Hausdorff space Y, the *free abelian topological group* over Y is the free abelian group $A(Y)$ endowed with the unique Hausdorff group topology for which

(1) the mapping $\eta : Y \to A(Y)$, which maps the topological space Y onto a basis of $A(Y)$, becomes a topological embedding and
(2) for every continuous mapping $f : Y \to G$, where G is an abelian Hausdorff group, there is a unique continuous group homomorphism $\tilde{f} : A(Y) \to G$ which satisfies $f = \tilde{f} \circ \eta$.

(See for instance Chapter 7 in [1].) Recall also that a *k_ω-space* is a Hausdorff topological space X which carries the weak topology with respect to an increasing sequence of compact subspaces whose union is X.

Theorem 4.1. [6, Theorem 2.8] *If* $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ *is an extension of topological abelian groups where* H *is compact, and* $Y \subseteq G$ *is a subspace of* G *which is zero-dimensional and a* k_ω*-space, then there is a partial continuous cross section for* E *with domain* Y*, that is, there is a continuous mapping* $\rho : Y \to X$ *such that* $\pi \circ \rho$ *is the inclusion* $Y \hookrightarrow G$*.*

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "\imath"] & X \arrow[r, "\pi"] & G \arrow[r] & 0 \\ & & & Y \arrow[ul, "\rho"] \arrow[u, "\imath_Y"'] & \end{tikzcd}$$

In particular if G *itself is a zero-dimensional* k_ω*-space then* E *has a globally continuous cross section.*

Corollary 4.2. *Let* H *be a compact abelian group and* $A(Y)$ *the free topological abelian group on a zero-dimensional* k_ω*-space* Y*. Then* $\operatorname{Ext}(A(Y), H) = 0$.

Proof. Fix an extension $0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} A(Y) \to 0$. Consider the canonical inclusion mapping $\imath_Y : Y \to A(Y)$. By Theorem 4.1 there exists a continuous mapping $\rho : Y \to X$ with $\pi \circ s = \imath_Y$.

$$\begin{tikzcd} 0 \arrow[r] & H \arrow[r, "\imath"] & X \arrow[r, "\pi"] & A(Y) \arrow[r] & 0 \\ & & & Y \arrow[ul, "\rho"] \arrow[u, "\imath_Y"'] & \end{tikzcd}$$

By the universal property of $A(Y)$ the continuous mapping ρ extends to a continuous homomorphism $S : A(Y) \to X$. Since $A(Y)$ is algebraically the free abelian group over Y, it is clear that $\pi \circ S = id_{A(Y)}$. By Proposition 1.9, the extension splits. $\square$

5. Quasi-homomorphisms and extensions

Given any mapping $\omega : G \to H$ where G and H are abelian groups, we denote by Δ_ω the associated mapping defined by $[(x, y) \in G \times G \mapsto \Delta_\omega(x, y) = \omega(x+y) - \omega(x) - \omega(y) \in H]$. (Note that mappings like Δ_ω are usually called *2-coboundaries* in homological algebra.)

Definition 5.1. [9] Let G and H be topological abelian groups. A mapping $\omega : G \to H$ is said to be a *quasi-homomorphism* if $\omega(0) = 0$ and $\Delta_\omega : G \times G \to H$ is continuous at $(0, 0)$.

It is clear that all mappings $\omega : G \to H$ which are continuous at zero, as well as all homomorphisms, are quasi-homomorphisms. (Indeed, if ω is a homomorphism, its coboundary is identically zero.)

Definition 5.2. Let G and H be topological abelian groups. A quasi-homomorphism $\omega : G \to H$ is said to be *approximable* if there exist a homomorphism $a : G \to H$ and a mapping $f : G \to H$ continuous at zero such that $\omega = a + f$.

Proposition 5.3. *Let $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$ be an extension of topological abelian groups. Assume that E splits algebraically and admits a cross section ρ which is continuous at zero.*

Let P be a homomorphism $P : X \to H$ which satisfies $P \circ \imath = id_H$. Then the mapping $\omega : G \to H$ defined by $\omega = P \circ \rho$ is a quasi-homomorphism. Moreover, the extension E splits if and only if ω is approximable.

Proof. The following diagram may be useful:

$$0 \longrightarrow H \underset{P}{\overset{\imath}{\rightleftarrows}} X \underset{\rho}{\overset{\pi}{\rightleftarrows}} G \longrightarrow 0$$

Slightly abusing notation, we will denote by $\imath^{-1}$ the inverse of the corestriction of $\imath$ to its image $\imath(H)$. Note that $\imath^{-1} : \imath(H) \to H$ is a topological isomorphism. On the other hand, an additive left inverse P for $\imath$ exists by Proposition 1.3, and clearly $P \restriction_{\imath(H)} = \imath^{-1}$.

Let us see that ω is a quasi-homomorphism. It is clear that $\omega(0) = 0$. Moreover,

$$\begin{aligned}
\Delta_\omega(x, y) &= P(\rho(x + y)) - P(\rho(x)) - P(\rho(y)) \\
&= P(\rho(x + y) - \rho(x) - \rho(y)) \\
&= \imath^{-1}(\rho(x + y) - \rho(x) - \rho(y))
\end{aligned}$$

since $\rho(x + y) - \rho(x) - \rho(y) \in \mathrm{Ker}\pi = \imath(H)$. Since $\imath^{-1}$ is continuous and ρ is continuous at zero, we deduce that Δ_ω is continuous at $(0, 0)$.

Assume that E splits. Let $S : G \to X$ be a continuous homomorphism such that $\pi \circ S = id_G$ (Proposition 1.9). Note that for every $g \in G$ we have $(P \circ \rho - P \circ S)(g) = P(\rho(g) - S(g)) = \imath^{-1}(\rho(g) - S(g))$ since $\rho(g) - S(g) \in \mathrm{Ker}\pi = \imath(H)$. This clearly implies that $\omega - P \circ S = P \circ \rho - P \circ S$ is continuous at zero, and in particular ω is approximable.

Conversely, assume that $\omega = P \circ \rho$ is approximable. Let $a : G \to H$ be a homomorphism such that $P \circ \rho - a = f$ is continuous at zero. Note that every $x \in X$ can be expressed as $x = \rho(\pi(x)) + (x - \rho(\pi(x))) = \rho(\pi(x)) + \imath(\imath^{-1}(x - \rho(\pi(x))))$ since $x - \rho(\pi(x)) \in \mathrm{Ker}\pi = \imath(H)$. Applying P on both sides we obtain $P(x) = (a + f)(\pi(x)) + \imath^{-1}(x - \rho(\pi(x)))$. This suggests the definition of $\tilde{P} : X \to H$ as $\tilde{P}(x) = P(x) - a(\pi(x)) = f(\pi(x)) + \imath^{-1}(x - \rho(\pi(x)))$ for every $x \in X$. From the expression $\tilde{P}(x) = P(x) - a(\pi(x))$ it easily follows that $\tilde{P}$ is a homomorphism and a left inverse for $\imath$. From $\tilde{P}(x) = f(\pi(x)) + \imath^{-1}(x - \rho(\pi(x)))$ it is clear that $\tilde{P}$ is continuous at zero, hence globally continuous. By Proposition 1.9 we deduce that E splits. □

This correspondence goes in the other direction, too: to every quasi-homomorphism ω one can associate an extension E satisfying the hypothesis of Proposition 5.3 in such a way that one can recover ω from E by the mechanism described in that Proposition. The construction is not difficult; the reader can find the details in [9] or [4, Chapter 6].

Proposition 5.4. *Let G and H be topological abelian groups and let $\omega : G \to H$ be a quasi-homomorphism. There are an extension of topological abelian groups $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} G \to 0$, a homomorphism $P : X \to H$ such that $P \circ \imath = id_H$, and a cross section ρ for E which is continuous at zero, such that $\omega = P \circ \rho$.*

When looking for sufficient conditions for a given class of extensions to split, the requirement of algebraic splitting is a natural place to start. The hypothesis of existence of a cross section continuous at zero is more restrictive, but in any case we see that quasi-homomorphisms can be used as a tool to study many of the extensions one is likely to encounter, especially when dealing with metrizable groups. This is convenient for several reasons; note for instance that unlike the concept of an extension, that of a quasi-homomorphism from a topological abelian group to another does not depend on some other, undefined third group. Since the quasi-homomorphisms representing splitting extensions are exactly the approximable ones, this two-way correspondence can be used to prove statements of the form $\mathrm{Ext}(G, H) = 0$ without actually dealing with the extensions themselves. The argument of the proof of Theorem 6.3 below includes solving the countable case by an application of this device (Corollary 6.2).

Let us briefly mention that it makes sense to consider quasi-homomorphisms $\omega : G \to H$ for which the associated coboundary Δ_ω is continuous not only at $(0,0)$ but on a neighborhood of the origin, which may be the whole product $G \times G$ in some cases. These concepts and properties are explored in [7]. The extra continuity requirements on Δ_ω are naturally linked to the analogous ones on the cross section which is available for the associated extension (Proposition 5.4). In particular the examples given in Proposition 3.4 and Proposition 3.5 can be easily translated in terms of non-approximable quasihomomorphisms ω which have locally or globally continuous coboundaries Δ_ω. If ω is an approximable quasi-homomorphism ω whose coboundary satisfies one of these stronger continuity properties, then in the corresponding decomposition $\omega = a + f$ (as in Definition 5.2) one can assume that f is continuous on a neighborhood of zero or even (if Δ_ω is globally continuous) the whole G.

The notion of a quasi-homomorphism can be also regarded as a natural, simultaneous generalization of those of a continuous mapping and a homomorphism, which is worth studying on its own. For instance, it is remarkable that quasi-homomorphisms with globally continuous coboundary between Polish groups satisfy the closed graph theorem.

Theorem 5.5. [7, Corollary 22] *Let G and H be Polish abelian groups and $\omega : G \to H$ be a quasi-homomorphism such that Δ_ω is continous on $G \times G$. If the graph of ω is closed in $G \times H$ then ω is continuous.*

We end this section with a few general remarks on the concept of a quasi-homomorphism and some related notions one can borrow from the algebraic theory. As we mentioned above, all extensions representable by quasi-homomorphisms are algebraically trivial, and from the definition of a quasi-homomorphism itself it can be seen that there is no natural algebraic counterpart of such a concept. This makes the notion sharper in a sense but on the other hand it is a departure from the approach we have taken up to this point, where the corresponding algebraic concepts and results were incorporated into the theory. Quasi-homomorphisms were introduced as a natural generalization of quasi-linear maps ([16, 13]), which play a fundamental role in the study of extensions ("twisted sums") of topological vector spaces and the solution of the three-space problem for local convexity. Of course, in such a linear context algebraic splitting is not an issue, while in the realm of topological abelian groups the situation is quite different.

But one can also widen the focus and present the concept of a quasi-homomorphism as the particularization of some general construction for the algebraically trivial case. Such a construction would deal with cocycles and coboundaries as in homological algebra, only with topology added in a natural way. We will not ellaborate further on this approach; some relevant references are e. g. [14] or [10].

6. Extensions of products of locally compact groups by $\mathbb{R}$ or $\mathbb{T}$

We end this survey with a nontrivial generalization of Proposition 2.2.

The following result is Proposition 1.8 in [6]. The main elements of the proof are the same as that of [9, Theorem 1(a)]; it relies on Hyers-type theorems on stability of homomorphisms. The analogous property for splitting extensions remains open.

Proposition 6.1. *If H is either $\mathbb{R}$ or $\mathbb{T}$ and every quasi-homomorphism $\omega_i : G_i \to H$ ($i \in I$) is approximable, then every quasi-homomorphism $\omega : \prod_{i\in I} G_i \to H$ is approximable.*

Corollary 6.2. *Let H be either $\mathbb{R}$ or $\mathbb{T}$. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of metrizable groups. If $\operatorname{Ext}(G_n, H) = 0$ for every $n \in \mathbb{N}$ then $\operatorname{Ext}(\prod_{n\in\mathbb{N}} G_n, H) = 0$.*

Proof. Let $E : 0 \to H \xrightarrow{\imath} X \xrightarrow{\pi} \prod_{n\in\mathbb{N}} G_n \to 0$ be an extension. Since H is divisible and metrizable and $\prod_{n\in\mathbb{N}} G_n$ is metrizable, E is representable by a quasi-homomorphism $\omega : \prod_{n\in\mathbb{N}} G_n \to H$ (Proposition 5.3). Since $\operatorname{Ext}(G_n, H) = 0$ for every $n \in \mathbb{N}$, in particular every quasi-homomorphism $\omega_n : G_n \to H$ is approximable (Proposition 5.4). By Proposition 6.1, ω is approximable and hence E splits. □

The following result generalizes this property to the nonmetrizable, noncountable case, provided the groups in the product are locally compact.

Theorem 6.3. [6, Corollary 3.14] *Let H be either $\mathbb{R}$ or $\mathbb{T}$. Let $(G_i)_{i\in I}$ be a family of locally compact groups. Then* $\operatorname{Ext}(\prod_{i\in I} G_i, H) = 0$.

Proof. The following is just a sketch of the argument; see Section 3 in [6] for details. Consider the family of admissible subgroups of $\prod_{i\in I} G_i$ that have the form $\prod_{i\in I} N_i$ where $N_i \leq G_i$ is compact and such that G_i/N_i is metrizable for every $i \in I$ and nontrivial for countably many $i \in I$. This is actually a cofinal family of admissible subgroups of $G = \prod_{i\in I} G_i$, so it suffices (Theorem 2.6) to check that $\operatorname{Ext}(G/P, H) = 0$ for every P in this family. Each G/P turns out to be expressable as the product of countably many metrizable groups G_i/N_i satisfying $\operatorname{Ext}(G_i/N_i, H) = 0$, so we can apply Corollary 6.2. □

This result was actually proved under the following weaker assumptions [6, Theorem 3.13]:

(a) $G = \prod_{i\in I} G_i$ where each G_i is a dense subgroup of a dually separated, Čech-complete group such that both $\operatorname{Ext}(G_i, \mathbb{R}) = 0$ and $\operatorname{Ext}(G_i, \mathbb{T}) = 0$ for each $i \in I$, and
(b) H is an arbitrary product of copies of $\mathbb{R}$ and $\mathbb{T}$.

References

[1] A. Arhangel'skii and M. Tkachenko, Topological Groups and Related Structures. Atlantis Studies in Mathematics, Volume 1. Atlantis Press/World Scientific, 2008.

[2] D. L. Armacost, The Structure of Locally Compact Abelian Groups. Pure and Applied Mathematics, 68. Marcel Dekker, Inc. New York and Basel, 1981.

[3] W. Banaszczyk, Additive subgroups of topological vector spaces. Lecture Notes in Mathematics, 1466, Springer-Verlag, Berlin, Heidelberg, New York, 1991.

[4] H. J. Bello, Extensions of Topological Abelian Groups (Ph. D. dissertation) University of Navarra. `http://dadun.unav.edu/handle/10171/41550`

[5] H. Bello, M. J. Chasco, X. Domínguez, *Extending topological abelian groups by the unit circle.* Abst. Appl. Anal. (2013), Article ID 590159 (9 pages). `http://dx.doi.org/10.1155/2013/590159`

[6] H. Bello, M. J. Chasco, X. Domínguez, M. Tkachenko, *Splittings and cross sections in topological groups.* J. Math. Anal. Appl. 435 (2016), 1607–1622.

[7] H. Bello, M. J. Chasco, X. Domínguez, *Cross sections and pseudo-homomorphisms of topological abelian groups.* Topology Appl. 228 (2017), 285–293.

[8] C. Bessaga, A. Pelczynski, Selected topics in infinite-dimensional topology. Mathematical Monographs, Vol. 58. PWN, Warsaw 1975.

[9] F. Cabello, *Quasi-homomorphisms.* Fundam. Math. 178 (2003), no. 3, 255–270.

[10] U. Cattaneo, *Abelian extensions of topological vector groups.* J. Funct. Anal. 35 (1980), no. 2, 143–152.

[11] S. Dierolf, U. Schwanengel, *Examples of locally compact non-compact minimal topological groups.* Pacific J. Math. 82 (1979), no. 2, 349–355.

[12] D. Dikranjan, An introduction to topological groups. `https://users.dimi.uniud.it/~dikran.dikranjan/ITG.pdf`

[13] P. Domański, *On the splitting of twisted sums, and the three-space problem for local convexity.* Studia Math. 82 (1985), no. 2, 155–189.

[14] S. Hu, *Cohomology theory in topological groups.* Michigan Math. J. 1 (1952), 11-59.

[15] N. J. Kalton, *The three space problem for locally bounded F-spaces.* Comp. Math. 37 (1978), 243-276.

[16] N. J. Kalton, N. T. Peck and J. W. Roberts, An F-space sampler. London Mathematical Society Lecture Note Series, 89, Cambridge Univ. Press, Cambridge, 1984.

[17] E. Michael, *Continuous selections II.* Ann. of Math. 64 (1956), no. 2, 562–580.

[18] E. Michael, *Selected Selection Theorems.* Amer. Math. Monthly 63 (1956), no. 4, 233–238.

[19] Ribe, M., *Examples for the nonlocally convex three space problem.* Proc. Amer. Math. Soc. 237 (1979), 351-355.

[20] Roberts, J. W., *A nonlocally convex F-space with the Hahn-Banach approximation property.* Banach spaces of analytic functions. Springer Lecture Notes 604, Berlin-Heidelberg-New York, 1977, 76–81.

[21] T. C. Stevens, *Decisive subgroups of analytic groups.* Trans. Amer. Math. Soc. 274 (1982), no. 1, 101–108.

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