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Markov-like generalized inverse systems and their limits

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1 Abstract

Markov interval maps were introduced by S. Holte [5] in 2002 and she showed that any two inverse limits with Markov interval bonding maps with the same pattern were homeomorphic. In 2013 I. Banič and T. Lunder [1] extended the notation from continuous maps to set-valued functions, called generalized Markov interval functions, and applied the theory of generalized inverse limits with set-valued functions. In this note we introduce Markov-like functions as a generalization of generalized Markov interval functions and show that any two generalized inverse limits with Markov-like bonding functions having same pattern are homeomorphic. Consequently we can give a generalization of [1].

2 Definition and Notation

Definition 2.1. For any $n \in \mathbb{N}$, let X_n be a compact space and let 2^{X_n} be the collection of all nonempty closed sets of X_n. Let $f_n : X_{n+1} \to 2^{X_n}$. A *generalized inverse system* is defined as a sequence of pairs X_n and f_n, which is denoted by $(X_n, f_n)_{n\in\mathbb{N}}$. The *generalized inverse limit* $\varprojlim \{X_n, f_n\}$ of an inverse system $(X_n, f_n)_{n\in\mathbb{N}}$ is defined by

$$\varprojlim \{X_n, f_n\} := \{(x_1, x_2, \dots) \in \prod_{n=1}^{\infty} X_n \mid \ x_n \in f_n(x_{n+1}) \ \text{for any } n \in \mathbb{N}\}.$$

In the case that $X_n = X$ and $f_n = f$ for each $n \in \mathbb{N}$, we write the inverse limit by $\varprojlim \{X, f\}$.

$\varprojlim \{X_n, f_n\}$ is compact if f_n is upper-semi continuous for all $n \in \mathbb{N}$. Moreover, if $f_n(x) \in C(X_{n+1})$ for all $x \in X_n$, where $C(X_{n+1})$ is the collection of all nonempty subcontinua of X_{n+1}, $\varprojlim \{X_n, f_n\}$ is a continuum.

Definition 2.2. Fix $m \in \mathbb{N}_{\geq 2}$. Let $\mathbb{I} = [a_1, a_m]$ be the closed interval. Let $A := a_1 < a_2 < \cdots < a_m$ be a finite partition of $\mathbb{I}$ and put $\mathbb{I}_j = [a_j, a_{j+1}]$ for each $j = 1, \ldots, m-1$.

A set-valued function $f : \mathbb{I} \to 2^{\mathbb{I}}$ having a surjective graph is *Markov-like* with respect to A if the following statements are satisfied.

(1) For all $j = 1, 2, \ldots m$, there exist $\frac{s_j}{2}$ mutually disjoint closed intervals (they can be degenerate) $[a_{r_1(j)}, a_{r_2(j)}], \ldots, [a_{r_{s_j - 1}(j)}, a_{r_{s_j}(j)}]$ such that

$$f(a_j) = \bigcup_{k=1}^{\frac{s_j}{2}} [a_{r_{2k-1}(j)}, a_{r_{2k}(j)}], \text{ and}$$
$$a_{r_l(j)} \in A \quad \text{for} \quad \text{each} \quad l \in \{1, 2, \ldots, s_j\}.$$

(2) Let define $G_j(f) := \{(y, x) \in G(f) \mid x \in \text{Int}\,(\mathbb{I}_j)\}$ for each $j = 1, 2, \ldots, m-1$. Then, there are $n_f\,(j)$ strictly monotone continuous functions $f_j^1, f_j^2, \ldots, f_j^{n_f(j)}$ having mutually disjoint graphs defined on Int $(\mathbb{I}_j)$ such that for each $1 \leq l \leq n_f(j)$

$$\lim_{x \downarrow a_j} f_j^l(x) \in f\,(a_j) \cap A, \quad \lim_{x \uparrow a_{j+1}} f_j^l(x) \in f\,(a_{j+1}) \cap A, \text{ and}$$
$$G_j\,(f) = \bigcup_{l=1}^{n_f(j)} G(f_j^l).$$

Definition 2.3. Let $\mathbb{I} = [a_1, a_m]$ and $\mathbb{J} = [b_1, b_m]$ be closed intervals and $A : a_1 < a_2 < \cdots < a_m$ and $B : b_1 < b_2 < \cdots < b_m$ be partitions of $\mathbb{I}$ and $\mathbb{J}$ respectively.

A Markov-like function $f : \mathbb{I} \to 2^{\mathbb{I}}$ with respect to A and a Markov-like function $g : \mathbb{J} \to 2^{\mathbb{J}}$ with respect to B have *the same pattern* if the following conditions are satisfied.

(3) For any $j = 1, 2, \ldots m$,

$$f(a_j) \supseteq [a_{r_1(j)}, a_{r_2(j)}] \quad \Leftrightarrow \quad g(b_j) \supseteq [b_{r_1(j)}, b_{r_2(j)}].$$

(4) For any $j \in \{1, 2, \ldots, m\}$, $n_f(j) = n_g(j)$ and there exists a bijection $\phi_j : \{1, 2, \ldots, n_f(j)\} \to \{1, 2, \ldots, n_g(j)\}$ such that

$$\lim_{x \downarrow a_j} f_j^k(x) = a_{l_1(j)} \quad \Leftrightarrow \quad \lim_{y \downarrow b_j} g_j^{\phi_j(k)}(y) = b_{l_1(j)},$$
$$\lim_{x \uparrow a_{j+1}} f_j^k(x) = a_{l_2(j)} \quad \Leftrightarrow \quad \lim_{y \uparrow b_{j+1}} g_j^{\phi_j(k)}(y) = b_{l_2(j)}.$$

3 The Main Theorem

Theorem 3.1. Let $\mathbb{I} = [a_1, a_m]$ and $\mathbb{J} = [b_1, b_m]$ be closed intervals and $A : a_1 < a_2 < \cdots < a_m$ and $B : b_1 < b_2 < \cdots < b_m$ be partitions of $\mathbb{I}$ and $\mathbb{J}$ where $m \geq 2$ respectively. Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be sequences of Markov-like functions with respect to A and B respectively. Then if for every $n \in \mathbb{N}$, f_n and g_n have the same pattern, two generalized inverse limits $\varprojlim \{\mathbb{I}, f_n\}$ and $\varprojlim \{\mathbb{J}, g_n\}$ are homeomorphic.

Here we state an idea of the proof. First we explain the reason why we may assume that both $\mathbb{I}$ and $\mathbb{J}$ are the unit interval $[0, 1]$.

Lemma 3.2. Let $\mathbb{I} = [a_1, a_m]$ be a closed interval and $A : a_1 < \cdots < a_m$ be a partition of $\mathbb{I}$, where $m \geq 2$. Let $\mathbb{J} = [0, 1]$. Let $f : \mathbb{I} \to 2^{\mathbb{I}}$ be a Markov-like function with respect to A. Suppose that $h : \mathbb{I} \to \mathbb{J}$ is a piecewise linear homeomorphism such that

$$h(a_1) = 0, \quad h(a_m) = 1, \text{ and}$$

$$h \text{ is non-differentiable at a point } x \in \mathbb{I} \implies x \in A.$$

Let define $b_i = h(a_i)$ for each $i = 1, \ldots, m$ and a partition $B : b_1 < \cdots < b_m$ of $\mathbb{J}$. Then there is a Markov-like function $g : \mathbb{J} \to 2^{\mathbb{J}}$ with respect to B such that

$$2^h \circ f = g \circ h,$$

where $2^h : 2^{\mathbb{I}} \to 2^{\mathbb{J}}$ is the induced homeomorphism by h, and

$$f \text{ and } g \text{ have the same pattern.}$$

Lemma 3.3. Let $\mathbb{I}$ and $\mathbb{J}$ be closed intervals. Let $\{f_n : \mathbb{I} \to 2^{\mathbb{I}}\}$ and $\{g_n : \mathbb{J} \to 2^{\mathbb{J}}\}$ be sequences of set-valued functions and let $\{h_n : \mathbb{I} \to \mathbb{J}\}$ be a sequence of homeomorphisms such that

$$2^{h_n} \circ f_n = g_n \circ h_{n+1} \quad \text{for each } n \in \mathbb{N},$$

where $2^{h_n} : 2^{\mathbb{I}} \to 2^{\mathbb{J}}$ is the induced homeomorphism by h_n. Then the generalized inverse limits $\varprojlim \{\mathbb{I}, f_n\}$ and $\varprojlim \{\mathbb{J}, g_n\}$ are homeomorphic.

Theorem 3.4. Let $\mathbb{I} = [a_1, a_m]$ be a closed interval and $A : a_1 < \cdots < a_m$ be a partition of $\mathbb{I}$, where $m \geq 2$. Let $f : \mathbb{I} \to 2^{\mathbb{I}}$ be a Markov-like function with respect to A. Suppose that $h : \mathbb{I} \to \mathbb{J} = [0, 1]$ be a piecewise linear homeomorphism such that

$$h(a_1) = 0, \; h(a_m) = 1, \text{ and}$$

$$h \text{ is non-differentiable at a point } x \in \mathbb{I} \implies x \in A.$$

Let define $b_i = h(a_i)$ for each $i = 1, \dots, m$ and a partition $B : b_1 < \cdots < b_m$ of $\mathbb{J}$. Then there exists a sequence $\{g_n\}$ of Markov-like functions with respect to B such that the generalized inverse limits $\varprojlim \{\mathbb{I}, f_n\}$ and $\varprojlim \{\mathbb{J}, g_n\}$ are homeomorphic.

Outline of the proof of Theorem 3.1.

Step 1. From Theorem 3.4, we can assume both $\mathbb{I}$ and $\mathbb{J}$ are the unit interval $[0, 1]$.

Let $h : \mathbb{I} \to \mathbb{J}$ be a piecewise linear homeomorphism such that $h(\mathbb{I}_j) = \mathbb{J}_j$ for all $j = 1, 2, \dots, m-1$.

Step 2. For any point $\mathbf{x} = (x_1, x_2, \dots) \in \varprojlim \{\mathbb{I}, f_n\}$, there exists exactly one point $\mathbf{y} = (y_1, y_2, \dots) \in \varprojlim \{\mathbb{J}, g_n\}$ with $y_1 = h(x_1)$ and satisfying the following properties for each $i \in \mathbb{N}$:

(1)-(i) $x_i \in \text{Int } (\mathbb{I}_j) \quad \Leftrightarrow \quad y_i \in \text{Int } (\mathbb{J}_j)$,

(2)-(i) $x_i = a_j \quad \Leftrightarrow \quad y_i = b_j$,

(3)-(i) $x_{i-1} = f^k_{i-1,\, j}(x_i) \quad \Leftrightarrow \quad y_{i-1} = g^{\phi_{i-1,\, j}(k)}_{i-1,\, j}(y_i)$.

For any $\mathbf{x} \in \varprojlim \{\mathbb{I}, f_n\}$, choosing the point $\mathbf{y} \in \varprojlim \{\mathbb{J}, g_n\}$ of Step 2, we can define the function

$$H : \varprojlim \{\mathbb{I}, f_n\} \to \varprojlim \{\mathbb{J}, g_n\}.$$

Step 3. We show that H is continuous.

We will provide some notations and lemmas to show that H is continuous.
Fix $i \in \mathbb{N}$. For any $j \in \{1, 2, \dots, m-1\}, k \in \{1, 2, \dots, n_{f_i}(j)\}$, let

$$\overline{f^k_{i,\, j}}(w) = \begin{cases} \lim\limits_{x \downarrow a_j} f^k_{i,\, j}(x) & \text{if } w = a_j \\ f^k_{i,\, j}(w) & \text{if } w \in \text{Int } (\mathbb{I}_j) \\ \lim\limits_{x \uparrow a_{j+1}} f^k_{i,\, j}(x) & \text{if } w = a_{j+1}. \end{cases}$$

Similarly, for any $j \in \{1, 2, \dots, m-1\}, k \in \{1, 2, \dots, n_{g_i}(j)\}$, let

$$\overline{g^k_{i,\, j}}(z) = \begin{cases} \lim\limits_{y \downarrow b_j} g^k_{i,\, j}(y) & \text{if } z = b_j \\ g^k_{i,\, j}(z) & \text{if } z \in \text{Int } (\mathbb{J}_j) \\ \lim\limits_{y \uparrow b_{j+1}} g^k_{i,\, j}(y) & \text{if } z = b_{j+1}. \end{cases}$$

Then $\overline{g_{i,\,j}^{k}}$ is a homeomorphism from $\mathbb{J}_j$ to a closed interval having endpoints in B.

Lemma 3.5. For any $\mathbf{x} \in \varprojlim \{\mathbb{I}, f_n\}$ and $i \in \mathbb{N}$, there exists $\delta_i > 0$ such that if $\mathbf{x}' \in \varprojlim \{\mathbb{I}, f_n\}$, $d(\mathbf{x}, \mathbf{x}') < \delta_i$, then

$$x_{i+1}, x'_{i+1} \in \mathbb{I}_j \text{ for some } j \in \{1, \dots, m-1\}$$

and one of the following statements hold.

(1) $x_{i+1} = x'_{i+1}, \quad x_{i+1}, x'_{i+1} \in A$,

(2) $(x_i, x_{i+1}), (x'_i, x'_{i+1}) \in G\left(\overline{f_{i,\,j}^{k_1}}\right)$ for some $k_1 \in \{1, \dots, n_{f_i}(j)\}$,

(3) $(x_i, x_{i+1}), (x'_i, x'_{i+1}) \in G\left(\overline{f_{i,\,j-1}^{k_2}}\right)$ for some $k_2 \in \{1, \dots, n_{f_i}(j-1)\}$.

Lemma 3.6. Choose $\mathbf{x}, \mathbf{x}' \in \varprojlim \{\mathbb{I}, f_n\}$ and let $\mathbf{y} = H(\mathbf{x}), \mathbf{y}' = H(\mathbf{x}')$. Suppose $x_{i+1}, x'_{i+1} \in \mathbb{I}_j$ for some $j \in \{1, \dots, m-1\}$. Then we have the following.

(1) if $x_{i+1} = x'_{i+1}$ and $x_{i+1}, x'_{i+1} \in A$, $y_{i+1} = y'_{i+1}$,

(2) if $(x_i, x_{i+1}), (x'_i, x'_{i+1}) \in G\left(\overline{f_{i,\,j}^{k_1}}\right)$ for some $k_1 \in \{1, \dots, n_{f_i}(j)\}$,

$$(y_i, y_{i+1}), (y'_i, y'_{i+1}) \in G\left(\overline{g_{i,\,j}^{\phi_{i,\,j}(k_1)}}\right),$$

(3) if $(x_i, x_{i+1}), (x'_i, x'_{i+1}) \in G\left(\overline{f_{i,\,j-1}^{k_2}}\right)$ for some $k_2 \in \{1, \dots, n_{f_i}(j-1)\}$,

$$(y_i, y_{i+1}), (y'_i, y'_{i+1}) \in G\left(\overline{g_{i,\,j-1}^{\phi_{i,\,j}(k_2)}}\right).$$

Definition 3.7. Fix $i \in \mathbb{N}$. For any $(y_i, y_{i+1}) \in G(g_i)$, the subset $G_{(y_i, y_{i+1})}(g_i)$ of $G(g_i)$ is defined to satisfy the following condition

$$(y'_i, y'_{i+1}) \in G_{(y_i, y_{i+1})}(g_i)$$

if $(y'_i, y'_{i+1}) \in G(g_i)$ and one of the following statements hold.

1. $y_{i+1} = y'_{i+1}$,
2. $(y_i, y_{i+1}), (y'_i, y'_{i+1}) \in G\left(\overline{g_{i,\,j}^{k_1}}\right)$ for some $j \in \{1, \dots, m-1\}$ and $k_1 \in \{1, \dots, n_{f_i}(j)\}$,
3. $(y_i, y_{i+1}), (y'_i, y'_{i+1}) \in G\left(\overline{g_{i,\,j-1}^{k_2}}\right)$ for some $j \in \{2, \dots, m-1\}$ and $k_2 \in \{1, \dots, n_{f_i}(j-1)\}$.

Lemma 3.8. Fix $i \in \mathbb{N}$. For any $(y_i, y_{i+1}) \in G(g_i)$, and $\epsilon > 0$, there exists a $\delta > 0$ such that

$$(y'_i, y'_{i+1}) \in G_{(y_i,y_{i+1})}(g_i), \; |y_i - y'_i| < \delta \;\Rightarrow\; |y_{i+1} - y'_{i+1}| < \epsilon.$$

Lemma 3.9. Fix $n \in \mathbb{N}$ and $(y_1, y_2, \dots) \in \mathbb{I}^{\mathbb{N}}$ with $(y_i, y_{i+1}) \in G(g_i)$ for $1 \le i \le n-1$. For any $\epsilon > 0$, there exists $\delta_n > 0$ such that for any $(y'_1, \dots, y'_n, \dots) \in \mathbb{I}^{\mathbb{N}}$ with $(y'_i, y'_{i+1}) \in G_{(y_i,y_{i+1})}(g_i)$ for $1 \le i \le n-1$, the following statement is true.

$$|y_1 - y'_1| < \delta_n \;\Rightarrow\; |y_{i+1} - y'_{i+1}| < \epsilon \text{ for } 1 \le i \le n-1.$$

We return to the proof that H is continuous.
Fix $\mathbf{x} \in \varprojlim \{\mathbb{I}, f_n\}$ and let $\mathbf{y} = H(\mathbf{x})$. Fix any $\epsilon > 0$ and choose $n_\epsilon \in \mathbb{N}$ with $\sum_{i=n_\epsilon}^{\infty} 2^{-i} < \frac{\epsilon}{2}$. From Lemma 3.5 and Lemma 3.6, there exists $\delta_{n_\epsilon} > 0$ such that

$$d(\mathbf{x}, \mathbf{x}') < \delta_{n_\epsilon} \Rightarrow (\pi_i \circ H(\mathbf{x}'), \pi_{i+1} \circ H(\mathbf{x}')) \in G_{(y_i,y_{i+1})}(g_i) \quad \text{for } 1 \le i \le n_\epsilon - 1.$$

Moreover, from Lemma 3.9, there exists $\eta_{n_\epsilon} > 0$ such that for any $(y'_1, \dots, y'_{n_\epsilon}, \dots) \in \mathbb{I}^{\mathbb{N}}$ with $(y'_i, y'_{i+1}) \in G_{(y_i,y_{i+1})}(g_i)$ for $1 \le i \le n_\epsilon - 1$,

$$|y_1 - y'_1| < \eta_{n_\epsilon} \;\Rightarrow\; |y_{i+1} - y'_{i+1}| < \frac{\epsilon}{2n_\epsilon} \text{ for } 1 \le i \le n_\epsilon - 1.$$

Since $h : \mathbb{I} \to \mathbb{I}$ is continuous, there exists $\delta'_{n_\epsilon} > 0$ such that

$$d(\mathbf{x}, \mathbf{x}') < \delta'_{n_\epsilon} \;\Rightarrow\; |y_1 - \pi_1 \circ H(\mathbf{x}')| = |h(x_1) - h(x'_1)| < \min\left\{\frac{\epsilon}{2n_\epsilon}, \eta_{n_\epsilon}\right\}.$$

Let $\delta = \min\{\delta_{n_\epsilon}, \delta'_{n_\epsilon}\}$. Then

$$d(\mathbf{x}, \mathbf{x}') < \delta \;\Rightarrow\; |y_i - \pi_i \circ H(\mathbf{x}')| < \frac{\epsilon}{2n_\epsilon} \text{ for } 1 \le i \le n_\epsilon.$$

Therefore

$$\begin{aligned} d(\mathbf{y}, H(\mathbf{x}')) &= \sum_{i=1}^{\infty} 2^{-i} |y_i - \pi_i \circ H(\mathbf{x}')| \\ &= \sum_{i=1}^{n_\epsilon} 2^{-i} |y_i - \pi_i \circ H(\mathbf{x}')| + \sum_{i=n_\epsilon+1}^{\infty} 2^{-i} |y_i - \pi_i \circ H(\mathbf{x}')| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, H is continuous. The same proof can be applied to the inverse map of H. Therefore we have that H is a homeomorphism.

4 Some examples

In this section we show that some examples of Markov-like functions and their generalized inverse limits. Here we suppose that $\mathbb{I}$ means the unit interval $[0,1]$.

Example 4.1. For $f : \mathbb{I} \to 2^{\mathbb{I}}$, assume that there is a strictly monotone continuous function $g : \mathbb{I} \to \mathbb{I}$ such that $(0,0), (1,1) \in G(g)$ and $G(f) = G(g)$. Then, $\varprojlim \{\mathbb{I}, f\}$ is an arc.

Proof. Let $\pi_1 : \varprojlim \{\mathbb{I}, f\} \to \mathbb{I}$ be the projection map to the first coordinate. Then π_1 is a homeomorphism. Therefore, $\varprojlim \{\mathbb{I}, f\}$ is an arc having endpoints $\{(0,0,\dots),(1,1,\dots)\}$. □

Example 4.2. For $f : \mathbb{I} \to 2^{\mathbb{I}}$, assume that there is a strictly monotone continuous function $g : \mathbb{I} \to \mathbb{I}$ such that $(0,1), (1,0) \in G(g)$ and $G(f) = G(g)$. Then, by the same proof of Example 4.1, $\varprojlim \{\mathbb{I}, f\}$ is an arc having endpoints $(0,1,0,1,\dots)$ and $(1,0,1,0,\dots)$.

Example 4.3. Fix $n \in \mathbb{N}_{\geq 2}$. Suppose that $f_1, \dots, f_n : [0,1] \to [0,1]$ are strictly monotone continuous functions such that

$$i \neq j \quad \Rightarrow \quad G(f_i) \cap G(f_j) = \{(0,0),(1,1)\}.$$

Let $f : \mathbb{I} \to 2^{\mathbb{I}}$ be defined by

$$G(f) = \bigcup_{i=1}^{n} G(f_i).$$

Then, $\varprojlim \{\mathbb{I}, f\}$ is a union of uncountable arcs. All arcs have same endpoints and they are pairwise disjoint on each point without their endpoints.

Example 4.4. Fix $n \in \mathbb{N}_{\geq 2}$. Suppose that $g_1, \dots, g_n : [0,1] \to [0,1]$ are strictly monotone continuous functions such that

$$i \neq j \quad \Rightarrow \quad G(g_i) \cap G(g_j) = \{(0,1),(1,0)\}.$$

Let $g : \mathbb{I} \to 2^{\mathbb{I}}$ be defined by

$$G(g) = \bigcup_{i=1}^{n} G(g_i).$$

Then, $\varprojlim \{\mathbb{I}, g\}$ is a union of uncountable arcs. All arcs have same endpoints and they are pairwise disjoint on each point without their endpoints.

The next example show that there are two Markov-like functions f and g such that they do not have the same pattern but their generalized inverse limits are homeomorphic.

Example 4.5. Let l, m be distinct natural numbers greater than two. Suppose that $f_1, \ldots, f_l, g_1, \ldots, g_m : [0,1] \to [0,1]$ are strictly monotone continuous functions such that

$$\begin{aligned} i \neq j \quad &\Rightarrow \quad G(f_i) \cap G(f_j) = \{(0,0), (1,1)\}, \\ i' \neq j' \quad &\Rightarrow \quad G(g_{i'}) \cap G(g_{j'}) = \{(0,0), (1,1)\}. \end{aligned}$$

Let define set-valued functions $f, g : \mathbb{I} \to 2^{\mathbb{I}}$ by

$$\begin{aligned} G(f) &= \bigcup_{i=1}^{l} G(f_i), \\ G(g) &= \bigcup_{j=1}^{m} G(g_j). \end{aligned}$$

Then f and g do not have the same pattern but their generalized inverse limits $\varprojlim \{\mathbb{I}, f\}$ and $\varprojlim \{\mathbb{I}, g\}$ are homeomorphic.

Proof. Let $\Lambda_l := \prod_{i \in \mathbb{N}} \{1, \ldots, l\}$ and $\Lambda_m := \prod_{j \in \mathbb{N}} \{1, \ldots, m\}$. Take a homeomorphism $\phi : \Lambda_l \to \Lambda_m$. For each $s = (s_1, s_2, \ldots) \in \Lambda_l$ let denote

$$\begin{aligned} L_s &:= \left\{ \mathbf{x} \in \prod_{k=1}^{\infty} \mathbb{I} \;\middle|\; (x_k, x_{k+1}) \in G(f_{s_k}) \quad \text{for each } k \in \mathbb{N} \right\}, \\ N_{\phi(s)} &:= \left\{ \mathbf{y} \in \prod_{k=1}^{\infty} \mathbb{I} \;\middle|\; (y_k, y_{k+1}) \in G\left(g_{\phi(s)_k}\right) \quad \text{for each } k \in \mathbb{N} \right\}. \end{aligned}$$

Since $L_s, N_{\phi(s)}$ are arcs having endpoints $\{(0,0,\ldots), (1,1,\ldots)\}$, there is a homeomorphism $h_s : L_s \to N_{\phi(s)}$ such that

$$\begin{aligned} h_s((0,0,\ldots)) &= (0,0,\ldots), \\ h_s((1,1,\ldots)) &= (1,1,\ldots). \end{aligned}$$

As seen in Example 4.3,

$$\begin{aligned} \varprojlim \{\mathbb{I}, f\} &= \bigcup_{s \in \Lambda_l} L_s, \\ \varprojlim \{\mathbb{I}, g\} &= \bigcup_{t \in \Lambda_m} N_t \\ &= \bigcup_{s \in \Lambda_l} N_{\phi(s)}. \end{aligned}$$

Hence we can define $H : \varprojlim \{\mathbb{I}, f\} \to \varprojlim \{\mathbb{I}, g\}$ by

$$H(\mathbf{x}) = h_s(\mathbf{x}) \quad \text{if} \quad \mathbf{x} \in L_s, \ s \in \Lambda_l.$$

Since ϕ is a homeomorphism, H is continuous and bijective. Therefore H is a homeomorphism. □

In the end we give an example of a generalized inverse limit with a Markov-like function which have interesting topological properties (c.f.[2], [3] and [6]). We need the following known fact.

Definition 4.6. a continuum X is a *triod* if there is a subcontinuum $A \subseteq X$ such that $X \backslash A$ have no less than three components.

Theorem 4.7. ([4]) Plane cannot include uncountable mutually disjoint triods.

Example 4.8. ([2],[3], [6]) Let $g : \mathbb{I} \to \mathbb{I}$ be a strictly monotone continuous function with $g(0) = 1, g(1) = 0$. Let define the Markov-like function $f : \mathbb{I} \to 2^{\mathbb{I}}$ by

$$f(x) = \begin{cases} [0,1] & \text{if } x = 0 \\ \{g(x)\} & \text{if } x \in (0,1]. \end{cases}$$

Then the generalized inverse limit $\varprojlim \{\mathbb{I}, f\}$ is a one-dimensional non-planer continuum.

Proof. We note that $\varprojlim \{\mathbb{I}, f\}$ is a continuum.
From Theorem 3.1, we may assume that $g(x) = 1 - x$. Let

$$A := \left\{ \mathbf{x} \in \prod_{j \in \mathbb{N}} \mathbb{I} \;\middle|\; x_j = 1 - x_{j+1} \ \text{ for each } j \in \mathbb{N} \right\}.$$

For each $i \in \mathbb{N}$ put

$$B_i := \left\{ \mathbf{x} \in \prod_{j \in \mathbb{N}} \mathbb{I} \;\middle|\; x_{i+1} = 0, \ x_j = 1 - x_{j+1} \ (j < i), \ x_j \in f(x_{j+1}) \ \ (j \geq i+1) \right\}.$$

Then we can see that

$$\varprojlim \{\mathbb{I}, f\} = A \ \cup \ \bigcup_{i=1}^{\infty} B_i.$$

First we show that $\varprojlim \{\mathbb{I}, f\}$ is one-dimensional. By Example 4.2, A is an arc with endpoints $\mathbf{p} = (0,1,0,1,\dots), \mathbf{q} = (1,0,1,0,\dots) \in \prod_{j=1}^{\infty} \{0,1\}$. For each $i \in \mathbb{N}$ put

$$\pi_{\langle 1,i \rangle}(B_i) = \left\{ (x_1, x_2, \dots, x_i) \in \prod_{j=1}^{i} \mathbb{I} \;\middle|\; x_j = 1 - x_{j+1} \ \text{ for } 1 \leq j \leq i-1 \right\},$$

$$\pi_{\langle i+1,\infty\rangle}(B_i) = \left\{ (x_{i+1}, x_{i+2}, \dots) \in \prod_{j=i+1}^{\infty} \mathbb{I} \;\middle|\; x_{i+1} = 0,\ x_j \in f(x_{j+1}) \text{ for } j \geq i+1 \right\}.$$

Then $\pi_{\langle 1,i\rangle}(B_i)$ is an arc having endpoints $p_i = (0,1,0,1,\dots), q_i = (1,0,1,0,\dots) \in \prod_{j=1}^{n} \mathbb{I}$. On the other hand, since $f^{-1}(0) = \{0,1\}$ and $f^{-1}(1) = \{0\}$, $\pi_{\langle i+1,\infty\rangle}(B_i) \subseteq \prod_{j=i+1}^{\infty} \{0,1\}$. Moreover, it is seen easily that $\pi_{\langle i+1,\infty\rangle}(B_i)$ is perfect. Hence $\pi_{\langle i+1,\infty\rangle}(B_i)$ is a Cantor set. Therefore B_i is a one-dimensional compact set as a product space of an arc and a Cantor set. Since A is also a one-dimensional compact set, by the countable sum theorem, $\varprojlim \{\mathbb{I}, f\}$ is one-dimensional.

Next we show that $\varprojlim \{\mathbb{I}, f\}$ is not planar. For the proof we precisely describe the subset $B_i \subset \varprojlim \{\mathbb{I}, f\}$. For each $i \in \mathbb{N}$, we denote the Cantor set $\pi_{\langle i+1,\infty\rangle}(B_i)$ by C_i. Put the endpoints p_i, q_i of $\pi_{\langle 1,i\rangle}(B_i)$ and let

$$D_i := \{p_i\} \times C_i \subseteq \prod_{j=1}^{i} \{0,1\} \times C_i,$$

$$E_i := \{q_i\} \times C_i \subseteq \prod_{j=1}^{i} \{0,1\} \times C_i.$$

By $\mathcal{B}_i$ we denote the collection of arc-components of B_i. Then

(1) Each element of $\mathcal{B}_i$ is an arc having one endpoint in D_i and the other endpoint in E_i,

(2) For each $c \in C_i$, there exists an element of $\mathcal{B}_i$ joining $(p_i, c) \in D_i$ and $(q_i, c) \in E_i$.

(3) $D_{2i} = D_{2i+1} \cup D_{2i+2}$, $E_{2i-1} = E_{2i} \cup E_{2i+1}$ for each $i \in \mathbb{N}$.

(4) Let

$$C_0^0 := \left\{ \mathbf{x} \in \prod_{j=1}^{\infty} \mathbb{I} \;\middle|\; x_1 = 0,\ x_j \in f(x_{j+1}) \text{ for each } j \in \mathbb{N} \right\},$$

$$C_0^1 := \left\{ \mathbf{x} \in \prod_{j=1}^{\infty} \mathbb{I} \;\middle|\; x_1 = 1,\ x_j \in f(x_{j+1}) \text{ for each } j \in \mathbb{N} \right\}.$$

Then $C_0^0 = D_1 \cup D_2$ and $C_0^1 = E_1$.

(5) $D_i \cap D_j = \emptyset$ for any distinct odd numbers i, j.
$E_i \cap E_j = \emptyset$ for any distinct even numbers i, j.

(6) $\bigcap_{n\in\mathbb{N}} D_{2n} = \{\mathbf{p}\}$, $\bigcap_{n\in\mathbb{N}} E_{2n-1} = \{\mathbf{q}\}$.

(7) $D_2 = \left(\bigcup_{n\in\mathbb{N}} D_{2n+1}\right) \cup \{\mathbf{p}\}$, $E_1 = \left(\bigcup_{n\in\mathbb{N}} E_{2n}\right) \cup \{\mathbf{q}\}$.

Let $\mathbf{v}$ be a point of E_4 and let α be an arc in $\mathcal{B}_4$ from $\mathbf{v}$ to a point of D_4. Because $E_4 \subseteq E_3 \subseteq E_1$, there are arcs β and γ in $\mathcal{B}_3$ and $\mathcal{B}_1$ respectively, having $\mathbf{v}$ as an endpoint. Let $T_{\mathbf{v}} = \alpha \cup \beta \cup \gamma$. Since D_1, D_3 and D_4 are pairwise mutually exclusive, $T_{\mathbf{v}}$ is a triod. If $\mathbf{v}$ and $\mathbf{w}$ are two different points of E_4, $T_{\mathbf{v}} \cap T_{\mathbf{w}} = \emptyset$. Because E_4 is uncountable, $\varprojlim \{\mathbb{I}, f\}$ contains uncountably many mutually disjoint triods. Therefore $\varprojlim \{\mathbb{I}, f\}$ is non-planer by Theorem 4.7. □

References

[1] Iztok Banič and Tjaša Lunder, Inverse limits with generalized Markov interval functions, Bull. Mal. Math. Sci. Soc. **39** (2013), 839-848.

[2] Iztok Banič , Matevž Črepnjak, Matej Merhar, Uroš Milutinovič, Towards the complete classification of generalized ten maps inverse limits, Topology Appl. **160** (2013), 63-73.

[3] Judy Kennedy and Van Nall, Dynamical properties of shift maps on inverse limits with a set valued function, Ergod. Th. & Dynam. Sys. (2016).

[4] R. L. Moore, Foundations of Point set theory, American Math. Sci. Colloquium Pub. (1962).

[5] Sarah E. Holte, Inverse limits of Markov interval map, Topology Appl. **123** (2002), 421-427.

[6] W. T. Ingram, An Introduction to Inverse Limits with Set-valued Functions, Springer (2012).

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