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Weak and Strong Convergence Theorems for a Finite Family of Demimetric Mappings with Variational Inequality Problems in Hilbert Spaces

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Abstract. In this article, using a new nonlinear mapping called demimetric, we prove weak and strong convergence theorems for finding a common element of the set of common fixed points of a finite family of such new demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space. Using the results, we obtain well-known and new strong convergence theorems in a Hilbert space.

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. For a mapping $U : C \to H$, we denote by $F(U)$ the set of fixed points of U. Let k be a real number with $0 \leq k < 1$. A mapping $U : C \to H$ is called a k-strict pseudo-contraction [5] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then we have that, for $x \in C$ and $q \in F(U)$,

$$\|Ux - q\|^2 \leq \|x - q\|^2 + k\|x - Ux\|^2.$$

From $\|Ux - q\|^2 = \|Ux - x\|^2 + \|x - q\|^2 + 2\langle Ux - x, x - q\rangle$, we have that

$$\|Ux - x\|^2 + \|x - q\|^2 + 2\langle Ux - x, x - q\rangle \leq \|x - q\|^2 + k\|x - Ux\|^2.$$

Therefore, we have that

$$2\langle x - Ux, x - q\rangle \geq (1-k)\|x - Ux\|^2 \tag{1.1}$$

for all $x \in C$ and $q \in F(U)$. A mapping $U : C \to H$ is called generalized hybrid [10] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping U is called $(\alpha,\ \beta)$-generalized hybrid. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive, i.e.,

$$\|Ux - Uy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

It is nonspreading [11, 12] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Ux - Uy\|^2 \leq \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [21] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Ux - Uy\|^2 \leq \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [7]. If U is generalized hybrid and $F(U) \neq \emptyset$, then we have that, for $x \in C$ and $q \in F(U)$,

$$\alpha\|q - Ux\|^2 + (1-\alpha)\|q - Ux\|^2 \leq \beta\|q - x\|^2 + (1-\beta)\|q - x\|^2$$

and hence $\|Ux - q\|^2 \leq \|x - q\|^2$. From this, we have that

$$2\langle x - q, x - Ux\rangle \geq \|x - Ux\|^2. \tag{1.2}$$

On the other hand, there exists such a mapping in a Banach space. Let E be a smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then, for the metric resolvent J_λ of B for $\lambda > 0$, we have from [19] that, for any $x \in E$ and $q \in B^{-1}0$,

$$\langle J_\lambda x - q, J(x - J_\lambda x)\rangle \geq 0.$$

Then we get

$$\langle J_\lambda x - x + x - q, J(x - J_\lambda x)\rangle \geq 0$$

and hence

$$\langle x - q, J(x - J_\lambda x)\rangle \geq \|x - J_\lambda x\|^2, \tag{1.3}$$

where J is the duality mapping on E. Motivated by (1.1), (1.2) and (1.3), Takahashi [23] introduced a new nonlinear mapping as follows: Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let k be a real number with $k \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called k-demimetric if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, J(x - Ux)\rangle \geq (1-k)\|x - Ux\|^2,$$

where J is the duality mapping on E. According to the definition, we get that a k-strict pseudo-contraction U with $F(U) \neq \emptyset$ is k-demimetric, an $(\alpha,\ \beta)$-generalized hybrid mapping U with $F(U) \neq \emptyset$ is 0-demimetric and the metric resolvent J_λ with $B^{-1}0 \neq \emptyset$ is (-1)-demimetric.

In this article, using this new nonlinear mapping called demimetric, we prove weak and strong convergence theorems for finding a common element of the set of common fixed points of a finite family of such new demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space. Using the results, we obtain well-known and new strong convergence theorems in a Hilbert space.

2 Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We have from [20] that for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \tag{2.1}$$

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. A mapping $T : C \to H$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. If $T : C \to H$ is nonexpansive, then $F(T)$ is closed and convex; see [8, 20]. For a nonempty, closed and convex subset D of H, the nearest point projection of H onto D is denoted by P_D, that is, $\|x - P_D x\| \leq \|x - y\|$ for all $x \in H$ and $y \in D$. Such a mapping P_D is called the metric projection of H onto D. We know that the metric projection P_D is firmly nonexpansive; $\|P_D x - P_D y\|^2 \leq \langle P_D x - P_D y, x - y \rangle$ for all $x, y \in H$. Furthermore, $\langle x - P_D x, y - P_D x \rangle \leq 0$ holds for all $x \in H$ and $y \in D$; see [18, 20]. Using this inequality and (2.3), we have that

$$\|P_D x - y\|^2 + \|P_D x - x\|^2 \leq \|x - y\|^2, \quad \forall x \in H, \ y \in D. \tag{2.4}$$

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. For $\alpha > 0$, a mapping $A : C \to H$ is called α-inverse strongly monotone if

$$\langle x - y, Ax - Ay \rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

If A is α-inverse-strongly monotone and $0 < \lambda \leq 2\alpha$, then $I - \lambda A : C \to H$ is nonexpansive. In fact, we have that for all $x, y \in C$,

$$\begin{aligned}
\|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay \rangle + \lambda^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\lambda\alpha\|Ax - Ay\|^2 + \lambda^2\|Ax - Ay\|^2 \\
&= \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2.
\end{aligned}$$

Thus, $I - \lambda A : C \to H$ is nonexpansive; see [1, 16, 20] for more results of inverse-strongly monotone mappings. The variational inequalty problem for $A : C \to H$ is to find a point $u \in C$ such that

$$\langle Au, x - u\rangle \geq 0, \quad \forall x \in C. \tag{2.5}$$

The set of solutions of (2.5) is denoted by $VI(C, A)$. We also have that, for any $\lambda > 0$, $u = P_C(I - \lambda A)u$ if and only if $u \in VI(C, A)$. In fact, let $\lambda > 0$. Then, for $u \in C$,

$$\begin{aligned} u = P_C(I - \lambda A)u &\Longleftrightarrow \langle (I - \lambda A)u - u, u - y\rangle \geq 0, \quad \forall y \in C \\ &\Longleftrightarrow \langle -\lambda Au, u - y\rangle \geq 0, \quad \forall y \in C \\ &\Longleftrightarrow \langle Au, u - y\rangle \leq 0, \quad \forall y \in C \\ &\Longleftrightarrow \langle Au, y - u\rangle \geq 0, \quad \forall y \in C \\ &\Longleftrightarrow u \in VI(C, A). \end{aligned}$$

In the case when a Banach space E is a Hilbert space, the definition of a demimetric mapping is as follows: Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 1)$. A mapping $U : C \to H$ with $F(U) \neq \emptyset$ is called k-demimetric if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, x - Ux\rangle \geq (1 - k)\|x - Ux\|^2.$$

The following lemma which was essentially proved in [23] is important and crucial in the proof of our main result. For the sake of completeness, we give the proof.

Lemma 2.1 ([23]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $k \in (-\infty, 1)$ and let U be a k-demimetric mapping of C into H. Then $F(U)$ is closed and convex.*

Proof. Let us show that $F(U)$ is closed. For a sequence $\{q_n\}$ such that $q_n \to q$ and $q_n \in F(U)$, we have from the definition of U that

$$2\langle q - q_n, q - Uq\rangle \geq (1 - k)\|q - Uq\|^2.$$

From $q_n \to q$, we have $0 \geq (1 - k)\|q - Uq\|^2$. From $1 - k > 0$, we have $\|q - Uq\|^2 = 0$ and hence $q = Uq$. This implies that $F(U)$ is closed.

Let us prove that $F(U)$ is convex. Let $p, q \in F(U)$ and set $x = \alpha p + (1 - \alpha)q$, where $\alpha \in [0, 1]$. Then we have

$$\begin{aligned} 2\|x - Ux\|^2 &= 2\langle x - Ux, x - Ux\rangle \\ &= 2\langle \alpha p + (1 - \alpha)q - Ux, x - Ux\rangle \\ &= 2\langle \alpha p + (1 - \alpha)q - (\alpha Ux + (1 - \alpha)Ux), x - Ux\rangle \\ &= 2\alpha\langle p - Ux, x - Ux\rangle + 2(1 - \alpha)\langle q - Ux, x - Ux\rangle \\ &= 2\alpha\langle p - x + x - Ux, x - Ux\rangle + 2(1 - \alpha)\langle q - x + x - Ux, x - Ux\rangle \\ &\leq \alpha(k - 1)\|x - Ux\|^2 + 2\alpha\|x - Ux\|^2 \\ &\quad + (1 - \alpha)(k - 1)\|x - Ux\|^2 + 2(1 - \alpha)\|x - Ux\|^2 \\ &= (k - 1)\|x - Ux\|^2 + 2\|x - Ux\|^2 \end{aligned}$$

and hence $0 \leq (k - 1)\|x - Ux\|^2$. We have from $0 > k - 1$ that $\|x - Ux\| \leq 0$ and hence $x = Ux$. This means that $F(U)$ is convex. □

The following lemma is used in the proof of our main result.

Lemma 2.2 ([26]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 1)$ and let T be a k-demimetric mapping of C into H such that $F(T)$ is nonempty. Let λ be a real number with $0 < \lambda \leq 1 - k$ and define $S = (1-\lambda)I + \lambda T$. Then S is a quasi-nonexpansive mapping of C into H.*

Proof. It is obvious that $F(T) = F(S)$. Since T be a k-demimetric mapping of C into H, we have that for any $x \in C$ and $z \in F(S)$,

$$\begin{aligned}
2\langle x-z, x-Sx\rangle &= 2\langle x-z, x-(1-\lambda)x-\lambda Tx\rangle = 2\lambda\langle x-z, x-Tx\rangle \\
&\geq \lambda(1-k)\|x-Tx\|^2 = \lambda^2\frac{1-k}{\lambda}\|x-Tx\|^2 \\
&= \frac{1-k}{\lambda}\|\lambda x-\lambda Tx\|^2 = \frac{1-k}{\lambda}\|x-Sx\|^2 \\
&\geq \frac{\lambda}{\lambda}\|x-Sx\|^2 = \|x-Sx\|^2.
\end{aligned}$$

Then S is a 0-demimetric mapping. Furthermore, we have from (2.3) that for any $x \in C$ and $z \in F(S)$,

$$\begin{aligned}
&\|x-Sx\|^2 \leq 2\langle x-z, x-Sx\rangle \\
&\iff \|x-Sx\|^2 \leq \|x-Sx\|^2 + \|x-z\|^2 - \|Sx-z\|^2 \\
&\iff \|Sx-z\|^2 \leq \|x-z\|^2 \\
&\iff \|Sx-z\| \leq \|x-z\|.
\end{aligned}$$

Therefore, S is quasi-nonexpansive. □

3 Main Results

In this section, we first prove a weak convergence theorem of Mann's type iteration for finding a common element of the set of common fixed points for a finite family of demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called demiclosed if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup w$ and $x_n - Ux_n \to 0$, $w = Uw$ holds. For example, if C is a nonempty, closed and convex subset of H and T is a nonexpansive mapping of C of H, then T is demiclosed; see [20].

Theorem 3.1 ([13]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{k_1, \ldots, k_M\} \subset (-\infty, 1)$ and $\{\mu_1, \ldots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of k_j-demimetric and demiclosed mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-inverse strongly monotone mappings of C into H. Assume that*

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i)) \neq \emptyset.$$

For any $x_1 = x \in C$, define $\{x_n\}$ as follows:

$$\begin{cases} z_n = \sum_{j=1}^{M} \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^{N} \sigma_i P_C(I - \eta_n B_i)x_n, \\ x_{n+1} = P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n), \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\} \subset (0,\infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0,1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $a, b, c \in \mathbb{R}$ satisfy the following conditions:

(1) $0 < a \le \lambda_n \le \min\{1-k_1, \dots, 1-k_M\}$, $0 < b \le \eta_n \le 2\min\{\mu_1, \dots, \mu_N\}$;
(2) $\sum_{j=1}^{M} \xi_j = 1$ *and* $\sum_{i=1}^{N} \sigma_i = 1$;
(3) $0 < c \le \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequence $\{x_n\}$ converges weakly to a point $z_0 \in \cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i))$, where $z_0 = \lim_{n\to\infty} P_{\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C,B_i))} x_n$.

Next, we prove a strong convergence theorem of Halpern's type iteration for finding a common element of the set of common fixed points for a finite family of demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 3.2 ([24]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{k_1, \dots, k_M\} \subset (-\infty, 1)$ and $\{\mu_1, \dots, \mu_N\} \subset (0,\infty)$. Let $\{T_j\}_{j=1}^{M}$ be a finite family of k_j-demimetric and demiclosed mappings of C into H and let $\{B_i\}_{i=1}^{N}$ be a finite family of μ_i-inverse strongly monotone mappings of C into H. Assume that*

$$\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i)) \neq \emptyset.$$

Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. For $x_1 = x \in C$, let $\{x_n\} \subset C$ be a sequence generated by

$$\begin{cases} z_n = \sum_{j=1}^{M} \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^{N} \sigma_i P_C(I - \eta_n B_i)x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n)\big(P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n)\big), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\} \subset (0,\infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\} \subset (0,1)$ and $a, b, c \in \mathbb{R}$ satisfy the following conditions:

(1) $0 < a \le \lambda_n \le \min\{1-k_1, \dots, 1-k_M\}$, $0 < b \le \eta_n \le 2\min\{\mu_1, \dots, \mu_N\}$;
(2) $\sum_{j=1}^{M} \xi_j = 1$ *and* $\sum_{i=1}^{N} \sigma_i = 1$;
(3) $0 < c \le \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$;
(4) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{i=1}^{\infty} \delta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i))$, where $z_0 = P_{\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C,B_i))} u$.

Using the hybrid method by Nakajo and Takahashi [17], we can also prove a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 3.3 ([2]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{k_1, \dots, k_M\} \subset (-\infty, 1)$ and $\{\mu_1, \dots, \mu_N\} \subset (0,\infty)$. Let $\{T_j\}_{j=1}^{M}$ be a finite* *family of k_j-demimetric and demiclosed mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-inverse strongly monotone mappings of C into H. Assume that*

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i)) \neq \emptyset.$$

Let $x_1 \in C$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n, \\ y_n = \alpha_n x_n + \beta_n z_n + \gamma_n w_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0,1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $a, b, c \in \mathbb{R}$ satisfy the following conditions:

(1) $0 < a \leq \lambda_n \leq \min\{1 - k_1, \dots, 1 - k_M\}$, $0 < b \leq \eta_n \leq 2\min\{\mu_1, \dots, \mu_N\}$;
(2) $\sum_{j=1}^M \xi_j = 1$ *and* $\sum_{i=1}^N \sigma_i = 1$;
(3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))$, where $z_0 = P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C,B_i))} x_1$.

Using the shrinking projection method [25], we finally prove a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of demimetric mappings and the set of common solutions of variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 3.4 ([26]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{k_1, \dots, k_M\} \subset (-\infty, 1)$ and $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of k_j-demimetric and demiclosed mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-inverse strongly monotone mappings of C into H. Assume that*

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i)) \neq \emptyset.$$

Let $x_1 \in C$ and $C_1 = C$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n, \\ y_n = \alpha_n x_n + \beta_n z_n + \gamma_n w_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0,1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ and $a, b, c \in \mathbb{R}$ satisfy the following conditions:

(1) $0 < a \leq \lambda_n \leq \min\{1 - k_1, \dots, 1 - k_M\}$, $0 < b \leq \eta_n \leq 2\min\{\mu_1, \dots, \mu_N\}$;
(2) $\sum_{j=1}^M \xi_j = 1$ *and* $\sum_{i=1}^N \sigma_i = 1$;
(3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))$, *where* $z_0 = P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C,B_i))} x_1$.

4 Applicationss

In this section, we apply Theorems 3.1, 3.2, 3.3 and 3.4 to obtain well-known and new strong convergence theorems in Hilbert spaces. We know the following lemmas obtained by Marino and Xu [15] and Kocourek, Takahashi and Yao [10]; see also [27, 28].

Lemma 4.1 ([15, 27]). *Let* H *be a Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* k *be a real number with* $0 \le k < 1$ *and* $U : C \to H$ *be a* k*-strict pseudo-contraction. If* $x_n \rightharpoonup z$ *and* $x_n - Ux_n \to 0$, *then* $z \in F(U)$.

Lemma 4.2 ([10, 28]). *Let* H *be a Hilbert space, let* C *be a nonempty, closed and convex subset of* H *and let* $U : C \to H$ *be generalized hybrid. If* $x_n \rightharpoonup z$ *and* $x_n - Ux_n \to 0$, *then* $z \in F(U)$.

Using Theorem 3.1, we obtain the following weak convergence results.

Theorem 4.3. *Let* H *be a Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. *Let* $\{B_i\}_{i=1}^N$ *be a finite family of* μ_i*-inverse strongly monotone mappings of* C *into* H. *Assume that* $\cap_{i=1}^N VI(C, B_i) \neq \emptyset$. *For any* $x_1 = x \in C$, *define* $\{x_n\}$ *as follows:*

$$\begin{cases} w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n, \\ x_{n+1} = \alpha_n x_n + \gamma_n w_n, \end{cases}$$

where $\{\eta_n\} \subset (0, \infty)$, $\{\sigma_1, \dots, \sigma_N\} \subset (0, 1)$, $\{\alpha_n\}, \{\gamma_n\} \subset (0, 1)$ *and* $b, c \in \mathbb{R}$ *satisfy the following conditions:*

(1) $0 < b \le \eta_n \le 2 \min\{\mu_1, \dots, \mu_N\}$;
(2) $\sum_{i=1}^N \sigma_i = 1$;
(3) $0 < c \le \alpha_n, \gamma_n < 1$ *and* $\alpha_n + \gamma_n = 1$.

Then $\{x_n\}$ *converges weakly to* $z_0 \in \cap_{i=1}^N VI(C, B_i)$, *where* $z_0 = \lim_{n\to\infty} P_{\cap_{i=1}^N VI(C,B_i)} x_n$.

Proof. The identity mapping I is a $\frac{1}{2}$-demimetric mapping of C into H. Putting $T_j = I$ for all $j \in \{1, \dots, M\}$ and $\lambda_n = \frac{1}{2}$ for all $n \in \mathbb{N}$ in Theorem 3.1, we have that $z_n = x_n$ for all $n \in \mathbb{N}$. Furthermore, replacing $\beta_n + \gamma_n$ by γ_n, we have the desired result from Theorem 3.1. □

Theorem 4.4. *Let* H *be a Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* $\{T_j\}_{j=1}^M$ *be a finite family of generalized hybrid mappings of* C *into* H *and let* $\{U_i\}_{i=1}^N$ *be a finite family of nonexpansive mappings of* C *into* H. *Assume that*

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N F(U_i)) \neq \emptyset.$$

For any $x_1 = x \in C$, *define* $\{x_n\}$ *as follows:*

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j((1 - \lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i P_C((1 - \eta_n)I + \eta_n U_i)x_n, \\ x_{n+1} = P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n), \end{cases}$$

where $\{\lambda_n\},\{\eta_n\} \subset (0,\infty)$, $\{\xi_1,\dots,\xi_M\},\{\sigma_1,\dots,\sigma_N\} \subset (0,1)$, $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \subset (0,1)$ *and* $a,b,c \in \mathbb{R}$ *satisfy the following conditions:*

(1) $0 < a \le \lambda_n \le 1$, $0 < b \le \eta_n \le 1$;
(2) $\sum_{j=1}^M \xi_j = 1$ *and* $\sum_{i=1}^N \sigma_i = 1$;
(3) $0 < c \le \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequence $\{x_n\}$ *converges weakly to a point* $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N F(U_i))$, *where* $z_0 = \lim_{n\to\infty} P_{\cap_{j=1}^M F(T_j)\cap(\cap_{i=1}^N F(U_i))} x_n$.

Proof. Since T_j is generalized hybrid, T_j is 0-demimetric. Furthermore, from Lemma 4.2 T_j is demiclosed. Since U_i is nonexpansive, $B_i = I - U_i$ is a $\frac{1}{2}$-inverse strongly monotone mapping. We also have from $\cap_{i=1}^N F(U_i) \neq \emptyset$ that

$$\cap_{i=1}^N VI(C, I - U_i) = \cap_{i=1}^N F(P_C U_i) = \cap_{i=1}^N F(U_i).$$

Therefore, we have the desired result from Theorem 3.1. □

Using Theorem 3.2, we can prove a strong convergence theorem for a finite family of strict pseudo-contractions in a Hilbert space.

Theorem 4.5. *Let* H *be a Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* $\{k_1,\dots,k_M\} \subset [0,1)$ *and let* $\{T_j\}_{j=1}^M$ *be a finite family of* k_j*-strict pseudo-contractions of* C *into* H. *Let* $\{u_n\}$ *be a sequence in* C *such that* $u_n \to u$. *Assume that* $\cap_{j=1}^M F(T_j) \neq \emptyset$. *For any* $x_1 = x \in C$, *define* $\{x_n\}$ *as follows:*

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n)\big(P_C(\alpha_n x_n + \beta_n z_n)\big), \end{cases}$$

where $a,c \in \mathbb{R}$, $\{\lambda_n\} \subset (0,\infty)$, $\{\xi_1,\dots,\xi_M\} \subset (0,1)$ *and* $\{\alpha_n\},\{\beta_n\},\{\beta_n\} \subset (0,1)$ *satisfy the following conditions:*

(1) $0 < a \le \lambda_n \le \min\{1-k_1,\dots,1-k_M\}$;
(2) $\sum_{j=1}^M \xi_j = 1$;
(3) $0 < c \le \alpha_n, \beta_n < 1$ *and* $\alpha_n + \beta_n = 1$;
(4) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{i=1}^\infty \delta_n = \infty$.

Then $\{x_n\}$ *converges strongly to* $z_0 \in \cap_{j=1}^M F(T_j)$, *where* $z_0 = P_{\cap_{j=1}^M F(T_j)} u$.

Proof. Since T_j is a k_j-strict pseud-contraction of C into H such that $F(T_j) \neq \emptyset$, T_j is k_j-demimetric. Furthermore, from Lemma 4.1, T_j is demiclosed. Furthermore, if $B_i = 0$ for all $i \in \{1,\dots,N\}$ in Theorem 3.2, then B_i is a 1-inverse strongly monotone mapping. Putting $\eta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.2, we have that $w_n = x_n$ for all $n \in \mathbb{N}$. Furthermore, replaceing $\beta_n + \gamma_n$ by β_n. we have the desired result from Theorem 3.2. □

Using Theorem 3.3, we prove a strong convergence theorem for a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 4.6. *Let* H *be a Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* $\{\mu_1,\dots,\mu_N\} \subset (0,\infty)$. *Let* $\{B_i\}_{i=1}^N$ *be a finite family of* μ_i*-inverse strongly monotone mappings of* C *into* H. *Assume that* $\cap_{i=1}^N VI(C,B_i) \neq \emptyset$. *Let* $x_1 \in C$. *Let* $\{x_n\}$ *be a sequence* *generated by*

$$\begin{cases} w_n = \sum_{i=1}^{N} \sigma_i P_C(I - \eta_n B_i)x_n, \\ y_n = \alpha_n x_n + \gamma_n w_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $b, c \in \mathbb{R}$, $\{\eta_n\} \subset (0, \infty)$, $\{\sigma_1, \ldots, \sigma_N\} \subset (0,1)$ *and* $\{\alpha_n\}, \{\gamma_n\} \subset (0,1)$ *satisfy the following conditions:*

(1) $0 < b \leq \eta_n \leq 2\min\{\mu_1, \ldots, \mu_N\}$;
(2) $\sum_{i=1}^{N} \sigma_i = 1$;
(3) $0 < c \leq \alpha_n, \gamma_n < 1$ *and* $\alpha_n + \gamma_n = 1$.

Then $\{x_n\}$ *converges strongly to* $z_0 \in \cap_{i=1}^{N} VI(C, B_i)$, *where* $z_0 = P_{\cap_{i=1}^{N} VI(C,B_i)} x_1$.

Proof. The identity mapping I is a $\frac{1}{2}$-demimetric mapping of C into H. Putting $T_j = I$ for all $j \in \{1, \ldots, M\}$ and $\lambda_n = \frac{1}{2}$ for all $n \in \mathbb{N}$ in Theorem 3.3, we have that $z_n = x_n$ for all $n \in \mathbb{N}$. Furthermore, replacee $\beta_n + \gamma_n$ by γ_n. Thus, we have the desired result from Theorem 3.3. □

Using Theorem 3.4, we prove a strong convergence theorem for a finite family of generalized hybrid mappings and a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 4.7. *Let* H *be a Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* $\{\mu_1, \ldots, \mu_N\} \subset (0, \infty)$. *Let* $\{T_j\}_{j=1}^{M}$ *be a finite family of generalized hybrid mappings of* C *into* H *and let* $\{B_i\}_{i=1}^{N}$ *be a finite family of* μ_i*-inverse strongly monotone mappings of* C *into* H. *Assume that*

$$\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i)) \neq \emptyset.$$

Let $x_1 \in C$ *and* $C_1 = C$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = \sum_{j=1}^{M} \xi_j((1 - \lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^{N} \sigma_i P_C(I - \eta_n B_i)x_n, \\ y_n = \alpha_n x_n + \beta_n z_n + \gamma_n w_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \ldots, \xi_M\}, \{\sigma_1, \ldots, \sigma_N\} \subset (0,1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0,1)$ *and* $a, b, c \in \mathbb{R}$ *satisfy the following conditions:*

(1) $0 < a \leq \lambda_n \leq 1$, $0 < b \leq \eta_n \leq 2\min\{\mu_1, \ldots, \mu_N\}$;
(2) $\sum_{j=1}^{M} \xi_j = 1$ *and* $\sum_{i=1}^{N} \sigma_i = 1$;
(3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$.

Then the sequence $\{x_n\}$ *converges strongly to a point* $z_0 \in \cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C, B_i))$, *where* $z_0 = P_{\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} VI(C,B_i))} x_1$.

Proof. Since T_j is a generalized hybrid mapping of C into H such that $F(T_j) \neq \emptyset$, from (1.2), T_j is 0-demimetric. Furthermore, from Lemma 4.2, T_j is demiclosed. Therefore, we have the desired result from Theorem 3.4. □

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