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Geometry of anisotropic surface energy

Miyuki Koiso

Institute of Mathematics for Industry,
Kyushu University

1 Introduction

We discuss a variational problem for piecewise-smooth hypersurfaces in the $(n+1)$-dimensional euclidean space $\mathbb{R}^{n+1}$. Our functional is a natural generalization of area for surfaces. It is an anisotropic energy for hypersurfaces which is an integral of an energy density that depends on the direction of the normal to the considered hypersurface. Since in the special case where the energy density is constant one, our energy functional is the surface area, and so the variational problem of such an energy with enclosed-volume constrained has minimal hypersurfaces and hypersurfaces with constant mean curvature (CMC hypersurfaces) as solutions of a special case. It is known that any embedded closed (compact without boundary) CMC hypersurface in $\mathbb{R}^{n+1}$ is a round n-sphere ([1]), and any CMC surface with genus 0 in $\mathbb{R}^3$ is a round 2-sphere ([6]). However, for our variational problem, such uniqueness results are not true in general (Theorems 1.1, 1.2). On the other hand, any stable closed CMC hypersurface in $\mathbb{R}^{n+1}$ is a round n-sphere ([3]). Here a critical point of a variational problem is said to be stable if the second variation of the corresponding energy functional is always nonnegative. For our variational problem, if we assume that the energy density function is of C^3 and convex (for definition, see §2), such a uniqueness result also holds (Theorem 1.3). In this article, we explain these results precisely.

Let $\gamma : S^n \to \mathbb{R}_{>0}$ be a positive C^3- function on the unit sphere $S^n = \{\nu \in \mathbb{R}^{n+1} \mid \|\nu\| = 1\}$ in $\mathbb{R}^{n+1}$. Let $M = \cup_{i=1}^k M_i$ be an n-dimensional oriented connected compact C^∞ manifold, and $X : M \to \mathbb{R}^{n+1}$ be a piecewise-C^3 immersion. This means that X is continuous on M and it is a C^3 immersion on each M_i, where M_i is an n-dimensional submanifold of M with smooth boundary. We sometimes say that X is a piecewise smooth hypersurface. Denote by $\mathcal{S}(X)$ the set of singularities of X, and let $\nu : M \setminus \mathcal{S}(X) \to S^n$ be the unit normal vector field along $X|_{M\setminus\mathcal{S}(X)}$. We can think that ν is defined on each M_i. An anisotropic energy $\mathcal{F}_\gamma(X)$ of X is defined as follows.

$$\mathcal{F}_\gamma(X) := \sum_{i=1}^{k} \int_{M_i} \gamma(\nu)\, dM, \tag{1}$$

where dM is the n-dimensional volume form of M induced by X. Such an energy was introduced by J. W. Gibbs (1839-1903) in order to model the shape of small crystals, and it is used as a mathematical model of anisotropic surface energy ([18],[19]). In the special case where $\gamma \equiv 1$, $\mathcal{F}_\gamma(X)$ is the usual n-dimensional volume of the piecewise-immersed hypersurface X. Another special case gives surface area in the Lorentz-Minkowski space ([7]).

The $(n+1)$-dimensional (algebraic) volume $V(X)$ enclosed by X is given by

$$V(X) := \frac{1}{n+1}\sum_{i=1}^{k}\int_{M_i}\langle X, \nu\rangle \, dM.$$

For any positive number $V > 0$, among all closed hypersurfaces in $\mathbb{R}^{n+1}$ enclosing the same $(n+1)$-dimensional volume V, there exists a unique (up to translation in $\mathbb{R}^{n+1}$) minimizer $W(V)$ of $\mathcal{F}_\gamma$ (Wulff's theorem. cf.[17]). Here a closed hypersurface means that the boundary (having tangent space almost everywhere) of a set of positive Lebesgue measure. Therefore, $W(V)$ is the solution of the isoperimetric problem for the functional $\mathcal{F}_\gamma$. The minimizer $W(V_0)$ for $V_0 := (n+1)^{-1}\displaystyle\int_{S^n}\gamma(\nu)\, dS^n$ is called the Wulff shape (for γ) (the standard definition of the Wulff shape will be given in §2), and we will denote it by W or W_γ. When $\gamma \equiv 1$, W is the unit sphere S^n. All $W(V)$ are homothetic to W. It is known that W is convex but not necessarily smooth. On the other hand, for a given convex set $\tilde{W}$ having the origin of $\mathbb{R}^{n+1}$ as an interior point, there exists a continuous function $\gamma : S^n \to \mathbb{R}_{>0}$ such that $\tilde{W}$ is the Wulff shape for γ. However, such γ is not unique. The "smallest" γ is called the convex integrand for $\tilde{W}$ (or, simply, convex) (for the precise definition, see §2).

Each equilibrium hypersurfaces of the functional $\mathcal{F}_\gamma$ for $(n+1)$-dimensional volume-preserving variations has constant anisotropic mean curvature. Here the anisotropic mean curvature Λ of a piecewise C^r ($r \geq 2$) hypersurface $X : M \to \mathbb{R}^{n+1}$ is defined as

$$\Lambda := \frac{1}{n}(-\mathrm{div}_M D\gamma + nH\gamma),$$

where $D\gamma$ is the gradient of γ and H is the mean curvature of X. In fact, we have

Proposition 1.1 (Euler-Lagrange equations, Koiso [8]. For $n = 2$, see B. Palmer [15]). *A piecewise C^r ($r \geq 2$) immersion $X : M = \sum_{i=1}^{k} M_i \to \mathbf{R}^{n+1}$ is a critical point of the anisotropic energy $\mathcal{F}_\gamma(X) = \int_M \gamma(\nu)\, dM$ for $(n+1)$-dimensional volume-preserving variations if and only if*

(i) The anisotropic mean curvature Λ of X is constant on M, and

(ii) $(\xi\circ\nu|_{M_i} - \xi\circ\nu|_{M_j})(\zeta) \in T_\zeta M_i \cap T_\zeta M_j = T_\zeta(\partial M_i \cap \partial M_j)$ at any $\zeta \in \partial M_i \cap \partial M_j$, here $\xi\circ\nu = D\gamma + \gamma(\nu)\nu : M \to \mathbf{R}^{n+1}$ is called the Cahn-Hoffman field for X or the anisotropic Gauss map of X, and the tangent space of a submanifold of $\mathbb{R}^{n+1}$ is naturally identified with a linear subspace of $\mathbb{R}^{n+1}$.

In this article, we call a piecewise C^r ($r \geq 2$) immersion X a CAMC (constant anisotropic mean curvature) hypersurface if it satisfies (i) and (ii) in Proposition 1.1. A CAMC hypersurface is said to be stable if the second variation of the energy for any $(n+1)$-dimensional volume-preserving variation is nonnegative.

In general, the Wulff shape and CAMC hypersurfaces are not smooth. When the Wulff shape is a smooth strictly convex hypersurface (which is equivalent to the condition that γ is uniformly convex, see §2), then any CAMC hypersurface $X : M \to \mathbb{R}^{n+1}$ is also an immersion. And in this case, if a closed CAMC hypersurface X satisfies either one of the following conditions (i)-(iii), then it is a homothety of the Wulff shape: (i) X is embedded ([5]). (ii) X is stable ([14]). (iii) $n = 2$ and the genus of M is zero ([10], [2]). In this paper, we show that if γ is not uniformly convex, such a uniqueness result for embedded CAMC hypersurfaces in $\mathbb{R}^{n+1}$ is not always true, the uniqueness for CAMC surfaces with genus zero in $\mathbb{R}^3$ is not always true either:

Theorem 1.1 (Koiso [8]). *There exists a C^3 function $\gamma : S^n \to \mathbb{R}^+$ which is not a convex integrand such that there exists a closed embedded CAMC hypersurface for γ which is not (any homothety and translation of) the Wulff shape.*

Theorem 1.2 (Koiso [8]). *There exists a C^3 function $\gamma : S^2 \to \mathbb{R}^+$ which is not a convex integrand such that there exists a closed embedded CAMC surface with genus zero for γ which is not (any homothety and translation of) the Wulff shape.*

We conjecture that, for any C^3 function $\gamma : S^n \to \mathbb{R}^+$ which is not a convex integrand, there exists a closed embedded CAMC hypersurface for γ which is not (any homothety and translation of) the Wulff shape.

As for the uniqueness of stable closed CAMC surfaces, we obtain the following result.

Theorem 1.3 (Koiso [8]). *Assume $\gamma : S^2 \to \mathbb{R}^+$ is of C^3 and it is the convex integrand of its Wulff shape W. Then, any closed stable CAMC surface in $\mathbb{R}^3$ for γ is (up to translation and homothety) W.*

It is expected that Theorem 1.3 can be generalized to hypersurfaces in $\mathbb{R}^{n+1}$.

We should remark again that, although it is natural and important to study variational problems for anisotropic surface energy for which equilibrium surfaces have singular points, it has not yet done sufficiently well. As for planer curves, F. Morgan [11] proved that, if $\gamma : S^1 \to \mathbb{R}_{>0}$ is continuous and convex, then any closed equilibrium rectifiable curve for $\mathcal{F}_\gamma$ in $\mathbb{R}^2$ is (up to translation and homothety) a covering of the Wulff shape (see also [12]). About uniqueness of stable closed equilibria (not necessary the energy minimum) in $\mathbb{R}^3$, B. Palmer [15] proved the following result.

Theorem 1.4 (B. Palmer [15]). Let $\gamma : S^2 \to \mathbb{R}_{>0}$ be a convex integrand of C^3. Assume that the Wulff shape $W_\gamma \subset \mathbb{R}^3$ is a piecewise C^2 surface whose principal curvatures are bounded by a positive constant from below. Let $X : M \to \mathbb{R}^3$ be a closed embedded piecewise smooth CAMC surface. We assume that the Cahn-Hoffman field $\tilde{\xi} : M \setminus \mathcal{S}(X) \to \mathbb{R}^3$ can be extended to M continuously. Then, if X is stable, then it is (up to translation and homothety) the Wulff shape W_γ.

The mapping $\tilde{\xi}$ in Theorem 1.4 is defined as follows. For a C^1 function $\gamma : S^n \to \mathbb{R}_{>0}$, denote by $D\gamma$ the gradient of γ on S^n. Then, by using the Cahn-Hoffman field $\xi : S^n \to \mathbb{R}^{n+1}$, $\xi(\nu) := D\gamma(\nu) + \gamma(\nu)\nu$, on S^n, the Cahn-Hoffman field $\tilde{\xi} : M \setminus \mathcal{S}(X) \to \mathbb{R}^{n+1}$ for a piecewise C^1 hypersurface X is defined by $\tilde{\xi} := \xi \circ \nu$. They say that the origin of the Cahn-Hoffman field is the so-called capillary vector formulation of interface energies introduced by John W. Cahn and David W. Hoffman (1972).

By using the Cahn-Hoffman field, we explain an application of our results mentioned above to the anisotropic mean curvature flow. Let $\gamma : S^n \to \mathbb{R}_{>0}$ be of C^3 with Cahn-Hoffman field $\xi : S^n \to \mathbb{R}^{n+1}$. Let $X : M^n \to \mathbb{R}^{n+1}$ be an embedded piecewise C^2 hypersurface with (not necessary constant) anisotropic mean curvature Λ. Consider the anisotropic mean curvature flow

$$X_t : M \to \mathbb{R}^{n+1}, \quad \frac{\partial}{\partial t} X_t = \Lambda \tilde{\xi}.$$

Our non-uniqueness result (Theorem 1.2) implies that there exists a C^3 function $\gamma : S^2 \to \mathbb{R}^+$ such that there exists a closed embedded self-similar shrinking solution with genus zero for γ other than the Wulff shape. We should remark that, in contrast with our result, the round sphere is the only closed embedded self-similar shrinking solution of mean curvature flow in $\mathbb{R}^3$ with genus zero ([4]).

Finally we give an important remark about the convexity of the Wulff shape in our main results. In the above theorems, we assumed that the integrand $\gamma : S^n \to \mathbb{R}^+$ is of C^3. This assumption implies that the Wulff shape W_γ has singularities in general, but at any regular point $p \in W_\gamma$ the principal curvatures of W_γ for the inward normal are all positive (Theorem 2.1). If a Wulff shape has a flat face, then the integrand $\gamma : S^n \to \mathbb{R}^+$ has a point $\nu \in S^n$ where γ is not differentiable (cf. [13]). It is our important future work to study such case.

This article is organized as follows. In §2 we give definitions of the Wulff shape, the Cahn-Hoffman field, the anisotropic mean curvature, and their fundamental properties. In §3 we give outlines of the proofs of our main theorems.

2 Preliminaries

2.1 Wulff shape, convexity of the integrand

Let $\gamma : S^n \to \mathbb{R}^+$ be a continuous function. Set

$$W[\gamma] := \cap_{\nu \in S^n} \Big\{ X \in \mathbb{R}^{n+1} \mid \langle X, \nu \rangle \leq \gamma(\nu) \Big\}. \tag{2}$$

Then $W[\gamma]$ is a convex set which is not smooth in general. $W[\gamma]$ is often called the Wulff shape for γ. However, in this article we call the boundary W_γ of $W[\gamma]$ the Wulff shape (for γ):

$$W_\gamma := W := \partial\left(\cap_{\nu\in S^n}\left\{X \in \mathbb{R}^{n+1} \mid \langle X, \nu\rangle \leq \gamma(\nu)\right\}\right). \tag{3}$$

Definition 2.1. For $\gamma \in C^0(S^n, \mathbb{R}^+)$, the set $\{\gamma(\nu)\nu \ ; \ \nu \in S^n\} \subset \mathbb{R}^{n+1}$ is called the Wulff plot of γ.

Example 2.1. Let $n = 1$. For $\nu = (\nu_1, \nu_2) \in S^1 \subset \mathbb{R}^2$, define $\gamma(\nu) := |\nu_1| + |\nu_2|$. Then the Wulff shape is the square and the Wulff plot is the dotted curve in Figure 1.

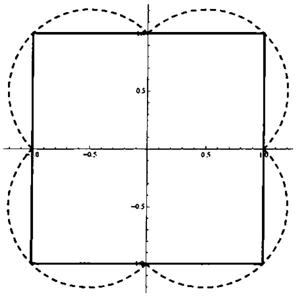

Figure 1: The Wulff shape (solid curve) and the Wulff plot (dotted cuve) for γ in Example 2.1

Example 2.2. Let $n = 1$. For $\nu = (\nu_1, \nu_2) \in S^1 \subset \mathbb{R}^2$, define $\gamma(\nu) := 4\nu_1^3 - 3\nu_1 + 1.2$. Then the image of the Cahn-Hoffman field (see Definition 2.4) is given by Figure 2. The Wulff shape is its subset that is the convex solid curve.

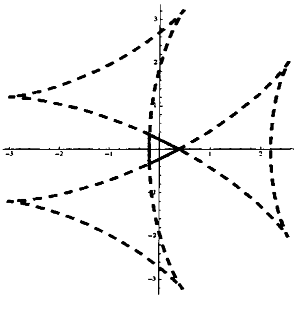

Figure 2: The image of the Cahn-Hoffman field for γ in Example 2.2. The Wulff shape is the convex solid curve.

The homogeneous extension $\overline{\gamma} : \mathbb{R}^{n+1} \to \mathbb{R}_{\geq 0}$ of γ is defined by

$$\overline{\gamma}(rX) = r\gamma(X), \qquad \forall X \in S^n,\ \forall r \geq 0.$$

If $\overline{\gamma}$ is convex (that is, $\overline{\gamma}(X+Y) \leq \overline{\gamma}(X) + \overline{\gamma}(Y)$, $X, Y \in \mathbb{R}^{n+1}$) and has the following symmetry $\overline{\gamma}(-X) = \overline{\gamma}(X)$, then $\overline{\gamma}$ defines a norm in $\mathbb{R}^{n+1}$. In this case, consider the dual norm

$$\overline{\gamma}^*(Y) = \sup\{Y \cdot Z \mid \overline{\gamma}(Z) \leq 1\}$$

of $\overline{\gamma}$. Then the unit sphere

$$\{Y \in \mathbb{R}^{n+1} \mid \overline{\gamma}^*(Y) = 1\}$$

of $\overline{\gamma}^*$ coincides with the Wulff shape W.

Definition 2.2. A continuous map $\gamma : S^n \to \mathbb{R}_{>0}$ is called a convex integrand if the Wulff plot of the map

$$1/\gamma : S^n \to \mathbb{R}_{>0}, \quad 1/\gamma(\nu) := \gamma(\nu)^{-1},\ \forall \nu \in S^n$$

is convex.

Proposition 2.1. *Assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of C^2. Then the following (i) - (iii) are equivalent.*

(i) γ is a convex integrand.

(i) $\overline{\gamma}(v_1 + v_2) \leq \overline{\gamma}(v_1) + \overline{\gamma}(v_2)$ holds for all $v_1, v_2 \in \mathbb{R}^{n+1}$.

(ii) $D^2\gamma + \gamma \cdot 1$ is positive-semidefinite, that is, the eigenvalues are all nonnegative, on the tangent space at each point in S^n.

Remark 2.1 ([17]). *(i) For any continuous $\gamma : S^n \to \mathbb{R}_{>0}$, there exists a unique convex integrand $\tilde{\gamma}$ such that $W_\gamma = W_{\tilde{\gamma}}$ holds.*

(ii) $\tilde{\gamma}$ is the smallest integrand having the same Wulff shape, that is

$$\tilde{\gamma}(\nu) = \min\{f(\nu) \mid f \in C^0(S^n, \mathbb{R}_{>0}), W_f = W_{\tilde{\gamma}}\}, \quad \forall \nu \in S^n$$

holds.

Remark 2.2 ([13]). *If a convex integrand γ is of C^1, then $W(\gamma)$ is strictly convex.*

Lemma 2.1. *For $\gamma \in C^3(S^n, \mathbb{R}_{>0})$, the following (i) and (ii) are equivalent.*

(i) W_γ is a closed strictly-convex smooth hypersurface, that is, all of the principal curvatures of W are positive for the inward-pointing unit normal.

(ii) The $n \times n$ matrix $D^2\gamma + \gamma \cdot 1$ is positive-definite, that is, the eigenvalues are all positive, on the tangent space at each point in S^n, where $D^2\gamma$ is the Hessian of γ and 1 is the unit matrix.

Definition 2.3. Assume $\gamma \in C^3$. γ is said to be uniformly convex if the matrix $D^2\gamma + \gamma 1$ is positive-definite at each point in S^n.

2.2 Cahn-Hoffman field

In this section, we give the definition of the Cahn-Hoffman field on S^n and its important properties. One of the most important properties of the Cahn-Hoffman field is that it gives a representation of the Wulff shape.

Definition 2.4. Assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of C^1. We call the continuous map $\xi : S^n \to \mathbb{R}^{n+1}$ defined as

$$\xi(\nu) := \xi_\gamma(\nu) := D\gamma + \gamma(\nu)\nu, \quad \nu \in S^n \tag{4}$$

the Cahn-Hoffman field on S^n (for γ).

Proposition 2.2 (Koiso [8]). *Assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of C^2. Then, the Cahn-Hoffman field ξ on S^n satisfies the following (i) and (ii), hence ξ is a C^1-(wave)front.*
(i)

$$\langle (d\xi)_\nu(u), \nu \rangle = 0, \quad \forall \nu \in S^n, \ \forall u \in T_\nu S^n. \tag{5}$$

(ii) The mapping

$$(\xi, id_{S^n}) : S^n \to \mathbb{R}^{n+1} \times S^n, \quad (\xi, id_{S^n})(\nu) := (\xi(\nu), \nu) \tag{6}$$

is a C^1-immersion.

Proposition 2.2 (i) implies the following Corollaries 2.1, 2.2.

Corollary 2.1. *Assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of C^2. Then, at any point $\nu \in S^n$ we may call the hyperplane perpendicular to ν the tangent hyperplane of ξ_γ at ν (or at $\xi_\gamma(\nu)$).*

Corollary 2.2. *Assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of C^2. Then, at any point $\nu \in S^n$ where ξ_γ is an immersion, ν itself gives a unit normal to ξ_γ.*

The following proposition gives an important relation between γ and its Cahn-Hoffman field.

Proposition 2.3. *Assume that $\gamma : S^n \to \mathbb{R}_{>0}$ is of C^2. Then, γ is the support function of ξ_γ, that is, $\gamma(\nu)$ is the distance between the origin of $\mathbb{R}^{n+1}$ and the tangent hyperplane of ξ_γ at the point $\xi_\gamma(\nu)$.*

Theorem 2.1 (Koiso [8]). *If $\gamma : S^n \to \mathbb{R}^+$ is of C^3, then the following (i) and (ii) hold.*

(i) The principal curvatures at any regular point of the Cahn-Hoffman field ξ never vanish.

(ii) For any singular point $\nu \in S^n$ of ξ, and for any smooth one-parameter family $\nu_t \in S^n$ with $\lim_{t\to\infty} \nu_t = \nu$ of regular points of ξ with principal curvatures $k_1(t), \cdots, k_n(t)$ which are continuous in t, the limit $\lim_{t\to\infty} |k_i(t)|$ is either ∞ or a nonzero real value, $(i = 1, \cdots, n)$.

We will give the relationship between the Cahn-Hoffman field and the Wulff shape. Denote by $\hat{W}_\gamma = \hat{W}$ the image of ξ_γ, that is

$$\hat{W}_\gamma := \hat{W} := \xi_\gamma(S^n).$$

Lemma 2.2. *For $\gamma \in C^1(S^n, \mathbb{R}_{>0})$, W_γ is the unique convex hypersurface determined by the following properties (i) and (ii).*

(i) $W_\gamma \subset \hat{W}_\gamma$.

(ii) The (open) domain bounded by W_γ contains the origin of $\mathbb{R}^{n+1}$.

Lemma 2.3. *For $\gamma \in C^1(S^n, \mathbb{R}_{>0})$, the following (i) and (ii) are equivalent.*

(i) γ is convex.

(ii) $W_\gamma = \hat{W}_\gamma$.

2.3 First variation formula and anisotropic mean curvature (cf. [9], [8])

First we consider a C^2 immersion $X : M_0 \to \mathbb{R}^{n+1}$ from an oriented compact connected n-dimensional C^∞ manifold M_0 with smooth boundary ∂M_0 to $\mathbb{R}^{n+1}$ with unit normal ν. Let

$$X_\epsilon = X + \epsilon(\eta + \psi\nu) + \mathcal{O}(\epsilon^2)$$

be a smooth variation of X, where η is the tangential component and $\psi\nu$ is the normal component of the variation vector field δX of X_ϵ. Then the first variation of the anisotropic energy $\mathcal{F}_\gamma$ is given as follows.

$$\begin{aligned} \delta\mathcal{F}_\gamma &:= \left.\frac{d\mathcal{F}_\gamma(X_\epsilon)}{d\epsilon}\right|_{\epsilon=0} \\ &= \int_{M_0} \psi(\operatorname{div}_{M_0} D\gamma - nH\gamma)\, dM_0 + \oint_{\partial M_0} -\psi\langle D\gamma, N\rangle + \gamma\langle \eta, N\rangle\, d\tilde{s}, \end{aligned} \tag{7}$$

where H is the mean curvature of X, dM_0 is the n-dimensional volume form of M_0 induced by X, N is the outward-pointing unit conormal along ∂M_0, $d\tilde{s}$ is the $(n-1)$-dimensional volume form of ∂M_0. Denote by R the $\pi/2$-rotation on the (N, ν)-plane, and by p the projection from $\mathbb{R}^{n+1}$ to the (N, ν)-plane. Then, we have ([8])

$$\delta\mathcal{F}_\gamma = \int_{M_0} \psi(\operatorname{div}_{M_0} D\gamma - nH\gamma)\, dM_0 - \oint_{\partial M_0} \langle \delta X, R(p(\xi \circ \nu))\rangle\, d\tilde{s}. \tag{8}$$

On the other hand the first variation of the $(n+1)$-dimensional volume enclosed by X_ϵ is well-known:

$$\delta V = \int_{M_0} \psi\, dM_0. \tag{9}$$

(8) with (9) gives the Euler-Lagrange equations in Proposition 1.1. Especially, if X is a critical point of $\mathcal{F}_\gamma$ for all $(n+1)$-dimensional volume-preserving variations,

$$\mathrm{div}_M D\gamma - nH\gamma = \text{constant} \quad \text{on } M_0, \tag{10}$$

which is the reason why

$$\Lambda := \frac{1}{n}(-\mathrm{div}_M D\gamma + nH\gamma)$$

is called the anisotropic mean curvature of X (cf. [16], [9]). It is shown that

$$\Lambda = -\frac{1}{n}\mathrm{trace}_M(D^2\gamma + \gamma 1) \circ d\nu = -\frac{1}{n}\mathrm{trace}_M\, d(\xi \circ \nu) \tag{11}$$

holds (cf. [9]). X is called a hypersurface with constant anisotropic mean curvature (CAMC) if Λ is constant.

Remark 2.3. (i) In the special case where $\gamma \equiv 1$, $\Lambda = H$.

(ii) At points where (γ is of C^2 and) γ is uniformly convex, by (11), the equation "Λ = constant" is elliptic.

Let $\nu : M \setminus \mathcal{S}(X) \to S^n$ be the Gauss map of a piecewise C^2 immersion $X : M = M^n \to \mathbb{R}^{n+1}$ with singular set $\mathcal{S}(X)$ (the set of singularities of X). Then, for any point $p \in M \setminus \mathcal{S}(X)$, there is a point $\xi(\nu)$ in $\hat{W}_\gamma$ where ν gives the normal to $\hat{W}_\gamma$, and

$$\Lambda = -\frac{1}{n}\mathrm{trace}(d(\xi \circ \nu)) \tag{12}$$

holds. Since ξ^{-1} gives the unit normal vector field ν_ξ for the Cahn-Hoffman field $\xi : S^n \to \mathbb{R}^{n+1}$, we have the following:

Proposition 2.4. *The anisotropic mean curvature of the Cahn-Hoffman field $\xi : S^n \to \mathbb{R}^{n+1}$ is -1 at any regular point. Hence, particularly the anisotropic mean curvature of the Wulff shape (for the outward-pointing unit normal) is -1 at any regular point.*

Proposition 2.4 immediately gives the following:

Corollary 2.3. *Cahn-Hoffman field is a critical point of $\mathcal{F}_\gamma$ for $(n+1)$-dimensional volume-preserving variations.*

3 Idea of proofs of the main theorems

Proof of Theorems 1.1, 1.2. Example 2.2 gives an example. In fact, each of the three closed dotted curves in Figure 2 is a closed CAMC curve which is not any homothety of the Wulff shape. And it is easy to get a higher dimensional example by using rotation. □

In order to prove Theorem 1.3, we need the following lemmas.

Lemma 3.1. *Assume $\gamma : S^n \to \mathbb{R}^+$ is of C^3 and the convex integrand of its Wulff shape W. Then the Gauss curvature of W is bounded below by a positive constant.*

Proof of Lemma 3.1. From Theorem 2.1, the absolute values of the principal curvatures of any regular point of the Cahn-Hoffman field ξ are bounded by a positive constant from below. Hence the Gauss curvatures at regular points of W are bounded below by a positive constant. □

Lemma 3.2 (Koiso [8]). *Assume $\gamma : S^n \to \mathbb{R}^+$ is of C^3 and the convex integrand of its Wulff shape W. Let $X : M \to \mathbb{R}^{n+1}$ be a closed piecewise C^3 CAMC hypersurface with unit normal $\nu : M \setminus \mathcal{S}(X) \to S^n$, here $\mathcal{S}(X)$ is the set of singularities of X. Then, Cahn-Hoffman field $\tilde{\xi} := \xi \circ \nu : M \setminus \mathcal{S}(X) \to \mathbb{R}^{n+1}$, ($\xi := D\gamma(\nu) + \gamma(\nu)\nu$, $\xi : S^n \to \mathbb{R}^{n+1}$) can be extended to M.*

Idea of the proof of Theorem 1.3. Because of Lemmas 2.2, 3.1, and 3.2, we can use the idea in [15]. Let $\xi : S^2 \to \mathbb{R}^3$ be the Cahn-Hoffman field. Let $X : M \to \mathbb{R}^3$ be a closed piecewise C^3 CAMC surface. We consider the following variation X_t of X that preserves the enclosed volume:

$$X_t(u,v) := \mu(t) \cdot (X(u,v) + t\xi(\nu(u,v))), \quad (u,v) \in M. \cdot$$

Actually, by Lemma 3.2, each X_t gives a piecewise C^2 closed surface. By a long calculation, we can prove the following ([8]), which gives the desired result.

$$\left.\frac{d^2}{dt^2}\right|_{t=0} \mathcal{F}_\gamma(X_t) \geq 0. \iff X \text{ is a homothety of } W (\text{up to translation}).$$

□

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Institute of Mathematics for Industry
Kyushu University
744 Motooka, Nishi-ku
Fukuoka 819-0395, Japan
E-mail address: koiso@math.kyushu-u.ac.jp

九州大学　マス・フォア・インダストリ研究所　小磯 深幸