 

# ON SOME RESULTS FOR QUANTUM HYDRODYNAMICAL MODELS

PAOLO ANTONELLI, LARS ERIC HIENTZSCH, PIERANGELO MARCATI, AND HAO ZHENG

ABSTRACT. In this paper we review some recent results on the existence of finite energy weak solutions to a class of quantum hydrodynamics (QHD) system. Our approach is based on a polar factorization method. This method allows to overcome the mathematical difficulty arising in the classical WKB approach, to define the velocity field inside the vacuum regions. Our methods to show existence of finite energy weak solutions fully exploit the dispersive and the local smoothing properties of the underlying nonlinear Schrödinger evolution in order to establish suitable "a priori" bounds for the hydrodynamical quantities. We finally sketch some new results towards a purely hydrodynamic theory in 1-D and recent developments of a low Mach number analysis of Quantum Vortices.

## 1. INTRODUCTION

QHD systems are typically fluid dynamical equations in which quantum effects are non-negligible and must be taken into account in the description. They appear in various contexts, for example they are extensively used in the description of phenomena like superfluidity [53], Bose-Einstein condensation [21], quantum plasmas [38], or in the modeling of semiconductor devices [31]. Moreover such systems are also intimately related to the class of the so-called Korteweg fluids [13] where capillary effects are considered in the description.

The prototype QHD model is the following

$$\begin{cases} \partial_t\rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div}\left(\frac{J\otimes J}{\rho}\right) + \nabla p(\rho) = \frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases} \tag{1.1}$$

where $\rho$ is the mass (or charge) density, $J$ is the momentum (or current) density, $p(\rho)$ is a pressure term. The term on the right hand side takes into account the quantum effects of the fluid and is a nonlinear third order (dispersive) term. Under suitable regularity assumptions, it may also be written in different ways

$$\frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \frac{1}{4}\nabla\Delta\rho - \operatorname{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}) = \frac{1}{4}\operatorname{div}(\rho\nabla^2\log\rho).$$

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System (1.1) is Hamiltonian, whose energy

$$\mathcal{E} = \int \frac{1}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}\frac{|J|^2}{\rho} + f(\rho)\,dx, \tag{1.2}$$

is formally conserved along the flow of solutions. The internal energy density $f(\rho)$ is related to the pressure through the relation $p(\rho) = \rho f'(\rho) - f(\rho)$.

**Assumption 1.1.** *$f : [0,\infty) \to [0,\infty)$ satisfies $f \in \mathcal{C}^1([0,\infty)) \cap \mathcal{C}^2((0,\infty))$ and*

$$|f'(\rho) + 2\rho f''(\rho)| \lesssim 1 + \rho^{\gamma-1}, \tag{1.3}$$

*where $\gamma > 1$ for $d = 1,2$ and $1 < \gamma < 3$ if $d = 3$.*

As it will become clear in what follows the QHD system (1.1) is strictly related to the following nonlinear Schrödinger (NLS) equation

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + f'(|\psi|^2)\psi \tag{1.4}$$

and it is fundamental in our study to exploit the properties of solutions to (1.4) in order to infer some analogous properties for solutions to (1.1). In fact since the early days of quantum mechanics Madelung [60] proposed a hydrodynamical formulation alternative to the wave function dynamics given by the linear Schrödinger equation, in which the quantum system is described in terms of the probability density and the phase of the wave function. This analogy was later resumed by Landau [53, 50] to describe nonlinear phenomena in superfluidity. The most immediate way to point out the relation between (1.4) and (1.1) is the so called WKB ansatz, namely we express the wave function $\psi$ in terms of its modulus $\sqrt{\rho}$ and its phase $S$, $\psi = \sqrt{\rho}e^{iS}$. By plugging this ansatz inside equation (1.4) and by then separating the real and imaginary parts of the identity, after some small calculations we find out that formally $(\rho, S)$ satisfy the following system

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho\nabla S) = 0 \\ \partial_t S + \frac{1}{2}|\nabla S|^2 + f'(\rho) = \frac{1}{2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}. \end{cases}$$

If we further define $J = \rho\nabla S$ then we see that $(\rho, J)$ satisfy the QHD system (1.1).

However, the WKB formalism experiences a main mathematical problem as soon as vacuum regions appear in the fluid. It is straightforward to see that in the set $\{\psi = 0\} = \{\rho = 0\}$ the phase cannot be uniquely defined, hence this analogy breaks down. On the other hand, in the study of quantum fluids it is of great interest to consider the possible presence of vacuum, both from the mathematical and physical points of view. One of the most striking features of superfluids is the presence of quantized vortices, namely the circuitation of the velocity field along a closed curve around a vortex can attain only values which are integer multiples of the vorticity quantum. Such coherent objects are located exactly in the vacuum region [68], hence it is physically interesting to study solutions to (1.1) allowing the presence of quantized vortices in the nodal region $\{\rho = 0\}$.

The aim of our study is to develop a rigorous analysis of finite energy solutions to hydrodynamical equations for quantum fluids, which retain the relevant physical properties of the system we want to describe.

For this reason we are going to develop a polar factorization approach which is not limited by the presence of vacuum regions and overcomes the difficulties encountered through the WKB method. In this way we set up a self consistent theory based on two hydrodynamical quantities, namely $\sqrt{\rho}$ and $\Lambda = J/\sqrt{\rho}$, so that we do not need to define the velocity field in the nodal region.

QHD systems are widely studied in the mathematical literature. In [32] the authors show the existence of weak solutions by defining the moments (via the Madelung transform) associated to a $H^2$ solution for a (linear) Schrödinger equation. One of the main motivations to study QHD systems is the modeling of semiconductor devices [31], where (1.1) is augmented by an electrostatic potential and a dissipative (relaxation) term, which phenomenologically describes the collisions between electrons [12]. In [47] such model is considered on a bounded domain; by using energy methods and the WKB approach the authors prove local well-posedness of smooth solutions. In [57] the authors study the same system, by using tools from the theory of hyperbolic systems of conservation laws; under a subsonicity condition they can show the existence of global in time regular solutions. The analysis of regular solutions is also done in [41, 61, 42, 45] where the authors also study the asymptotic stability of stationary states. Regarding the uniqueness of the weak solutions it has been pointed out in [26], by using methods of convex integration, the existence of infinitely many weak solutions to the Euler–Korteweg system, satisying the energy inequality. Recently also a class of viscous quantum fluid dynamical systems was considered. Such models can be derived from the Wigner-Fokker-Planck equation [46] (see also the interesting review [43]). Global existence of finite energy weak solutions was proved in [44, 9, 52] (see also [56] where similar arguments are used to study the compressible Navier-Stokes system with degenerate viscosity).

Finally, we also mention that the quantum hydrodynamical approach could have some interesting applications in the field of quantum synchronization [7].

Our main focus will be on finite energy weak solutions for a class of hydrodynamical equations describing quantum fluids. To be more precise about our framework we are going to state the definition of finite energy weak solutions for the prototype system (1.1). However, even though this will not be explicitly written, for all other systems introduced in this paper we will always consider Definition 1.2 below with the changes due to the different terms present in the system.

**Definition 1.2** (Finite energy weak solutions). *Let $\rho_0, J_0 \in L^1_{loc}(\mathbf{R}^d)$, we say the pair $(\rho, J)$ is a finite energy weak solution to the Cauchy problem for (1.1) with initial data $\rho(0) = \rho_0$, $J(0) = J_0$, in the space-time slab $[0,T) \times \mathbf{R}^d$ if there exist two locally integrable functions $\sqrt{\rho} \in L^2_{loc}(0,T; H^1_{loc}(\mathbf{R}^d))$,$\Lambda \in L^2_{loc}(0,T; L^2_{loc}(\mathbf{R}^d))$ such that*

(i) $\rho := (\sqrt{\rho})^2, J := \sqrt{\rho}\Lambda$;

(ii) $\forall\, \eta \in C_0^\infty([0,T) \times \mathbf{R}^d)$,

$$\int_0^T \int_{\mathbf{R}^d} \rho \partial_t \eta + J \cdot \nabla \eta \, dx dt + \int_{\mathbf{R}^d} \rho_0(x) \eta(0,x)\, dx = 0;$$

(iii) $\forall \zeta \in C_0^\infty([0,T) \times \mathbf{R}^d; \mathbf{R}^d)$,

$$\int_0^T \int_{\mathbf{R}^d} J \cdot \partial_t \zeta + \Lambda \otimes \Lambda : \nabla \zeta + p(\rho) \operatorname{div} \zeta + \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} : \nabla \zeta$$
$$+ \frac{1}{4} \rho \Delta \operatorname{div} \zeta \, dx dt + \int_{\mathbf{R}^d} J_0(x) \cdot \zeta(0,x)\, dx = 0;$$

(iv) *(finite energy) the total energy defined by*

$$\mathcal{E}(t) := \int_{\mathbf{R}^d} \frac{1}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}|\Lambda|^2 + f(\rho)\,dx, \tag{1.5}$$

*is finite for almost every $t \in [0,T)$;*

(v) *(generalized irrotationality condition) for almost every $t \in (0,T)$*

$$\nabla \wedge J = 2\nabla\sqrt{\rho} \wedge \Lambda,$$

*holds in the sense of distributions.*

*We say $(\rho, J)$ is a global in time finite energy weak solution if we can take $T = \infty$ in the above definition.*

*Remark* 1.3. In the case of a smooth solution $(\rho, J)$, for which we can write $J = \rho v$, for some smooth velocity field $v$, the Generalized Irrotationality condition defined above is equivalent to $\rho\nabla \wedge v = 0$, i.e. the velocity field $v$ is irrotational $\rho\,dx$ almost everywhere. It shows that the previous definition is the right weak formulation of the classical irrotationality condition $\nabla \wedge v = 0$ valid away from vacuum in the WKB approach. The generalized irrotationality condition is motivated by physics. Indeed in the theory of superfluidity (as well as in Bose-Einstein condensates) the whole vorticity of the fluid is carried over only by quantized vortices, which are located in the nodal region $\{\rho = 0\}$. More precisely, the flow is irrotational outside the set $\{\rho = 0\}$ and in the vacuum the vorticity becomes singular. In this respect the solutions introduced in Definition 1.2 are more general than those obtained by using the WKB ansatz, since in the latter case the velocity field $v = \nabla S$ is always irrotational and there is no vacuum. On the other hand, quantized vortices have a very rich structure and they are intensively studied in the physics of superfluids [68].

1.1. **Notations.** We conclude this Introduction by fixing some notations. $A \lesssim B$ denotes $A \leqslant CB$, for some constant $C > 0$. $L^p(\mathbf{R}^d)$, with Lebesgue exponent $p \in [1,\infty]$, is the usual Lebesgue space, for $s \geqslant 0$ $H^s(\mathbf{R}^d)$ denotes the Sobolev space whose norm is defined by $\|f\|_{H^s} := \|\langle\cdot\rangle^s \hat{f}\|_{L^2}$, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and $\hat{f}$ is the Fourier transform of $f$; furthermore $H^\infty(\mathbf{R}^d) := \cap_{s\geqslant 0} H^s(\mathbf{R}^d)$. We also use the notation $\||\cdot|^r f\|_{L^p} := \left(\int |x|^{pr}|f(x)|^p\,dx\right)^{1/p}$. If $p \in [1,\infty]$ is a Lebesgue exponent we denote its dual by $p'$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. We often make use of mixed space-time Lebesgue of Sobolev spaces, for instance $L^q(I; L^r(\mathbf{R}^d))$, where $I$ is an aribtrary interval and $q, r$ two Lebesgue exponents; to shorten notations we write $L^q_t L^r_x(I \times \mathbf{R}^d)$ or even simply $L^q_t L^r_x$ when there is no ambiguity. Analogously for $L^q(I; W^{1,r}(\mathbf{R}^d))$ or $L^q_t W^{1,r}_x$. $\mathcal{C}(I; H^s(\mathbf{R}^d))$ denotes the space of continuous $H^s$-valued functions and $\mathcal{C}_0^\infty$ is the space of infinitely differentiable, compactly supported functions.

## 2. A short review on dispersive properties for Schrödinger equations

In this Section we recall some of the theory and tools on Schrödinger equations which will be used in this paper in order to infer suitable properties for solutions to our QHD systems. Such results are well established in the literature, we address the interested reader to various textbooks [18, 67, 59, 28], and the references therein, treating those arguments in a much more comprehensive way. First of all, we recall the definition of admissible pairs for the Schrödinger equation.

**Definition 2.1.** *The pair of Lebesgue exponents $(q,r)$ is called* admissible *if* $2 \leqslant q, r \leqslant \infty$ *and we have*

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad (q,r,d) \neq (2,\infty,2).$$

By using this definition we can introduce a whole class of spacetime estimates enjoyed by the free Schrödinger propagator, known as Strichartz estimates [66, 34, 71, 69, 49]. More precisely, the dispersive nature of the Schrödinger propagator implies regularizing effects on the evolution, which enjoys some further space-time integrability properties.

**Theorem 2.2** ([49]). *Let $(q,r)$ and $(q_1,r_1)$ be two arbitrary admissible pairs and let $I$ be any time interval (possibily unbounded), then the following estimates hold*

$$\|e^{\frac{i}{2}t\Delta}f\|_{L^q_tL^r_x(I\times\mathbf{R}^d)} \lesssim \|f\|_{L^2(\mathbf{R}^d)};$$

$$\|\int_0^t e^{\frac{i}{2}(t-s)\Delta}F(s)\,ds\|_{L^q_tL^r_x(I\times\mathbf{R}^d)} \lesssim \|F\|_{L^{q_1'}_tL^{r_1'}_x(I\times\mathbf{R}^d)};$$

Another regularizing effect coming from the dispersive properties of the Schrödinger propagator is the so called Kato smoothing effect [48], namely the solution to the linear Schrödinger equation is half derivative smoother, locally in space, than its initial datum [19, 65, 70]. Here we use the result in [19] which is more suited for our applications, see also Section 4.3 in [59].

**Theorem 2.3.** *Let $\psi$ be a solution to*

$$\begin{cases} i\partial_t\psi = -\frac{1}{2}\Delta\psi + F \\ \psi(0) = \psi_0 \in L^2(\mathbf{R}^d), \end{cases}$$

*with $F \in L^1(\mathbf{R}; L^2(\mathbf{R}^d))$, then we have*

$$\|\psi\|_{L^2([0,T];H^{1/2}_{loc}(\mathbf{R}^d))} \lesssim \|\psi_0\|_{L^2(\mathbf{R}^d)} + \|F\|_{L^1}$$

Next Theorem resumes most of well-posedness and stability properties for a class of nonlinear Schrödinger equations which will be used through our paper. Those results are now standard, based on Strichartz estimates and the conservation of energy they also hold in more general cases; the proof and a more comprehensive discussion can be found in many textbooks [18, 67, 59].

In order to give the well-posedness result below we are going to need the following Assumption on the internal energy $f(\rho)$. Let us notice that this has a natural equivalent counterpart given in terms of the pressure, see Assumption 4.1. Condition (2.1) below ensures the nonlinearity in (2.2) to be locally Lipschitz, which guarantees the local well-posedness of (2.2). Then the non-negativity of the internal energy, together with the conservation of energy, allows to extend the solution globally in time.

**Assumption 2.4.** $f : [0,\infty) \to [0,\infty)$ *satisfies* $f \in C^1([0,\infty)) \cap C^2((0,\infty))$ *and*

$$|f'(\rho) + 2\rho f''(\rho)| \lesssim 1 + \rho^{\gamma-1}, \tag{2.1}$$

*where $\gamma > 1$ for $d = 1, 2$ and $1 < \gamma < 3$ if $d = 3$.*

**Theorem 2.5.** *For any $\psi_0 \in H^1(\mathbf{R}^3)$ there exists a unique solution $\psi \in \mathcal{C}(\mathbf{R}; H^1(\mathbf{R}^3))$ to*

$$\begin{cases} i\partial_t\psi = -\frac{1}{2}\Delta\psi + f'(|\psi|^2)\psi \\ \psi(0) = \psi_0 \end{cases} \tag{2.2}$$

*and the solution depends continuously on the initial data, namely the map from $H^1(\mathbf{R}^3)$ to $\mathcal{C}(\mathbf{R}; H^1(\mathbf{R}^3))$ which associates $\psi_0 \mapsto \psi$ is a continuous map. Furthermore the total energy*

$$E[\psi(t)] = \int \frac{1}{2}|\nabla\psi(t,x)|^2 + f(|\psi(t,x)|^2)\,dx, \tag{2.3}$$

*is conserved along the flow of solutions.*

Next Proposition involves a regularization of the solution $\psi$ to (2.2) and it will be used in Section 4. Its proof is a straightforward consequence of two properties of nonlinear Schrödinger equations, the former one being the persistence of regularity and the latter one being the stability of the equation with respect to small perturbations. For a more detailed explanation we refer the reader to Section 3.7 in [67] and the references therein.

**Proposition 2.6.** *Let $\psi \in \mathcal{C}(\mathbf{R}; H^1(\mathbf{R}^d))$ be the solution constructed in Theorem 2.5 with initial datum $\psi(0) = \psi_0$. For any $\varepsilon > 0$ there exists $\psi^\varepsilon \in \mathcal{C}(\mathbf{R}; H^\infty(\mathbf{R}^d))$ such that $\psi^\varepsilon$ solves*

$$\begin{cases} i\partial_t\psi^\varepsilon = -\frac{1}{2}\Delta\psi^\varepsilon + f'(|\psi^\varepsilon|^2)\psi^\varepsilon + e^\varepsilon \\ \psi^\varepsilon(0) = \psi_0^\varepsilon, \end{cases} \tag{2.4}$$

*with $\psi_0^\varepsilon \in \mathcal{C}^\infty(\mathbf{R}^d)$, $\|\psi_0^\varepsilon - \psi_0\|_{H^1} \lesssim \varepsilon$. Furthermore, for any $0 < T < \infty$ and any admissible pairs $(q, r)$ and $(q_1, r_1)$ we have*

$$\|e^\varepsilon\|_{L^{q_1'}([0,T];W^{1,r_1'}(\mathbf{R}^d))} \lesssim \varepsilon,$$
$$\|\psi^\varepsilon - \psi\|_{L^q([0,T];W^{1,r}(\mathbf{R}^d))} \lesssim \varepsilon.$$

## 3. The polar decomposition approach

In this Section we review the polar factorisation method, which will be exploited to define the hydrodynamic quantities $(\sqrt{\rho}, \Lambda)$ and to set up the correspondence between the wave function dynamics and the hydrodynamical system. The main advantage of this approach with respect to the usual WKB method, for instance, is that vacuum regions are allowed in the theory. More precisely, we factorize the wave function $\psi$ in its amplitude $\sqrt{\rho} := |\psi|$ and its polar factor $\phi$, namely a function taking its values in the unitary disk $\{|z| \le 1\}$ of the complex plane, such that $\psi = \sqrt{\rho}\phi$. In the WKB setting the polar factor would be given by $\phi = e^{iS/\hbar}$, however this equality holds only in the complement of the null set of the wave function which is not a well defined smooth set, in general not even a closed set [16, 23].

Given any function $\psi \in H^1(\mathbf{R}^d)$ we define the set

$$P(\psi) := \{\phi \in L^\infty(\mathbf{R}^d) \;:\; \|\phi\|_{L^\infty} \le 1, \psi = \sqrt{\rho}\phi \text{ a.e. in } \mathbf{R}^d\},$$

where $\sqrt{\rho} := |\psi|$. The possible appearance of vacuum regions prevents the polar factors to be uniquely determined on the whole space. However, we have that

$|\phi|=1$ $\sqrt{\rho}\,dx$ a.e. in $\mathbf{R}^d$ and $\phi$ is uniquely defined $\sqrt{\rho}\,dx$ a.e. in $\mathbf{R}^d$.
Next Lemma points out the main properties of the polar factorization and expresses the hydrodynamic quantities in terms of the wave function and its polar factor. Moreover, it shows the stability of this approach in the natural space of finite energy states. Finally we see that any current density originated from a wave function $\psi \in H^1(\mathbf{R}^d)$ satisfies the generalized irrotationality condition.

**Lemma 3.1** ($H^1$ **Stability).** *Let $\psi \in H^1(\mathbf{R}^d)$, $\sqrt{\rho} := |\psi|$ be its amplitude and let $\phi \in P(\psi)$ be a polar factor associated to $\psi$. Then $\sqrt{\rho} \in H^1(\mathbf{R}^d)$ and we have $\nabla\sqrt{\rho} = \mathrm{Re}(\bar{\phi}\nabla\psi)$. Moreover, by defining $\Lambda := \mathrm{Im}(\bar{\phi}\nabla\psi)$, then $\Lambda \in L^2(\mathbf{R}^d)$ and the following identity holds*

$$\mathrm{Re}(\nabla\bar{\psi}\otimes\nabla\psi) = \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho} + \Lambda\otimes\Lambda, \quad \text{a.e. in } \mathbf{R}^d.$$

*Furthermore, if $\{\psi_n\} \subset H^1(\mathbf{R}^d)$ is such that $\|\psi_n - \psi\|_{H^1} \to 0$ as $n \to \infty$, then we have*

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \quad \Lambda_n \to \Lambda, \quad \text{in } L^2(\mathbf{R}^d),$$

*where $\sqrt{\rho_n} := |\psi_n|, \Lambda_n := \mathrm{Im}(\bar{\phi}_n\nabla\psi_n)$, $\phi_n$ being a unitary factor for $\psi_n$. Finally the current density*

$$J := \mathrm{Im}(\bar{\psi}\nabla\psi) = \sqrt{\rho}\Lambda,$$

*satisfies*

$$\nabla\wedge J = 2\nabla\sqrt{\rho}\wedge\Lambda, \quad \text{a.e. in } \mathbf{R}^d.$$

*Proof.* Let $\psi \in H^1(\mathbf{R}^d)$ and let us consider a sequence of smooth functions converging to $\psi$, $\{\psi_n\} \subset C_0^\infty(\mathbf{R}^d)$, $\psi_n \to \psi$ in $H^1(\mathbf{R}^d)$. For each $\psi_n$ we may define

$$\phi_n(x) := \begin{cases} \dfrac{\psi_n(x)}{|\psi_n(x)|} & \text{if } \psi_n(x) \neq 0 \\ 0 & \text{if } \psi_n(x) = 0. \end{cases}$$

The $\phi_n$'s are clearly polar factors for the wave functions $\psi_n$. Since $\|\phi_n\|_{L^\infty} \leq 1$, then (up to passing to subsequences) there exists $\phi \in L^\infty(\mathbf{R}^d)$ such that

$$\phi_n \overset{*}{\rightharpoonup} \phi, \quad L^\infty(\mathbf{R}^d).$$

It is easy to check that $\phi$ is indeed a polar factor for $\psi$. Since $\{\psi_n\} \subset C_0^\infty(\mathbf{R}^d)$, we have

$$\nabla\sqrt{\rho_n} = \mathrm{Re}(\bar{\phi}_n\nabla\psi_n), \quad \text{a.e. in } \mathbf{R}^d.$$

It follows from the convergence above

$$\nabla\sqrt{\rho_n} \rightharpoonup \nabla\sqrt{\rho}, \quad L^2(\mathbf{R}^d)$$
$$\mathrm{Re}(\bar{\phi}_n\nabla\psi_n) \rightharpoonup \mathrm{Re}(\bar{\phi}\nabla\psi), \quad L^2(\mathbf{R}^d),$$

thus $\nabla\sqrt{\rho} = \mathrm{Re}(\bar{\phi}\nabla\psi)$ in $L^2(\mathbf{R}^d)$ and consequently the equality holds a.e. in $\mathbf{R}^d$. Resuming we have proved that for any $\psi \in H^1(\mathbf{R}^d)$ we have

$$\nabla\sqrt{\rho} = \mathrm{Re}(\bar{\phi}\nabla\psi),$$

where $\phi$ is *the* polar factor given as the weak$-*$ limit in $L^\infty$ of the polar factors $\phi_n$. It turns out that this equality holds independently on the particular choice of the polar factor. Indeed, by Theorem 6.19 in [54] we have $\nabla\psi = 0$ for almost every $x \in \psi^{-1}(\{0\})$ and, on the other hand, $\phi$ is uniquely determined on $\{x \in \mathbf{R}^d : |\psi(x)| > 0\}$ almost everywhere. Consequently, for any $\phi_1, \phi_2 \in P(\psi)$, we have $\mathrm{Re}(\bar{\phi}_1\nabla\psi) = \mathrm{Re}(\bar{\phi}_2\nabla\psi) = \nabla\sqrt{\rho}$. The same argument applies for $\Lambda := \mathrm{Im}(\bar{\phi}\nabla\psi)$, so that this definition is not ambiguous. Again, from Theorem 6.19 in [54] and the uniqueness of $\phi\ \sqrt{\rho}\,dx-$a.e. in $\mathbf{R}^d$, we have

$$\begin{aligned}\mathrm{Re}(\nabla\bar{\psi}\otimes\nabla\psi) &= \mathrm{Re}\left((\phi\nabla\bar{\psi})\otimes(\bar{\phi}\nabla\psi)\right)\\ &= \mathrm{Re}(\phi\nabla\bar{\psi})\otimes\mathrm{Re}(\bar{\phi}\nabla\psi) - \mathrm{Im}(\phi\nabla\bar{\psi})\otimes\mathrm{Im}(\bar{\phi}\nabla\psi)\\ &= \nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}+\Lambda\otimes\Lambda,\end{aligned}$$

almost everywhere in $\mathbf{R}^d$. By taking the trace on both sides of the above equality we furthermore obtain

$$|\nabla\psi|^2 = |\nabla\sqrt{\rho}|^2 + |\Lambda|^2. \tag{3.1}$$

Similarly,

$$\begin{aligned}\nabla\bar{\psi}\wedge\nabla\psi &= (\phi\nabla\bar{\psi})\wedge(\bar{\phi}\nabla\psi)\\ &= 2i\nabla\sqrt{\rho}\wedge\Lambda, \quad \text{a.e. in } \mathbf{R}^d,\end{aligned}$$

and $\nabla\wedge J = \nabla\wedge(\mathrm{Im}(\bar{\psi}\nabla\psi)) = \mathrm{Im}(\nabla\bar{\psi}\wedge\nabla\psi)$. This implies that $J$ satisfies the generalized irrotationality condition of Definition 1.2.

Now we prove the second part of Lemma. Let $\{\psi_n\}\subset H^1(\mathbf{R}^d)$ be any sequence such that $\psi_n\to\psi$ in $H^1(\mathbf{R}^d)$. As before it is straightforward to prove

$$\begin{aligned}\mathrm{Re}(\bar{\phi}_n\nabla\psi_n) &\rightharpoonup \mathrm{Re}(\bar{\phi}\nabla\psi), \quad L^2\\ \mathrm{Im}(\bar{\phi}_n\nabla\psi_n) &\rightharpoonup \mathrm{Im}(\bar{\phi}\nabla\psi), \quad L^2.\end{aligned}$$

Moreover, from (3.1), the strong convergence of $\psi_n$ and the weak convergence for $\nabla\sqrt{\rho_n}, \Lambda_n$, we obtain

$$\begin{aligned}\|\nabla\psi\|_{L^2}^2 = \|\nabla\sqrt{\rho}\|_{L^2}^2 + \|\Lambda\|_{L^2}^2 &\leq \liminf_{n\to\infty}\left(\|\nabla\sqrt{\rho_n}\|_{L^2}^2 + \|\Lambda_n\|_{L^2}^2\right)\\ &= \lim_{n\to\infty}\left(\|\nabla\psi_n\|_{L^2}^2\right) = \|\nabla\psi\|_{L^2}^2.\end{aligned}$$

Hence, we obtain $\|\nabla\sqrt{\rho_n}\|_{L^2}\to\|\nabla\sqrt{\rho}\|_{L^2}$ and $\|\Lambda_n\|_{L^2}\to\|\Lambda\|_{L^2}$. Consequently, from the weak convergence in $L^2$ and the convergence of the $L^2$ norms we may infer the strong convergence

$$\nabla\sqrt{\rho_n}\to\nabla\sqrt{\rho}, \quad \Lambda_n\to\Lambda, \quad \text{in } L^2(\mathbf{R}^d).$$

□

The next Lemma will be used in Section 5 to deal with the non-Hamiltonian case. More precisely it will be used in the fractional step argument in order to construct an approximate solution to the system under consideration. Its proof is a straightforward consequence of the polar decomposition method and its stability property in $H^1$, however we will state it here as it will be handy to use in the fractional step argument. Moreover this can be further generalized in order to take into account a wider class of terms in the fractional step, see for example [6].

**Lemma 3.2.** *Let $\psi\in H^1(\mathbf{R}^d)$ and let $\varepsilon,\tau>0$ be two arbitrary (small) real numbers, $\alpha\geqslant 0$. Then there exists $\tilde{\psi}\in H^1(\mathbf{R}^d)$ such that if $\sqrt{\tilde{\rho}}:=|\tilde{\psi}|, \tilde{\Lambda}:=\mathrm{Im}(\bar{\tilde{\phi}}\nabla\tilde{\psi})$, with $\tilde{\phi}$ polar factor for $\tilde{\psi}$, then*

$$\begin{cases}\sqrt{\tilde{\rho}} = \sqrt{\rho}+r_\varepsilon\\ \tilde{\Lambda} = (1-\alpha\tau)\Lambda+\Gamma_\varepsilon\\ \nabla\tilde{\psi} = \nabla\psi - i\alpha\tau\hat{\phi}\Lambda + R_{\varepsilon,\tau},\end{cases} \tag{3.2}$$

*where* $\|\hat{\phi}\|_{L^\infty} \leq 1$ *and*

$$\|r_\varepsilon\|_{H^1} + \|\Gamma_\varepsilon\|_{L^2} \leq \varepsilon$$

*and*

$$\|R_{\varepsilon,\tau}\|_{L^2} \lesssim \varepsilon + \tau\|\nabla\psi\|_{L^2}.$$

## 4. Existence of solutions for the QHD system

A first application of the polar factorization Lemma 3.1 is the existence of global in time finite energy weak solutions for the Cauchy problem associated to system (1.1), complemented with the following initial data

$$\rho(0) = \rho_0, \qquad J(0) = J_0.$$

This first result comes directly from the polar factorization method and the stability properties for solutions to NLS equations stated in Theorem 2.5. This approach allows to construct a global in time finite energy weak solution to (1.1) under some quite general assumptions on the initial data and without smallness restrictions. The main drawback will be that, since we will exploit the underlying Schrödinger dynamics, the choice of our initial data must be consistent with an initial wave function. It is an interesting open question to determine the level of generality of such initial data within the class of finite energy data. Indeed, while the polar factorization Lemma allow us to define the hydrodynamical quantities starting from a wave function, the converse is in general not true, due to the presence of vacuum. Furthermore, also the generalized irrotationality condition would be important for the hydrodynamical quantities in order to be consistent with a wave function.
The assumption (2.4) has a natural equivalent counterpart in terms of the pressure function, since $p(\rho) = \rho f'(\rho) - f(\rho)$.

**Assumption 4.1.** *By the regularity assuptions on $f$ we get*

$$p \in \mathcal{C}([0,\infty)) \cap \mathcal{C}^1((0,\infty)).$$

*The non negativity of the internal energy yields to:*

$$0 \leq \rho \int^\rho \frac{p'(s)}{s}\, ds - p(\rho)$$

*and the locally Lipschitz condition can be stated equivalently*

$$|2p'(\rho) + \int^\rho \frac{p'(s)}{s}\, ds| \lesssim 1 + \rho^{\gamma-1}$$

*with $\gamma > 1$ for $d = 1, 2$ and $1 < \gamma < 3$ for $d = 3$.*

We remark that no monotonicity has been assumed on $p$.

**Theorem 4.2.** *Let $\psi_0 \in H^1(\mathbf{R}^d)$ and define the initial data for the QHD system (1.1) as $\rho_0 := |\psi_0|^2, J_0 := \operatorname{Im}(\bar{\psi}_0\nabla\psi_0)$. Then there exists a global in time finite energy weak solution such that $\sqrt{\rho} \in L^\infty(\mathbf{R}; H^1(\mathbf{R}^d))$, $\Lambda \in L^\infty(\mathbf{R}; L^2(\mathbf{R}^d))$, which conserves the energy at all times.*

*Proof.* Let $\psi_0 \in H^1(\mathbf{R}^d)$, from Theorem 2.5 we know there exists a unique solution $\psi \in \mathcal{C}(\mathbf{R}; H^1(\mathbf{R}^d))$ to (2.2) such that the energy (2.3) is a conserved quantity for all times. Let us fix $\varepsilon > 0$ sufficiently small, and let $\psi^\varepsilon$ be given from Proposition 2.6. We can define the following hydrodynamical quantities $\rho^\varepsilon := |\psi^\varepsilon|^2$, $J^\varepsilon :=$

$\mathrm{Im}(\bar{\psi}^\varepsilon \nabla \psi^\varepsilon)$. By differentiating $\rho^\varepsilon$ with respect to time and by using the fact that $\psi^\varepsilon$ is a solution to (2.4) we obtain an approximate continuity equation

$$\partial_t \rho^\varepsilon + \operatorname{div} J^\varepsilon = r^\varepsilon, \tag{4.1}$$

where $r^\varepsilon = 2\,\mathrm{Re}(\bar{e}^\varepsilon \psi^\varepsilon)$. Again by using (2.4) we can differentiate $J^\varepsilon$ with respect to time to find out the following identity

$$\partial_t J^\varepsilon + \operatorname{div}(\mathrm{Re}(\nabla \bar{\psi}^\varepsilon \otimes \nabla \psi^\varepsilon)) + \nabla p(\rho^\varepsilon) = \frac{1}{4}\nabla \Delta \rho^\varepsilon + G^\varepsilon,$$

where the error $G^\varepsilon$ is given by $G^\varepsilon = \mathrm{Re}(\bar{e}^\varepsilon \nabla \psi^\varepsilon - \bar{\psi}^\varepsilon \nabla e^\varepsilon)$. Here we used the fact that $p(\rho) = \rho f'(\rho) - f(\rho)$. Lemma 3.1 shows that the following bilinear identity

$$\mathrm{Re}(\nabla \bar{\psi}^\varepsilon \otimes \nabla \psi^\varepsilon) = \nabla\sqrt{\rho^\varepsilon} \otimes \nabla\sqrt{\rho^\varepsilon} + \Lambda^\varepsilon \otimes \Lambda^\varepsilon$$

holds true, which implies

$$\partial_t J^\varepsilon + \operatorname{div}(\Lambda^\varepsilon \otimes \Lambda^\varepsilon) + \nabla p(\rho^\varepsilon) = \frac{1}{4}\nabla\Delta\rho^\varepsilon - \operatorname{div}(\nabla\sqrt{\rho^\varepsilon} \otimes \nabla\sqrt{\rho^\varepsilon}) + G^\varepsilon, \tag{4.2}$$

i.e. $(\rho^\varepsilon, J^\varepsilon)$ is a global in time finite energy solution to (1.1), up to the errors $r^\varepsilon, G^\varepsilon$. We can now take (4.1) and (4.2) and put them in the weak formulation given in Definition 1.2. By taking the limit $\varepsilon \to 0$ in the weak formulation and by using the convergence stated in Proposition 2.6 we then infer that $(\rho, J)$, defined by $\rho := |\psi|^2, J := \mathrm{Im}(\bar{\psi}\nabla\psi)$, is a global in time finite energy weak solution to (1.1). Indeed the convergence in Proposition 2.6 also implies $\|\nabla\sqrt{\rho^\varepsilon} - \nabla\sqrt{\rho}\|_{L_t^\infty H_x^1} \to 0$ and $\|\Lambda^\varepsilon - \Lambda\|_{L_t^\infty L_x^2} \to 0$, by the stability of the polar factorization in $H^1$. Furthermore, again from Lemma 3.1, we have that $|\nabla\psi|^2 = |\nabla\sqrt{\rho}|^2 + |\Lambda|^2$, a.e. in $\mathbf{R}^d$, so that by considering the energy functional in (1.2) and (2.3) and by exploiting the conservation of energy for NLS, we have

$$\mathcal{E}(t) = E[\psi(t)] = E[\psi(0)] = \mathcal{E}(0).$$

Therefore the energy is conserved for the solution constructed before. □

We conclude this Section by noticing that in the hydrodynamical system (1.1) we can include also extra terms, as long as the corresponding wave function dynamics has a satisfactory well-posedness theory in the energy space and satisfies suitable stability properties. In particular it is possible to consider electrostatic potentials (see also next Section) which give, in the wave function dynamics, a nonlocal nonlinearity of Hartree type. The case with magnetic fields is more delicate and requires a finer analysis on the underlying wave functions dynamics; we address the interested reader to [1] for some results in this direction.

## 5. The QHD model for semiconductor devices

In the last Section we showed that, as long as the dynamics in the wave function framework is described by a nonlinear Schrödinger equation having suitable well-posedness and stability properties, we are able to construct a finite energy weak solution to the hydrodynamical system under very general assumptions on the initial data.

On the other hand, this is not always the case, as there are various QHD models in the literature whose analogue wave function dynamics does not have a satisfactory well-posedness theory in the space of energy. That is the case, for instance, when we incorporate some dissipative effects in the fluid model, or interaction with other flows like in the Landau's two-fluid theory of superfluidity [50]. A typical model falling in this class is the following one

$$\begin{cases} \partial_t\rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div}\left(\frac{J\otimes J}{\rho}\right) + \nabla p(\rho) + \rho\nabla V = \frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) - \alpha J \\ -\Delta V = \rho, \end{cases} \tag{5.1}$$

where now we also consider an electrostatic potential $V$ and the term $-\alpha J$ with $\alpha \geqslant 0$ on the right hand side of the equation for the momentum density introduces a (linear) dissipation in the system. Indeed, along the flow of solutions to (5.1) we formally have

$$\mathcal{E}(t) + \alpha\int_0^t\int |\Lambda(t',x)|^2\,dxdt' = \mathcal{E}(0), \tag{5.2}$$

where now the energy is given by

$$\mathcal{E}(t) = \int \frac{1}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}|\Lambda|^2 + f(\rho) + \frac{1}{2}|\nabla V|^2\,dx.$$

The main result we are going to present in this Section is the following Theorem. Here we focus only on the three dimensional case; for the study of (5.1) in the two-dimensional case we address to [4]. In this Section the arguments to prove the Theorem below are only sketched, the reader can find more details in [3].

**Theorem 5.1.** *Let $\psi_0 \in H^1(\mathbf{R}^3)$ and let us define $\rho_0 := |\psi_0|^2, J_0 := \operatorname{Im}(\bar{\psi}_0\nabla\psi_0)$. Then there exists a global in time finite energy weak solution to (5.1) such that $\sqrt{\rho} \in L^\infty(\mathbf{R}; H^1(\mathbf{R}^3))$, $\Lambda \in L^\infty(\mathbf{R}; L^2(\mathbf{R}^3))$, and we have the following energy inequality*

$$\mathcal{E}(t) \leqslant \mathcal{E}(0), \quad \textit{for a.e. } t > 0.$$

System (5.1) is widely studied in the mathematical literature [57, 47, 61, 41, 42, 45] because this is a key model for semiconductor devices [31]. The dissipative term was introduced phenomenologically [12] in order to describe the collisions between electrons in the semiconductor device. For this reason we often refer to it as the collisional term and system (5.1) with $\alpha = 0$ will be regarded as the non-collisional system.

Formally, the wave function dynamics analogue to (5.1) reads

$$\begin{cases} i\partial_t\psi = -\frac{1}{2}\Delta\psi + f'(|\psi|^2)\psi + V\psi + \widetilde{V}\psi \\ -\Delta V = \rho, \end{cases} \tag{5.3}$$

where the self-consistent potential $\widetilde{V}$ is given by

$$\widetilde{V} = \frac{\alpha}{2i}\log\left(\psi/\bar{\psi}\right).$$

Similar dissipative Schrödinger equations arise also in other contexts, see for example the Schrödinger-Langevin equation [51], however to our knowledge there is no well-posedness result in the energy space for the Cauchy problem associated to (5.3), due to the ill-posedness created by the potential $\widetilde{V}$.

For this reason, in order to show the existence of a finite energy weak solution to (5.1) we construct a sequence of approximating solutions based on an operator splitting argument.

**Definition 5.2.** *Let $\tau > 0$ be a small parameter, we say $\{(\rho^\tau, J^\tau)\}_{\tau>0}$ is a sequence of approximate solutions for the system (5.1) with initial data $(\rho_0, J_0) \in L^1_{loc}(\mathbf{R}^d)$ if there exist locally integrable functions $\sqrt{\rho^\tau} \in L^2_{loc}(0,T; L^2_{loc}(\mathbf{R}^d))$, $\Lambda^\tau \in L^2_{loc}(0,T; L^2_{loc}(\mathbf{R}^d))$ such that conditions (i), (iv), (v) in Definition 1.2 hold true for $(\rho^\tau, J^\tau)$, and the right hand sides of (ii) and (iii) (with the obvious changes due to the extra terms in the system), with $(\rho^\tau, J^\tau)$, are $o(1)$ as $\tau \to 0$.*

More precisely, let us fix a small parameter $\tau > 0$, then we split our evolutionary problem into two parts. In the former step we solve the non-collisional dynamics ($\alpha = 0$) by means of the associated Schrödinger-Poisson system, then in the latter one we update the quantities in order to take into account the dissipative term $-\alpha J$.
The main difficulty here is to implement the collisional step at the level of wave function description of the system. Indeed, as already explained in the previous Section, we deal with the non-collisional step by exploiting the polar factorization approach and solving the nonlinear Schrödinger-Poisson system, i.e. (5.3) with $\alpha = 0$. Furthermore, while it is possible to define the hydrodynamical quantities given a wave function, the converse is not true in general (see also the discussion before Theorem 4.2). Therefore we need to perform also the collisional step at a wave function level, in order to start again with an updated wave function in the next step. From the operator splitting we see that the collisional step would consist in just solving the following ODE

$$\begin{cases} \partial_t \rho = 0 \\ \partial_t J + \alpha J = 0. \end{cases}$$

However, this has to be translated to an updating for the (approximating) wave function; it consists in adjusting its phase. For this purpose we are going to use Lemma 3.2.
More precisely, fixed $\tau > 0$, we construct our approximate solution $\psi^\tau$ [1] in the following way. Let $\psi_0 \in H^1(\mathbf{R}^d)$, at first step $k = 0$ we solve

$$\begin{cases} i\partial_t \psi^\tau = -\frac{1}{2}\Delta\psi^\tau + f'(|\psi^\tau|^2)\psi^\tau + V^\tau\psi^\tau, & (t,x) \in [0,\tau) \times \mathbf{R}^d \\ -\Delta V^\tau = |\psi^\tau|^2, & (t,x) \in [0,\tau) \times \mathbf{R}^d \\ \psi^\tau(0) = \psi_0, & x \in \mathbf{R}^d. \end{cases} \tag{5.4}$$

Let us define the approximate solution by induction: we assume we already constructed $\psi^\tau$ in $[(k-1)\tau, k\tau) \times \mathbf{R}^d$, we want to construct $\psi^\tau$ in the next space-time slab $[k\tau, (k+1)\tau) \times \mathbf{R}^d$. We invoke Lemma 3.2 with $\psi = \psi^\tau(k\tau-)$, $\varepsilon = \tau 2^{-k}\|\psi_0\|_{H^1}$. The $\tilde{\psi}$ in Lemma will be the updated wave function:

$$\psi^\tau(k\tau+) := \tilde{\psi}.$$

As a consequence we obtain

$$\begin{aligned} \sqrt{\rho^\tau}(k\tau+) &= \sqrt{\rho^\tau}(k\tau-) + r_{k\tau} \\ \Lambda^\tau(k\tau+) &= (1-\alpha\tau)\Lambda^\tau(k\tau-) + \Gamma_{k,\tau}, \end{aligned}$$

[1] Strictly speaking the approximate solution is given by the hydrodynamic quantities $(\rho^\tau, J^\tau)$ associated to the wave function $\psi^\tau$. More precisely, $\psi^\tau$ is *not* an approximate solution for any equation (see Remark 5.8 below), however we will call it in this way because its moments $(\rho^\tau, J^\tau)$ are approximate solutions, in the sense of Definition 5.2 to system (5.1).

where $\|r_{k,\tau}\|_{H^1} + \|\Gamma_{k,\tau}\|_{L^2} \leqslant \tau 2^{-k}\|\psi_0\|_{H^1}$, and

$$\nabla\psi^\tau(k\tau+) = \nabla\psi^\tau(k\tau-) - i\alpha\tau\phi_k^\tau\Lambda^\tau(k\tau-) + R_{k,\tau}, \tag{5.5}$$

for some $\phi_k^\tau$ with $\|\phi_k^\tau\|_{L^\infty} \leq 1$ and

$$\|E_{k,\tau}\|_{L^2} \leq C\left(\tau\|\nabla\psi^\tau(k\tau-)\|_{L^2} + \tau 2^{-k}\|\psi_0\|_{H^1}\right) \leq \tau C E_0^{1/2}.$$

Now we can start again with the Cauchy problem associated to (5.4) on the space-time slab $[k\tau, (k+1)\tau) \times \mathbf{R}^d$, by considering

$$\psi(k\tau) = \psi^\tau(k\tau+)$$

as initial condition. Thus we define $\psi^\tau$ on $[k\tau, (k+1)\tau) \times \mathbf{R}^d$ to be this solution. With this procedure we construct iteratively $\psi^\tau$ on $[0, \infty) \times \mathbf{R}^d$. By means of the polar factorization we define $(\sqrt{\rho^\tau}, \Lambda^\tau)$ from $\psi^\tau$. Now we need to prove that the sequence of approximate solutions has a limit and that this limit actually solves (in the weak sense) the QHD system (5.1). First of all we show the consistency of approximate solutions, namely that if the sequence $\{(\sqrt{\rho^\tau}, \Lambda^\tau)\}$ has a strong limit, then this limit is a weak solution to (5.1).

**Theorem 5.3.** *Let us consider a sequence of approximate solutions $\{(\rho^\tau, J^\tau)\}$ constructed via the fractional step method, and let us assume there exist $\sqrt{\rho} \in L^2_{loc}(0,T; H^1_{loc}(\mathbf{R}^d))$, $\Lambda \in L^2_{loc}(0,T; L^2_{loc}(\mathbf{R}^d))$ such that*

$$\begin{aligned} \sqrt{\rho^\tau} &\to \sqrt{\rho} \quad \text{in } L^2_{loc}(0,T; H^1_{loc}(\mathbf{R}^d)) \\ \Lambda^\tau &\to \Lambda \quad \text{in } L^2_{loc}(0,T; L^2_{loc}(\mathbf{R}^d)). \end{aligned}$$

*The $\rho := (\sqrt{\rho})^2$, $J := \sqrt{\rho}\Lambda$ is a weak solution to (5.1) in $[0,T) \times \mathbf{R}^d$.*

It thus remains to prove that the sequence of approximate solutions has a strong limit, as stated in the hypothesis of the Theorem above. That is, we need to show some compactness properties for the family $\{\psi^\tau\}$, which will then imply the necessary compactness for $\{(\sqrt{\rho^\tau}, \Lambda^\tau)\}$ by means of the polar factorization. First of all, we show that $\{\psi^\tau\}$ is uniformly bounded in the energy space. More precisely the sequence $\{\psi^\tau\}$ satisfies an approximate version of (5.2).

**Lemma 5.4.** *Let $0 < \tau < 1$ and let $\psi^\tau$ be the approximate solutions constructed above. Then we have the following energy inequality*

$$E^\tau(t) \leq -\frac{\tau}{2}\sum_{k=1}^{[t/\tau]} \|\Lambda^\tau(k\tau-)\|^2_{L^2} + (1+\tau)E_0. \tag{5.6}$$

The above estimate provides the uniform (in $\tau > 0$) boundedness of $\{\psi^\tau\}$ in the space $L^\infty(\mathbf{R}_+; H^1(\mathbf{R}^d))$. This implies there exists (up to passing to subsequences), a weak limit $\psi \in L^\infty(\mathbf{R}_+; H^1(\mathbf{R}^d))$, $\psi^\tau \overset{*}{\rightharpoonup} \psi$ $L^\infty_t H^1_x$. Unfortunately, this is not sufficient to prove the consistency of approximate solutions: indeed the quadratic term

$$\mathrm{Re}(\nabla\bar{\psi}^\tau \otimes \nabla\psi^\tau),$$

appearing in the equation for the current density, could exhibit some concentration phenomena in the limit. We thus need to exploit the dispersive properties of the approximate solutions inherited from (5.4). For this purpose we first need the following Lemma, which express the gradient of the approximate solution $\psi^\tau$ at time $t$ in terms of the Schrödinger evolution group $U(t) = e^{i\frac{t}{2}\Delta}$.

**Lemma 5.5.** *Let $\psi^\tau$ be the approximate solution constructed above, then we have*

$$\begin{aligned}\nabla\psi^\tau(t) =&U(t)\nabla\psi_0 - i\int_0^t U(t-s)\nabla\mathcal{N}(\psi^\tau)(s)\,ds\\ &- i\tau\sum_{k=1}^{[t/\tau]} U(t-k\tau)\,[\phi_k^\tau\Lambda^\tau(k\tau-)] + \sum_{k=1}^{[t/\tau]} U(t-k\tau)r_{k,\tau},\end{aligned}\tag{5.7}$$

*where*

$$\mathcal{N}(\psi^\tau) = f'(|\psi^\tau|^2)\psi^\tau + V^\tau\psi^\tau,$$

*and $\phi_k^\tau, r_{k,\tau}$ are defined in* (5.5).

The Lemma above shows the importance of defining the updating step in the construction of the approximate solutions by means of Lemma 3.2. Indeed, this approximate updating allows us to write formula (5.7) in a quite neat way. For a more detailed discussion on this point we refer the reader to [3], Remark 21.
At this point we may use the Strichartz estimates for the Schrödinger semigroup. Formula (5.7) is the key point to exploit the dispersive estimates associated to the Schrödinger propagator and infer suitable a priori estimates on the sequence of approximating solutions. By applying the Strichartz estimates of Theorem 2.2 and the local smoothing estimates of Theorem 2.3 we can show the compactness for the sequence $\{\psi^\tau\}_{\tau>0}$. Next Proposition collects all the needed a priori estimates, their proof can be found in [3].

**Proposition 5.6.** *Let $0 < T < \infty$ be a finite time. Then for any admissible pair $(q, r)$ we have*

$$\|\nabla\psi^\tau\|_{L^q(0,T;L^r(\mathbf{R}^d))} \leq C(E_0, \|\rho_0\|_{L^1}, T).$$

*Furthermore,*

$$\|\nabla\psi^\tau\|_{L^2([0,T];H^{1/2}_{loc}(\mathbf{R}^d))} \leq C(E_0, \|\rho_0\|_{L^1}, T).$$

Having those estimates at hand we can now use a Aubin-Lions type Lemma in order to extract a subsequence (which we will also call $\{\psi^\tau\}$) which has a strong limit. More precisely we make use of a result by Rakotoson and Temam [63] (see [3] for more details).

**Theorem 5.7.** *For any finite time $0 < T < \infty$, the sequence $\nabla\psi^\tau$ is relatively compact in $L^2(0, T; L^2_{loc}(\mathbf{R}^d))$. More precisely, there exists $\psi \in L^2(0, T; H^1_{loc}(\mathbf{R}^d))$ such that*

$$\psi = s-\lim_{\tau\to 0}\psi^\tau, \quad \text{in } L^2(0, T; H^1_{loc}(\mathbf{R}^d)).$$

*As a consequence,*

$$\begin{aligned}\sqrt{\rho^\tau} &\to \sqrt{\rho} &&\text{in } L^2(0,T;H^1_{loc}(\mathbf{R}^d))\\ \Lambda^\tau &\to \Lambda &&\text{in } L^2(0,T;L^2_{loc}(\mathbf{R}^d)).\end{aligned}$$

By combining the Theorem above and Theorem 5.3, we know that $(\sqrt{\rho}, \Lambda)$ satisfy (5.1) in the weak sense, in $[0, T] \times \mathbf{R}^d$, for any finite $0 < T < \infty$. Moreover, it is easy to check that the energy for $(\sqrt{\rho}, \Lambda)$ is finite for almost every time: this follows directly from passing (5.6) to the limit as $\tau \to 0$. Furthermore let us recall $(\sqrt{\rho}, \Lambda)$ are the hydrodynamic quantities associated to $\psi \in L^\infty(\mathbf{R}_+; H^1(\mathbf{R}^d))$, hence by the polar decomposition Lemma they also satisfy the generalized irrotationality condition. We can thus say that $(\sqrt{\rho}, \Lambda)$ define a finite energy weak solution to the QHD system (5.1), and this proves Theorem 5.1.

*Remark* 5.8. We should remark here that, despite the fact $\psi$ is the strong limit of the sequence $\{\psi^\tau\}$ and the hydrodynamic quantities $(\sqrt{\rho}, \Lambda)$ associated to $\psi$ solve the QHD system (5.1), it is not clear if the wave function $\psi$ solve any nonlinear Schrödinger equation. Indeed, while for $\nabla\psi^\tau$ we can write the Duhamel's formula (5.7), we don't have a similar expression for $\psi^\tau$. In any case, even regarding formula (5.7) it is not clear whether the second line has a limit as $\tau \to 0$.

The analysis given in this Section to deal with the dissipative term $-\alpha J$ is useful not only in the study of the QHD model for semiconductor devices, but this is also a starting point to attack a more general class of hydrodynamical systems, related to the Landau's two-fluid model [50]. This is a system which described superfluid phenomena at finite temperatures, where the quantum (superfluid) flow is coupled to a normal flow, described by a classical viscous fluid. A partial result in this direction was given in [5] but there are many interesting open questions on this topic.

## 6. A priori dispersive estimates for the 1D QHD system

So far the main method to study QHD systems is based on the underlying wave functions dynamics and the dispersive properties enjoyed by the Schrödinger propagator $U(t) = e^{\frac{i}{2}t\Delta}$. It is an interesting problem to see if general solutions to QHD systems (namely, not only those ones generated from a wave function) enjoyed suitable a priori (dispersive) estimates. In the existing literature some results in this direction were already proved in [10, 11] also considering the more general case of Korteweg fluids, but they all require further regularity on the solutions and furthermore it is necessary to consider the mass density to be uniformly bounded away from zero. In this Section we try to sketch some alternative arguments, which will appear in details in [8], to address such questions. Here we will focus on the one-dimensional case, some generalizations of those arguments will be the subject of a subsequent paper. Furthermore, again for the sake of simplicity in the exposition, in this Section we will assume the internal energy density to satisfy a power law, i.e. we consider $f(\rho) = \frac{1}{\gamma}\rho^\gamma$, with $1 < \gamma < \infty$.

Our approach is based on monotonicity formulae, namely the study of a class of functionals for which it is possible to show they are non-increasing in time. Similar functional are already used in the context of nonlinear Schrödinger equations, where they are proven to be useful in order to yield informations about the long time behavior of solutions. The advantage of using those functionals is that they may be written in terms of hydrodynamical quantities.

The conserved quantities, such as the total mass, momentum and energy, already imply uniform bounds on the hydrodynamical quantities, namely

$$\sqrt{\rho} \in L^\infty(\mathbf{R}; H^1(\mathbf{R})), \quad \Lambda \in L^\infty(\mathbf{R}; L^2(\mathbf{R})).$$

Moreover, some functionals also yield dispersive properties of solutions to the QHD system (1.1). The first one we take into consideration is associated to the so called pseudo-conformal vector field [35]

$$P = x + it\partial_x.$$

It is well known that $P$ commutes with the Schrödinger operator,

$$[P, i\partial_t + \frac{1}{2}\partial_{xx}] = 0.$$

This implies that, if $u$ is a solution to $i\partial_t u + \frac{1}{2}\partial_{xx}u = 0$, then so is $Pu$, hence

$$\|Pu(t)\|_{L^2} = \|Pu(0)\|_{L^2} = \||\cdot|u(0)\|_{L^2}.$$

By using the identity

$$Pf(t,x) = (it)e^{i\frac{|x|^2}{2t}}\partial_x\left(e^{-i\frac{|x|^2}{2t}}f(x)\right),$$

then we have

$$\|\partial_x\left(e^{-i\frac{|x|^2}{2t}}u(t)\right)\|_{L^2} \leqslant t^{-1}\||\cdot|u(0)\|_{L^2}$$

and by using the Sobolev embedding $\|f\|_{L^\infty} \lesssim \|f\|_{L^2}^{1/2}\|\partial_x f\|_{L^2}^{1/2}$, one obtains a dispersive estimate for the free Schrödinger evolution

$$\|u(t)\|_{L^\infty} \lesssim t^{-1/2}\||\cdot|u(0)\|_{L^2}.$$

A similar approach holds also in the nonlinear case, indeed in [39, 40] this was used to obtain existence of solutions and to derive smoothing estimates.

We can now follow the approach by [39, 40] to infer suitable bounds for solutions to the QHD system. Indeed it is possible to write the functional on the right hand side of formula (2.5) in [39] in terms of hydrodynamical variables [2]. We define

$$\begin{aligned} H(t) =& t^2E(t) - t\int xJ(t,x)\,dx + \int\frac{|x|^2}{2}\rho(t,x)\,dx \\ =& \int\frac{t^2}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}|x\sqrt{\rho} - t\Lambda|^2 + t^2f(\rho)\,dx, \end{aligned} \tag{6.1}$$

by studying its time evolution and by using a similar analysis to [39], we can infer the following dispersive estimates. We omit the details of the proof and we refer to the incoming paper [8].

**Proposition 6.1.** *Let $(\rho, J)$ be a solution to the one-dimensional QHD system (1.1) with $f(\rho) = \frac{1}{\gamma}\rho^\gamma$, with $1 < \gamma < \infty$ such that the total energy is conserved for all times. Then the following a priori estimates hold true*

$$\|\partial_x\sqrt{\rho}(t)\|_{L^2} \lesssim t^{-\beta}\||\cdot|^2\rho(0)\|_{L^1}^{1/2}, \quad \beta := \min\{1, \frac{\gamma-1}{2}\}, \tag{6.2}$$

*and*

$$\int f(\rho(t,x))\,dx \lesssim t^{-(\gamma-1)}C(\||\cdot|^2\rho(0)\|_{L^1}). \tag{6.3}$$

Proposition 6.1 already gives some dispersive estimates for solutions to the QHD system. Clearly now it is not possible to use the abstract argument in [49] to infer the Strichartz estimates, but we can still exploit exploit (6.2) and (6.3) to infer a class of Strichartz-type estimates for $\sqrt{\rho}$. Unfortunately, the pseudo-conformal energy yields informations only on the mass density. The next step is to infer some analogue bounds also for the current density. To this end we need to consider some higher order energy functionals. We emphasize that this approach does not come straightforwardly from a linearization of the QHD, but they involve nonlinear functions of (derivatives of) the hydrodynamical quantities $(\sqrt{\rho}, \Lambda)$.

Below we state a recent result in this direction obtained in [8].

[2]Let us remark that this analogy was already pointed out in Appendix A of [17].

**Theorem 6.2.** *Let us assume that $\sqrt{\rho_0} \in H^2(\mathbf{R})$, $\partial_t\sqrt{\rho}(0) \in L^2(\mathbf{R})$, $\Lambda_0 \in L^2(\mathbf{R})$ and $\frac{1}{\sqrt{\rho_0}}|\Lambda_0|^2 \in L^2(\mathbf{R})$. Then for any, $0 < T < \infty$, the solution $(\rho, J)$, with these initial data satysfies*

$$\partial_t\sqrt{\rho} \in L^\infty([0,T]; L^2(\mathbf{R})), \quad \Lambda \in L^\infty([0,T] \times \mathbf{R}),$$
$$\partial_{xx}\rho \in L^\infty([0,T]; L^2(\mathbf{R})), \quad \partial_x J \in L^\infty([0,T]; L^2(\mathbf{R})).$$

## 7. The QHD system with non-trivial conditions at infinity

This Section is dedicated to the study of the QHD system (1.1) with non-trivial conditions at infinity, more specifically we impose

$$\rho \to 1, \quad \text{as} \quad |x| \to \infty.$$

The motivation for this study is two-fold: first of all it is a physically relevant case in the description of superfluidity close to the $\lambda-$point, see [36, 62]. Furthermore, this will be the starting point for a more detailed analysis on the dynamics of quantized vortices and a rigorous approach to quantum turbulence [68].

For this reason in this Section we mainly focus on the two dimensional settings, even if some of our results, like Theorems 7.1 and 7.5 for instance, hold also in the 3-D setting.

For the sake of clarity here we assume the pressure term to be quadratic, i.e. $p(\rho) = \frac{1}{2}\rho^2$, however a similar analysis holds with minor modifications also for a pressure term with general power law $p(\rho) \sim \rho^\gamma$, with $1 < \gamma < \infty$ if $d = 2$ and $1 < \gamma < 3$ if $d = 3$.

In this framework, the total energy associated to the system is given by

$$\mathcal{E} = \int \frac{1}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}|\Lambda|^2 + \frac{1}{2}(\rho - 1)^2\, dx. \tag{7.1}$$

Also in this case we are going to exploit the wave function dynamics associated to (1.1). If we consider the energy (7.1) written in terms of a wave function, we find the celebrated Ginzburg-Landau functional

$$\mathcal{E}_{GP}(\psi)(t) = \int_{\mathbf{R}^d} \frac{1}{2}\left|\nabla\psi\right|^2 + \frac{1}{2}\left(|\psi|^2 - 1\right)^2 dx,$$

whose associated Hamilton equation is given by the Gross-Pitaevskii equation

$$\begin{cases} i\partial_t\psi = -\frac{1}{2}\Delta\psi + \left(|\psi|^2 - 1\right)\psi \\ \psi(0) = \psi_0. \end{cases} \tag{7.2}$$

In what follows, firstly we establish the existence of finite energy weak solutions for the system (1.1) in $\mathbf{R}^d$, for $d = 2, 3$, in the sense of Definition 1.2. Indeed, let us remark that since condition (iv) is satisfied, then we have $\rho(t) \to 1$, as $|x| \to \infty$ for almost every time. The loss of integrability of $\rho$ due the fact that the density is non-vanishing at infinity requires to modify the Cauchy theory presented above. Subsequently, we generalise the existence result to a class of solutions that allows us to consider vortices in the system. The major difficulty consists in the fact that the associated energy for vortex solutions is in general infinite. Most of the arguments exposed here are only sketched in this Section and they will appear in more details in the forthcoming paper [2].

In [33], Gérard shows the well-posedness for the Cauchy-Problem (7.2) in the energy space, i. e.

$$\mathbb{E} = \{u \in H^1_{loc}(\mathbf{R}^d) \quad : \nabla u \in L^2(\mathbf{R}^d),\, |u|^2 - 1 \in L^2\}$$

In particular, the fact that finite energy solutions $\psi$ of equation (7.2) do not vanish at infinity implies that $\psi \notin L^p$ for any $1 \leq p < \infty$ and therefore the classical theory for the Cauchy Problem of nonlinear Schrödinger equations does not apply.

The first result we prove is the existence of finite energy weak solutions.

**Theorem 7.1.** *Let $d = 2, 3$ and $\psi_0 \in \mathbb{E}$, define*

$$\rho_0 = |\psi_0|^2, \quad J_0 = \operatorname{Im}\left(\overline{\psi_0}\nabla\psi_0\right).$$

*Then there exists a finite energy weak solution $(\rho, J)$ to (1.1) such that*

$$\rho(0) = \rho_0, \quad J(0) = J_0$$

*Moreover, the energy defined in (7.1) is conserved for all times.*

In Theorem 4.2 above, the proof of existence for finite energy weak solutions is based on a polar factor decomposition and on the well-posedness property for the underlying NLS equation. The main difficulty in proving Theorem 7.1 consists in setting up a suitable polar factorization for wave functions lacking integrability. As in [33], it will be useful to consider the following distance

$$d_{\mathbb{E}}(f, g) = \|\nabla f - \nabla g\|_{L^2} + \||f|^2 - |g|^2\|_{L^2}, \tag{7.3}$$

with $f, g \in \mathbb{E}$. Moreover, one has the inclusion

$$\mathbb{E} \subset X^1(\mathbf{R}^d) + H^1(\mathbf{R}^d),$$

where $X^k$ is the Zhidkov space defined for any integer $k$ by

$$X^k = \{u \in L^\infty(\mathbf{R}^d) : \quad \partial^\alpha u \in L^2,\, |\alpha| \leq k\}.$$

and the following inequality holds

$$\|u\|_{X^1+H^1} \leq C\left(1 + \sqrt{\mathcal{E}_{GP}(u)}\right), \tag{7.4}$$

see Lemma 1 in [33]. It is crucial to observe that any function $f$ in the energy space can be approximated by a sequence of smooth function $\{f_n\}_{n\in\mathbf{N}} \subset C^\infty$ such that $d(f_n, f) \to 0$. These properties enable us to derive the polar decomposition Lemma being the analogue of Lemma 3.1, where we exploit the metric structure of the energy space $\mathbb{E}$.

**Lemma 7.2** (Stability in $\mathbb{E}$). *Let $\psi \in \mathbb{E}$, let $\sqrt{\rho} := |\psi|$ be its amplitude and let $\phi \in P(\psi)$ be a polar factor associated to $\psi$. Then $\sqrt{\rho} \in L^2_{loc}$ and $\nabla\sqrt{\rho} = \operatorname{Re}\left(\bar{\phi}\nabla\psi\right)$. Moreover, if we define $\Lambda := \operatorname{Im}\left(\bar{\phi}\nabla\psi\right)$, then $\Lambda \in L^2_{loc}$ and the following identity holds*

$$\operatorname{Re}\left(\nabla\bar{\psi} \otimes \nabla\psi\right) = \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} + \Lambda \otimes \Lambda \quad a.\ e.\ in\, \mathbf{R}^d$$

*Furthermore, if $\{\psi_n\} \subset \mathbb{E}$ such that $d_{\mathbb{E}}(\psi_n, \psi) \to 0$, then the stability property holds*

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \quad \Lambda_n \to \Lambda, \quad in\, L^2(\mathbf{R}^d)$$

Once Lemma 7.2 is established, it is possible to exploit the well-posedness result for (7.2). In particular, by using the continuous dependence on the initial data and the persistence of regularity, it is possible to adapt the arguments of Section 4 and prove Theorem 7.1.

*Remark* 7.3. The existence result for finite energy weak solutions can also be derived for periodic domains $\mathbf{T}^d$. It is straightforward to notice that in this case the total mass and momentum are finite, for finite energy weak solutions. On the other hand, for the stability properties of the nonlinear Schrödinger equation, we need to exploit the Strichartz estimates on a compact manifold. The study of quantum vortex dynamics in this framework is also interesting from a physical point of view and furthermore this avoid the mathematical difficulty of dealing with configurations which in general have infinite energy (see below for a more detailed discussion).

We now turn our attention to the study of weak solutions to (1.1) with non-trivial conditions at infinity and with the presence of vortices.

It can be proven that wave functions that do not vanish at infinity and with non-trivial degree at infinity have infinite energy. Indeed, let us consider the particular case of a stationary vortex solution for the equation (7.2). More precisely, we consider a function of the type $u(x) = f(r)e^{\imath d\theta}$ with $r = |x|$ and $\theta$ is defined by $e^{i\theta} = \frac{x}{|x|}$. Here $d$ is an integer and represents the winding number of the quantum vortex. If $u$ is a stationary solution to the 2D GP equation (7.2), then the radial profile $f$ must satisfy

$$\begin{cases} f''(r) + \frac{1}{r}f'(r) - \frac{d^2}{r^2}f(r) + 2\left(1 - f(r)^2\right)f(r) = 0 \\ f(r) \to 1 \quad \text{as}\, r \to \infty. \end{cases}$$

It can be checked that $\nabla_x f \in L^2$ but $\nabla u \notin L^2$, therefore in particular $E(u) = \infty$. Multi-vortex configurations may be investigated by considering the product of several of these vortices. In[15] Bethuel and Smets study the problem by considering initial data of the type $\psi = U_0 + v$, where $U_0$ is a fixed vortex-configuration $U_0 = f(r)e^{\imath d\theta}$ or a multi-vortex configuration and $v$ a perturbation of this stationary solution. For this type of data the suitable function considered is $\psi = U_0 + u \in V + H^1$, where

$$V := \left\{U \in L^\infty : \quad \nabla|U| \in L^2,\, \nabla^k U \in L^2, \text{for any } k \geq 2,\, (|U|^2 - 1) \in L^2\right\}.$$

It is easy to check that $U_0 \in V$, for details we refer to [15]. It is shown that for any $\psi_0 \in V + H^1(\mathbf{R}^2)$ there exists a unique solution $t \mapsto \psi(t)$ of (7.2) such that $\psi(0) = \psi_0$ and $\psi(t) - U_0 \in C^0(\mathbf{R}, H^1(\mathbf{R}^2))$. Relying on this existence result for the underlying wave-function, we show that there exists a global weak solution to (1.1) with vortex initial data of the mentioned type. The first step consists in generalising the polar decomposition Lemma 7.2. The previous arguments suggest that in terms of hydrodynamic variables $\nabla\sqrt{\rho} \in L^2(\mathbf{R}^2)$ but $\Lambda \in L^2_{loc}(\mathbf{R}^2)$ due to the non-vanishing topological degree of $\psi$ at infinity.

**Lemma 7.4.** *Let $\psi \in V + H^1(\mathbf{R}^2)$, let $\sqrt{\rho} := |\psi|$ be its amplitude and let $\phi \in P(\psi)$ be a polar factor associated to $\psi$. Then $\sqrt{\rho} \in L^2_{loc}$ and $\nabla\sqrt{\rho} = \operatorname{Re}\left(\bar{\phi}\nabla\psi\right)$. Moreover, if we define $\Lambda := \operatorname{Im}\left(\bar{\phi}\nabla\psi\right)$, then $\Lambda \in L^2_{loc}$ and the following identity holds*

$$\operatorname{Re}\left(\nabla\bar{\psi} \otimes \nabla\psi\right) = \nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} + \Lambda \otimes \Lambda \quad a.\ e.\ in\ \mathbf{R}^d$$

*Furthermore, let $\psi \in V + H^1(\mathbf{R}^2)$ and fix a decomposition $\psi = U_0 + u$ such that $U_0 \in V$ and $u \in H^1(\mathbf{R}^2)$. If $\{u_n\} \subset H^1(\mathbf{R}^2)$ such that $u_n \to u$, the following stability property holds for any fixed $R > 0$,*

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \quad \Lambda_n \to \Lambda, \quad in\, L^2(B_R).$$

The Lemma 7.4 allows us to state the existence result for a solution of infinite energy with a small perturbation of a vortex configuration as initial data.

**Theorem 7.5.** *Let $\psi_0 = U_0 + u_0$ for a fixed vortex configuration $U_0 \in V$ and $u_0 \in H^1(\mathbf{R}^2)$. Define*

$$\rho_0 = |\psi_0|^2, \quad J_0 = \operatorname{Im}\left(\overline{\psi_0}\nabla\psi_0\right).$$

*Then there exists a global weak solution $(\rho, J)$ of (1.1) such that*

$$\rho(0) = \rho_0, \quad J(0) = J_0.$$

We stress that given $\psi \in V + H^1$, the decomposition $\psi = U_0 + v$ is in general not unique and in particular the position of single vortices is not. The considered solutions exhibit non-static vortices. Once the decomposition and in particular the reference configuration $U_0$ is fixed, one may introduce a renormalized energy as in [15] that is needed to show that the result is global in time. We postpone the quite technical discussion on the vortex configuration in the QHD setting to [2].

This previous discussion is oriented towards a rigorous analysis of quantum vortex dynamics in the context of QHD equations. When studying such coherent objects, it is useful to scale the fluid dynamical equations in such a way to consider

$$\begin{cases} \partial_t\rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div}\left(\frac{J\otimes J}{\rho}\right) + \frac{1}{\varepsilon^2}\nabla p(\rho) = \frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases} \tag{7.5}$$

in the same spirit of low Mach number limit, which focus on the dynamics of the incompressible fluid flow of the system. In this way the GP equation scales like

$$\begin{cases} i\partial_t\psi = -\frac{1}{2}\Delta\psi + \frac{1}{\varepsilon^2}\left(|\psi|^2 - 1\right)\psi \\ \psi(x,0) = \psi_0(x), \end{cases} \tag{7.6}$$

with a small parameter $\varepsilon > 0$. From a physical point of view $\varepsilon > 0$ is (proportional to) the characteristic core size of the vortex (healing length). The scaling for the equation (7.6) with small $\varepsilon > 0$ has been proposed in [36, 62] as model for superfluidity and is obtained after rescaling the equation (7.2). Heuristically, this suggests that $\rho = |\psi|^2$ is close to 1 and the region where $\rho$ is different from 1 is of size $\varepsilon$. This fact for instance can be easily checked for the typology of vortices considered previously. In the asymptotic regime as $\varepsilon \to 0$, the vortex cores shrink to point-vortices. In $d = 2$, the dynamics of vortices for (7.6) in the incompressible limit has been extensively studied in literature. It first appeared in the periodic setting in the paper [20] by Colliander and Jerrard and on bounded domains by Lin and Xin [58]. Both paper rely on variational methods for the minimization problem for Ginzburg-Landau energy functional and prove that the vortex dynamics for almost energy minimizing solutions in the incompressible limit is governed by the classical Kirchhoff-Onsager law for point vortices in $2d$. Later several different settings have been investigated, for example in [14] the authors study the vortex dynamics on the plane and they have to deal with infinite energy configurations.
When considering the quantum vortex dynamics in the QHD system (7.5), it is possible to give a more detailed picture. In [24] the low Mach number limit for the QHD in the periodic setting is studied. It is shown that for sufficiently regular but ill-prepared data of finite energy the asymptotic regime is described by the incompressible Euler equation. The proof uses the modulated energy method

and a decreasing relative entropy functional. On the whole space, Li, Lin and Wu [55] considered the asymptotic regime for well-prepared initial data of finite energy. Exploiting Strichartz estimates for the linear wave equation on the plane and the modulated energy method the authors obtain strong convergence towards a solution of the incompressible Euler equation.

Our goal is to weaken the regularity assumptions on the initial data (in order to consider vortex configurations) and to carry out the limit by a careful analysis of the acoustic waves. The presence of a second sound described through the dispersive properties of a fourth order operator can be seen also in a similar framework in [25]. On the other hand, one of the main difficulties of considering vortex solutions is that the $2d$ incompressible Euler equation does not admit a weak solution if the initial vorticity is given by point vortices, i. e. a sum of weighted Dirac delta. The most general existence result for the incompressible Euler equations has been derived by Delort [22] and requires that the vorticity satisfies $\omega_0 \in H^{-1}$ positive Radon measure. To overcome this difficulty we shall combine those tools with an analysis done in the same spirit as in [20, 56].

## References

[1] P. Antonelli, M. D'Amico, P. Marcati, *Nonlinear Maxwell-Schrödinger system and quantum magneto-hydrodynamics in 3-D*, Comm. Math. Sci. **15** (2017), 451–479.

[2] P. Antonelli, L.E. Hientzsch, P. Marcati, *The low Mach number limit for some solutions to the QHD system with applications to quantum vortex dynamics*, to appear.

[3] P. Antonelli, P. Marcati, *On the finite energy weak solutions to a system in Quantum Fluid Dynamics*, Comm. Math. Phys. **287** (2009), no 2, 657-686.

[4] P. Antonelli, P. Marcati, *The Quantum Hydrodynamics system in two space dimensions*, Arch. Rat. Mech. Anal. **203** (2012), 499–527.

[5] P. Antonelli, P. Marcati, *Finite Energy Global Solutions to a Two-Fluid Model Arising in Superfluidity*, Bull. Inst. Math. Academia Sinica **10**, no. 3 (2015), 349–373.

[6] P. Antonelli, P. Marcati, *Quantum hydrodynamics with nonlinear interactions*, Disc. Cont. Dyn. Sys. Ser. S **9**, no. 1 (2016), 1–13.

[7] P. Antonelli, P. Marcati, *A model of synchronization over quantum networks*, J. Phys. A **50**, no. 31 (2017), 315101.

[8] P. Antonelli, P. Marcati, H. Zheng, *On the stability of solutions to the one dimensional quantum hydrodynamics system*, to appear.

[9] P. Antonelli, S. Spirito, *Global existence of finite energy weak solutions of quantum Navier-Stokes equations*, Arch. Rat. Mech. Anal. **225**, no.3 (2017), 1161–1199.

[10] C. Audiard, *Dispersive smoothing for the Euler-Korteweg model*, SIAM J. Math. Anal. **44**, no. 4 (2012), 3018 – 3040.

[11] C. Audiard, B. Haspot, *Global well-posedness of the Euler-Korteweg system for small irrotational data*, Comm. Math. Phys. **351**, no. 1 (2017), 2017–247.

[12] G. Baccarani, M. Wordeman, *An investignation of steady state velocity overshoot effects in Si and GaAs devices*, Solid State Electron. **ED–29** (1982), 970–977.

[13] S. Benzoni-Gavage, *Propagating phase boundaries and capillary fluids*, available online at `http://math.univ-lyon1.fr/~benzoni/Levico.pdf`.

[14] F. Bethuel, R. Jerrard, D. Smets, *On the NLS dynamics for infinite energy vortex configurations on the plane*, Rev. Mat. Iberoamericana **24**, no. 2 (2008), 671–702.

[15] F. Bethuel, D. Smets, *A remark on the Cauchy problem for the 2D Gross-Pitaevskii equation with nonzero degree at infinity*, Diff. Int. Equ. **20**, no. 3 (2007), 325–338.

[16] B.Bojarski, P.Hajlasz, P.Strzelecki *Sard's theorem for mappings in Hölder and Sobolev spaces* Manuscripta Mathematica **118** (2005), 383–397.

[17] R. Carles, R. Danchin, J.-C. Saut, *Madelung, Gross-Pitaevskii and Korteweg*, Nonlinearity **25** (2012), 2843–2873.

[18] T. Cazenave, *Semilinear Schödinger Equations*, Courant Lecture Notes in Mathematics, vol. 10. New York University, Courant Institute of Mathematical Sciences, AMS, 2003.

[19] P. Constantin, J.-C. Saut, *Local smoothing properties of dispersive equations*, J. Amer. Math. Soc. **1** (1998), 413–439.

[20] J. Colliander, R. Jerrard, *Ginzburg-landau vortices: weak stability and Schrödinger equation dynamics*, Journal d'Analyse Mathématique. **77** (1999), 129–205.

[21] F. Dalfovo, S. Giorgini, L. Pitaevskii, S. Stringari, *Theory of Bose-Einstein condensation in trapped gases*, Rev. Mod. Phys., **71** (1999), 463–512.

[22] J. Delort *Existence de Nappes de Tourbillon en Dimension Deux.* Journal of the American Mathematical Society. **4** (1991), 553-586

[23] L. De Pascale *The Morse-Sard theorem in Sobolev spaces.* Indiana Univ. Math. J. **50** (2001), 1371–1386.

[24] D. Donatelli, P. Marcati, *Low Mach number limit for the quantum hydrodynamics system*, Research in the Mathematical Sciences. **3** (2016), 3–13.

[25] D. Donatelli, P. Marcati, Quasineutral limit, dispersion and oscillations for Korteweg type fluids, SIAM J. Math. Anal. **47** (2015), 2265–2282

[26] D. Donatelli, E. Feireisl, P. Marcati, *Well/ill posedness for the Euler- Korteweg-Poisson system and related problems*, Communications in Partial Differential Equations, **40** (2015), 1314–1335

[27] B. Ducomet, E. Feireisl, H. Petzeltova, I. Straskraba, *Global in time weak solutions for compressible barotropic self-gravitating fluids*, Disc. Cont. Dyn. Syst. A **11**, no. 1 (2004), 113–130.

[28] B. Erdogan, N. Tzirakis, *Dispersive partial differential equations: Wellposedness and applications.* London Mathematical Society Student Texts **86**. Cambridge University Press, Cambridge, 2016.

[29] E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford Lecture Series in Math **26** (2004).

[30] D. Fujiwara, *A construction of the fundamental equation for the Schrödinger equation*, Journ. Anal. Math. **35** (1979), 41–96.

[31] C. Gardner, *The quantum hydrodynamic model for semincoductor devices*, SIAM J. Appl. Math. **54** (1994), 409–427.

[32] I. Gasser, P. Markowich, *Quantum hydrodynamics, Wigner transforms and the classical limit*, Asympt. Anal. **14** (1997), 97–116.

[33] P. Gérard, *The Cauchy problem for the Gross-Pitaevskii equation*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis **23** (2006) no. 5, 765-779.

[34] J. Ginibre, G. Velo, *The global Cauchy problem for the non linear Schrödinger equation revisited*, Ann. I.H.P. sect. C **2**, no. 4 (1985), 309–327.

[35] J. Ginibre, G. Velo, *On a class of nonlinear Schrödinger equations I, II*, J. Funct. Anal. **32** (1979), 1–71.

[36] V. Ginzburg, L. Pitaevskii *On the theory of superfluidity*, Sov. Phys. JETP. **7** (1958), 858–861

[37] A. Griffin, T. Nikuni, E. Zaremba, *Bose-Condensed Gases at Finite Temperature*, 2009

[38] F. Haas, *Quantum plasmas: An hydrodynamic approach*, New York: Springer.

[39] N. Hayashi, K. Nakamitsu, M. Tsutsumi, *On solutions of the initial value problem for the nonlinear Schrödinger equations in one space dimension*, Math. Z. **192** (1986), 637–650.

[40] N. Hayashi, K. Nakamitsu, M. Tsutsumi, *On solutions of the initial value problem for the nonlinear Schrödinger equations*, J. Funct. Anal. **71** (1987), 218–245.

[41] F. Huang, H.-L. Li, A. Matsumura, *Existence and stability of steady-state of one-dimensional quantum hydrodynamic system for semiconductors*, J. Diff. Equ. **225** (2006), 1–25.

[42] F. Huang, H.-L. Li, A. Matsumura, S. Odanaka, *Well-posedness and stability of quantum hydrodynamics ofr semiconductors in* $\mathbb{R}^3$, in "Some problems on nonlinear hyperbolic equations and applications", Ser. Contemp. Appl. Math. CAM, vol. 15, p. 131–160. Higher Ed. Press, Beijing, 2010.

[43] A. Jüngel, *Dissipative quantum fluid models*, Riv. Mat. Univ. Parma **3** (2012), 217–290.

[44] A. Jüngel, *Global weak solutions to compressible Navier-Stokes equations for quantum fluids*, SIAM J. Math. Anal. **42**, no. 3 (2010), 1025–1045.

[45] A. Jüngel, H.-L. Li, A. Matsumura, *The relaxation-time limit in the quantum hydrodynamic equations for semiconductors*, J. Diff. Equ. **225**, no. 2 (2006), 440–464.

[46] A. Jüngel, J.L. López, J. Montejo-Gámez, *A new derivation of the quantum Navier-Stokes equations in teh Wigner-Fokker-plack approach*, J. Stat. Phys. **145**, no. 6 (2011), 1661–1673.
[47] A. Jüngel, M.C. Mariani, D. Rial, *Local existence of solutions to the transient quantum hydrodynamic equations*, Math. Mod. Meth. Appl. Sci. **12** (2002), 485.
[48] T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Adv. Math. Supp. Stud., Stud. in Apll. Math. **8** (1993), 93–128.
[49] M. Keel, T. Tao, *Enpoint Strichartz Estimates*, Amer. J. Math. **120** (1998), 955–980.
[50] I. Khalatnikov, *An Introduction to the Theory of Superfluidity*, 2000.
[51] M. Kostin, *On the Schrödinger-Langevin equation*, J. Chem. Phys. **57** (1972), 3589–3591.
[52] I. Lacroix-Violet, A. Vasseur, *Global weak solutions to the compressible quantum Navier-Stokes and its semi-classical limit*, preprint `arxiv.org/abs/1607.06646`.
[53] L. Landau, *Theory of the Superfluidity of Helium II*, Phys. Rev. **60**, 356 (1941).
[54] E. Lieb, M. Loss, *Analysis*, Graduate Studies in Mathematics, vol. 14, AMS, 2001.
[55] F. Li, C.-K. Lin, K.-C. Wu, *Asymptotic limit of the Gross-Pitaevskii equation with general initial data*, Science China Mathematics. **59** (2016), 1113–1126.
[56] J. Li, Z. Xin, *Global existence of weak solutions to the barotropic compressible navier-stokes flows with degenerate viscosities*, archived as `arxiv.org/abs/1504.06826`
[57] H-L. Li, P. Marcati, *Existence and asymptotic behavior of multi-dimensional quantum hydrodynamic model for semiconductors*, Comm. Math. Phys. **245** (2004), 215–247.
[58] F. Lin, J.X. Xin, *On the incompressible fluid limit and the vortex motion law of the nonlinear Schrödinger equation*, Communications in Mathematical Physics. **200** (1999), 249–274.
[59] F. Linares and G. Ponce, *Introduction to nonlinear dispersive equations*. Springer-Verlag, New York, 2009.
[60] E. Madelung, *Quantuentheorie in hydrodynamischer form*, Z. Physik **40** (1927), 322.
[61] S. Nishibata, M. Suzuki, *Initial boundary value problems for a quantum hydrodynamic model of semiconductors: Asymptotic behaviors and classical limits*, J. Diff. Equ. **244** (2008), 836–874.
[62] L. Pitaevskii *Vortex lines in an imperfect Bose gas*, Sov. Phys. JETP. **13** (1961), 451–454
[63] J. M. Rakotoson, R. Temam, *An Optimal Compactness Theorem and Application to Elliptic-Parabolic Systems*, Appl. Math. Letters **14** (2001), 303-306.
[64] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Mat. Pura Appl.**146** (1987), 65–96.
[65] P. Sjölin, *Regularity of solutions to the Schrödinger equation*, Duke Math. J. **55**, n.3 (1987), 699–715.
[66] R.S. Strichartz, *Restriction of Fourier transforms to quadratic surfaces and ecay of solutions of wave equations*, Duke Math. J. **44** (1977), 705–774.
[67] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*. CBMS Regional Conference Series in Mathematics, AMS 2006.
[68] M. Tsubota, *Quantized Vortices in Superfluid Helium and Bose-Einstein Condensates*, J. Phys.: Conf. Ser. **31** (2006) 88–94.
[69] Y. Tsutsumi, $L^2-$*solutions for nonlinear Schrödinger equations and nonlinear groups*, Funkc. Ekv. **30** (1987), 115–125.
[70] L. Vega, *Schödinger equations: pointwise convergence to the initial data*, Proc. AMS **102**, n.4 (1988), 874–878.
[71] K. Yajima, *Existence of solutions for Schrödinger evolution equations*, Comm. Math. Phys. **110** (1987), 415–426.
[72] K. Yajima, *On smoothing property of Schrödinger propagators*, Springer Lecture Notes in Math. vol. 1450 (1990), 20–35.

GRAN SASSO SCIENCE INSTITUTE, VIALE FRANCESCO CRISPI, 7, 67100 L'AQUILA
*E-mail address*: `paolo.antonelli@gssi.it`

GRAN SASSO SCIENCE INSTITUTE, VIALE FRANCESCO CRISPI, 7, 67100 L'AQUILA
*E-mail address*: `larseric.hientzsch@gssi.it`

GRAN SASSO SCIENCE INSTITUTE, VIALE FRANCESCO CRISPI, 7, 67100 L'AQUILA
*E-mail address*: `pierangelo.marcati@gssi.it`

GRAN SASSO SCIENCE INSTITUTE, VIALE FRANCESCO CRISPI, 7, 67100 L'AQUILA
*E-mail address*: `haozheng-zheng@gssi.it`