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Asymptotic behavior of radially symmetric solutions for the Burgers equation in several space dimensions

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1 Introduction

In the present article, we consider the asymptotic behavior of radially symmetric solutions of the multi-dimensional Burgers equation. Burgers equation in multi-dimensional space is written as

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u=\mu\Delta u, \quad t>0,\ x\in\mathbb{R}^{n}, \tag{1.1}$$

where $u=(u_{1}(t,x),\cdots,u_{n}(t,x))$ is a vector valued unknown function of $t>0$ and $x=(x_{1},\cdots,x_{n})$, and μ is a given positive constant.

In previous papers [1, 2], we investigated an initial boundary value problem for radially symmetric solutions of (1.1) on the exterior domain $|x|>r_{0}$ for some positive constant r_{0}, where the initial data, and the boundary and far-field conditions are prescribed. Introducing a new unknown variable $v(t,r)$ by $u=(x/r)v(t,r)$ with $r=|x|$, our problem for (1.1) can be rewritten with respect to r as follows:

$$\begin{cases} v_{t}+vv_{r}=\mu(v_{rr}+(n-1)(\frac{v}{r})_{r}), & r>r_{0},\quad t>0,\\ v(t,r_{0})=v_{-}, \quad \lim_{r\to+\infty}v(t,r)=v_{+}, & t>0,\\ v(0,r)=v_{0}(r), & r>r_{0}, \end{cases} \tag{1.2}$$

where the initial data v_{0} is assumed to satisfy the compatibility conditions $v_{0}(r_{0})=v_{-}$ and $\lim_{r\to\infty}v_{0}(r)=v_{+}$.

In a previous article [1], we considered the initial boundary value problem (1.2) for $n\geq 2$ for the cases in which the boundary and far field conditions satisfy (a): $v_{-}<v_{+}=0$, (b): $v_{-}<0<v_{+}$, and (c): $0=v_{-}<v_{+}$, and we showed that the asymptotic states of the time-global solution are given by a monotonically increasing stationary wave in case (a), a superposition of a monotonically increasing

stationary wave and a rarefaction wave in case (b), and a rarefaction wave in case (c). These results are similar to those for the 1-D Burgers equation investigated by Liu-Matsumura-Nishihara [5]. Note that in all cases the corresponding 1-D Riemann problem admits a single rarefaction wave. Here the monotonically increasing property of the stationary wave played an important role.

In [2], we considered case (d), in which $0 < v_- < v_+$, which was excluded in [1]. We first showed that if and only if $0 < v_- \leq 2(n-2)\mu/r_0$ and $v_+ = 0$, there exists a stationary wave, " that decreases monotonically" to zero as $r \to \infty$, which never occurs for the 1-D Burgers equation. Based on this result, we considered the initial boundary value problem (1.2) for $n \geq 3$ in case (d), and showed that the asymptotic state of the time-global solution is given by a superposition of a monotonically decreasing stationary wave and a rarefaction wave under the condition $0 < v_- < \mu/(2r_0)$. Note that in this case the corresponding 1-D Riemann problem still admits a single rarefaction wave, and the asymptotic state for the solution of 1-D Burgers equation is proved to be the rarefaction wave by Liu-Matsumura-Nishihara [5] and Nakamura [8].

In [3], we further investigate cases (e): $0 = v_+ < v_-$, (f): $0 < v_+ \leq v_-$ and (g): $v_- \leq v_+ < 0$. We show that the asymptotic states of the time-global solutions are still given by a monotonically decreasing stationary wave in case (e), and a linear superposition of a monotonically decreasing stationary wave and a rarefaction wave in case (f) under the assumption $0 < v_- < 2\mu/(r_0(1+\sqrt{(n-3)/(n-1)}))$. For case (g), we first show the existence of a non-monotonic stationary wave, which neither increases nor decreases monotonically. This never occurs in the case of the 1-D Burgers equation, and we show that this non-monotonic stationary wave is asymptotically stable. Here, we use the spatial weighted energy method because of the difficulty arising from the non-monotonic property of the stationary wave. Note that in case (g), the asymptotic state for the solution of the 1-D Burgers equation is known to be a monotonically increasing stationary wave (see Liu-Matsumura-Nishihara [5]). This suggests that, the 1-D Riemann problem for the non-viscous part can never classify all of the asymptotic states of the multi-dimensional Burgers equation. In particular, the remaining case (h): $0 > v_+ < v_-$ is open in general. Nevertheless, for the case in which $n = 3$, due to the specific structure of the 3-D equation, we can reduce the problem (1.2) to that for the plain 1-D Burgers equation, and we eventually succeed in obtaining the complete classification of asymptotic states, including a linear superposition of stationary wave and a viscous shock wave. Thus, we can exactly clarify the multi-dimensional effects on the asymptotic behaviors for the case in which $n = 3$.

Some Notation. We denote the usual Lebesgue space of square integrable functions over (r_0, ∞) by $L^2 = L^2((r_0, \infty))$, and denote the corresponding kth-order Sobolev space by $H^k, k = 1, 2, \ldots$. Further, we denote the space of functions $f \in H^1$ with $f(r_0) = 0$ by $H_0^1 = H_0^1((r_0, \infty))$. For $\beta > 0$, we also denote the first-order weighted Sobolev space, that is, the space of functions $(1+r)^{\beta/2} f \in H^k$, by $H^{k,\beta} = H^{k,\beta}((r_0,\infty))$. For an interval $I \subset R^1$ and a Banach space X, $C^k(I;X)$ denotes the space of k-times continuously differentiable X-valued functions on I, and $L^2(I;X)$ denotes the space of square integrable X-valued functions on I.

2 Main theorems

Before we state the main theorems, let us recall the stationary and rarefaction waves of (1.2). We call $\phi(r)$ the stationary wave of (1.2) if ϕ satisfies the stationary problem corresponding to (1.2):

$$\begin{cases} \left(\frac{1}{2}\phi^2\right)_r = \mu\left(\phi_{rr} + (n-1)\left(\frac{\phi}{r}\right)_r\right), & r > r_0, \\ \phi(r_0) = v_-, \quad \lim_{r\to+\infty}\phi(r) = v_+. \end{cases} \tag{2.1}$$

In what follows, we write the solution of (2.1) as $\phi_{v_-,v_+}(r)$ when we emphasize the boundary value of the stationary solution v_- and the far-field state v_+. The basic properties of the stationary wave are given as follows.

Proposition 2.1 (Case $v_+ = 0$). *Suppose $n \geq 3$, $0 < v_- \leq 2\mu(n-2)/r_0$, and $v_+ = 0$. Then the stationary problem* (2.1) *has a unique smooth solution $\phi(r)$ satisfying the following.*
(i) *If $v_- = 2\mu(n-2)/r_0$, then $\phi(r) = 2\mu(n-2)/r$, $r \geq r_0$.*
(ii) *If $0 < v_- < 2\mu(n-2)/r_0$, then $0 < \phi(r) \leq v_-$ and $\phi_r(r) < 0$, $r \geq r_0$.*
Moreover, ϕ satisfies $|\phi(r)| \sim (r+1)^{-n+1}$, $r \to \infty$.

The proof of Proposition 2.1 is clear because the solution of (2.1) is exactly given by the formula

$$\phi(r) = \phi_{v_-,0}(r) = \frac{v_-}{\left(1 - \frac{r_0 v_-}{2\mu(n-2)}\right)(r/r_0)^{n-1} + \frac{r_0 v_-}{2\mu(n-2)}(r/r_0)}. \tag{2.2}$$

Next, we state the non-monotonic stationary wave which is used in Section 3.

Proposition 2.2 (Case $v_+ < 0$). *Suppose $n \geq 3$, $v_- \leq v_+ < 0$. Then the stationary problem* (2.1) *has a unique smooth solution ϕ satisfying the following.*
(i) *There exists a negative constant $\nu_0 \in (v_+, 0)$ such that*

$$\phi(r) < \nu_0, \quad r \geq r_0. \tag{2.3}$$

(ii) *It holds that*

$$\phi(r) - \frac{\mu(n-1)}{r} < v_+, \quad r \geq r_0, \tag{2.4}$$

and $\phi - \mu(n-1)^2/(2r)$ is monotonically increasing, that is,

$$\phi_r(r) > -\frac{\mu(n-1)^2}{2r^2}, \quad r \geq r_0. \tag{2.5}$$

(iii) *It holds that*

$$|\phi(r) - v_+ - \frac{\mu(n-1)}{r}| \leq O(r^{-2}), \quad r \to \infty. \tag{2.6}$$

Next, for $0 \leq v_- < v_+$, we define the rarefaction wave ψ_{v_-,v_+} of (1.2) which connects constant states v_- to v_+ by $\psi_{v_-,v_+} = \hat{\psi}_{v_-,v_+}((r-r_0)/t)$ for $t > 0$, where

$$\hat{\psi}_{v_-,v_+}(\xi) := \begin{cases} v_-, & \xi \leq v_-, \\ \xi, & v_- \leq \xi \leq v_+, \\ v_+, & v_+ \leq \xi. \end{cases} \tag{2.7}$$

Now we are ready to state our first main theorem.

Theorem 2.3. *Suppose $n \geq 3$, $0 < v_- < 2\mu/(r_0(1+\sqrt{(n-3)/(n-1)}))$, and $0 \leq v_+$. Then we have the following results.*

(1) (Asymptotic stability) *Assume that $v_0 - v_+ \in H^1$. Then the initial-boundary value problem* (1.2) *has a unique time-global solution v satisfying*

$$v - v_+ \in C^0([0,\infty); H^1), \qquad v_r \in L^2(0,T;H^1), \quad T > 0,$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{r>r_0} \left| v(t,r) - \phi_{v_-,0}(r) - \hat{\psi}_{0,v_+}\left(\frac{r-r_0}{t}\right) \right| = 0.$$

(2) (Decay rate) *Further assume that $v_0 - v_+ \in H^1 \cap L^1$. Then the solution v satisfies the following decay rate estimates: if $v_+ > 0$, it holds that*

$$\|(v - \phi_{v_-,0} - \psi_{0,v_+})(t)\|_{H^1} \leq C(1+t)^{-\frac{1}{4}} \log^2(2+t), \quad t \geq 1, \tag{2.8}$$

and if $v_+ = 0$, it holds that

$$\|(v - \phi_{v_-,0})(t)\|_{H^1} \leq C(1+t)^{-\frac{1}{4}}, \quad t \geq 0. \tag{2.9}$$

As for the asymptotic stability of the non-monotonic stationary solution ϕ, we have the following:

Theorem 2.4. *Suppose that $n \geq 3$, $v_- \leq v_+ < 0$ and $v_0 - \phi \in H^{1,\frac{n-1}{2}}$. Then there exists a positive constant ϵ_0 such that if $\|v_0 - \phi\|_{H^{1,\frac{n-1}{2}}} \leq \epsilon_0$, then the initial-boundary value problem* (1.2) *has a unique time-global solution v satisfying*

$$v \in C^0([0,\infty); H^1), \quad v_r \in L^2(0,T;H^1), \quad T > 0,$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{r>r_0} |v(t,r) - \phi(r)| = 0.$$

Next, we state the theorem for the complete classification of asymptotic states for the space dimension $n = 3$. In this case, if we introduce a new unknown valuable V by

$$v(t,r) = \frac{2\mu}{r} + V(t,r), \tag{2.10}$$

then our original problem (1.2) for the 3-D Burgers equation is surprisingly reduced to the 1-D Burgers equation:

$$\begin{cases} V_t + VV_r = \mu V_{rr}, & t > 0,\ r > r_0, \\ V(t,r_0) = v_- - 2\mu/r_0 =: V_-, & t > 0, \\ \lim_{r\to+\infty} V(t,r) = v_+ =: V_+, & t > 0, \\ V(0,r) = V_0(r) := v_0(r) - 2\mu/r, & r > r_0. \end{cases} \tag{2.11}$$

Once the problem is reduced to the 1-D Burgers equation, all of the asymptotic behaviors have been classified in terms of the boundary and far field values $V_\pm$ by Liu-Matsumura-Nishihara [5], Nakamura [8], Liu-Nishihara [6], Nishihara [9], and Liu-Yu [7]. To state the results precisely, let us first recall the stationary wave solution Φ to the problem (2.11):

$$\begin{cases} \Phi\Phi_r = \mu\Phi_{rr}, \quad r > r_0, \\ \Phi(r_0) = V_-, \ \lim_{r\to+\infty} \Phi(r) = V_+. \end{cases} \tag{2.12}$$

An elementary calculation shows the following properties.

Proposition 2.5. *If and only if $V_+ \le 0$ and $V_- < |V_+|$, a solution of (2.12), except the trivial solution $\Phi \equiv 0$, uniquely exists and satisfies the following.*
(i) *For $V_- = V_+$, the solution is given by the constant state $\Phi(r) = V_- = V_+$.*
(ii) *For $V_- < V_+ = 0$, the solution is monotonically increasing and, given by*

$$\Phi(r) = \frac{V_-}{1 - \frac{V_-}{2\mu}(r - r_0)}. \tag{2.13}$$

(iii) *For $V_+ < 0$ and $V_- < V_+$ (*resp. *$V_+ < V_- < |V_+|$), the solution is monotonically increasing (*resp. *decreasing), and is given in both cases by*

$$\Phi(r) = \frac{V_+\left(1 - \frac{V_+ - V_-}{V_- + V_-}e^{V_+(r-r_0)/\mu}\right)}{1 + \frac{V_+ - V_-}{V_- + V_-}e^{V_+(r-r_0)/\mu}}. \tag{2.14}$$

We write the stationary wave Φ of (2.12) also as Φ_{V_-,V_+} when the boundary and the far field states are emphasized. Based on the result of Proposition 2.5, it is easy to see that, for the original stationary problem (2.1) with $n = 3$, except the trivial solution $\phi = 0$, the necessary and sufficient condition for the existence of the nontrivial stationary solution is $v_+ \leq 0$ and $v_- - 2\mu/r_0 < |v_+|$, and then the solution is given by the formula

$$\phi_{v_-,v_+}(r) = \frac{2\mu}{r} + \Phi_{v_- - 2\mu/r_0, v_+}(r). \tag{2.15}$$

Next, let us recall the viscous shock wave for the 1-D Burgers equation on the whole space $\mathbb{R}^1$ with the far field states $V_\pm$:

$$\begin{cases} V_t + VV_r - \mu V_{rr}, & t > 0,\ r \in \mathbb{R}^1, \\ \lim\limits_{r \to \pm\infty} V(t,r) = V_\pm, & t > 0. \end{cases} \tag{2.16}$$

We refer to a traveling wave solution of (2.16) with the form $V = \tilde{V}(\xi)$, $\xi = r - st$, as a viscous shock wave of (2.16), where $s \in \mathbb{R}^1$ is the shock speed. The viscous shock wave is known to exist uniquely up to the shift under the entropy condition $V_- > V_+$ and the Rankine-Hugoniot condition $-s(V_+ - V_-) + (V_+^2/2 - V_-^2/2) = 0$, that is, $s = (V_- + V_+)/2$, and has the properties $V_+ < \tilde{V}(\xi) < V_-$, $\tilde{V}_\xi(\xi) < 0$, $\xi \in \mathbb{R}^1$. In fact, the viscous shock wave with $\tilde{V}(0) = (V_- + V_+)/2$ is concretely given by

$$V = \tilde{V}(r - st) = \frac{V_+ + V_-}{2} + \frac{V_+ - V_-}{2} \tanh\Big(-\frac{(V_- - V_+)}{2\mu}(r - st)\Big). \tag{2.17}$$

In the case $s > 0$, that is $V_- + V_+ > 0$, note that the viscous shock wave $\tilde{V}$ is expected to be a good approximation of (2.11) because $\tilde{V}(r_0 - st)$ exponentially tends toward V_- as $t \to \infty$. We write the viscous shock wave $\tilde{V}$ in (2.17) also as $\tilde{V}_{V_-,V_+}$ when the far field states are emphasized

Now we are ready to state our second main theorem.

Theorem 2.6. *Suppose that $n = 3$. Then we have the following classification of the asymptotic states.*

(I) **Case** $v_- - 2\mu/r_0 < v_+ \leq 0$:
Assume that $v_0 - v_+ \in H^1$. Then the initial-boundary value problem (1.2) *has a unique time-global solution v satisfying*

$$v - v_+ \in C^0([0,\infty); H^1), \qquad v_r \in L^2(0,T;H^1), \quad T > 0,$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{r > r_0} \big|v(t,r) - \phi_{v_-,v_+}(r)\big| = 0.$$

(II) **Case** $v_- - 2\mu/r_0 < 0 < v_+$:
Assume that $v_0 - v_+ \in H^1$. Then the initial-boundary value problem (1.2) *has a unique time-global solution v satisfying*

$$v - v_+ \in C^0([0,\infty); H^1), \qquad v_r \in L^2(0,T;H^1), \quad T > 0,$$

and the asymptotic behavior

$$\lim_{t\to\infty}\sup_{r>r_0}\left|v(t,r)-\phi_{v_-,0}(r)-\hat{\psi}_{0,v_+}\left(\frac{r-r_0}{t}\right)\right|=0.$$

Here note that $\hat{\psi}_{0,v_+}=0$ *for* $v_+=0$.

(III) **Case** $0\le v_- - 2\mu/r_0 < v_+$:
Assume that $v_0 - v_+ \in H^1$. *Then the initial-boundary value problem* (1.2) *has a unique time-global solution* v *satisfying*

$$v-v_+\in C^0([0,\infty);H^1),\qquad v_r\in L^2(0,T;H^1),\quad T>0,$$

and the asymptotic behavior

$$\lim_{t\to\infty}\sup_{r>r_0}\left|v(t,r)-\frac{2\mu}{r}-\hat{\psi}_{v_--\frac{2\mu}{r_0},v_+}\left(\frac{r-r_0}{t}\right)\right|=0.$$

(IV) **Case** $v_- - 2\mu/r_0 > v_+,\ v_- + v_+ < 2\mu/r_0$:
Assume that $v_0 - 2\mu/r - v_+ \in H^1\cap L^1$. *Then there exists a positive constant* ϵ_0 *such that if* $\|\chi_0\|_{H^2}\le\epsilon_0$ *then the initial-boundary value problem* (1.2) *has a unique time-global solution* v *satisfying*

$$v-v_+\in C^0([0,\infty);H^1),\qquad v_r\in L^2(0,T;H^1),\quad T>0,$$

and the asymptotic behavior

$$\lim_{t\to\infty}\sup_{r>r_0}\left|v(t,r)-\phi_{v_-,v_+}(r)\right|=0,$$

where the function χ_0 *is defined by*

$$\chi_0(r)=\int_r^\infty(v_0(y)-2\mu/y-\phi_{v_-,v_+}(y))\,dy,\quad r\ge r_0.$$

(V) **Case** $v_- - 2\mu/r_0 > v_+,\ v_- + v_+ > 2\mu/r_0$:
Assume that $v_0 - 2\mu/r - v_+ \in H^2\cap L^1$ *and* $\int_{r_0}^\infty(v_0(r)-2\mu/r-v_+)\,dr>0$. *Then there exist positive constants* ϵ_0 *and* β_0 *such that if* $(1/d_0)+\|W_0\|_{H^{3,\beta_0}}<\epsilon_0$, *then the initial-boundary value problem* (1.2) *has a unique time-global solution* v *satisfying*

$$v-v_+\in C^0([0,\infty);H^2),\qquad v_r\in L^2(0,T;H^2),\quad T>0,$$

and the asymptotic behavior

$$\lim_{t\to\infty}\sup_{r>r_0}\left|v(t,r)-\left(2\mu/r+\tilde{V}_{v_--2\mu/r_0,v_+}(r-st-d(t))\right)\right|=0,$$

for a function $d(t) \in C^0([0,\infty))$ *which has a finite limit* $d_\infty = \lim_{t\to\infty} d(t)$. *Here* $s = (v_+ + v_- - 2\mu/r_0)/2 > 0$, *and* d_0 *and,* W_0 *are defined as*

$$\int_{r_0}^{\infty} (v_0(r) - 2\mu/r - \tilde{V}_{v_- - 2\mu/r_0, v_+}(r - d_0))dr = 0,$$

$$W_0(r) = \int_{r}^{\infty} (v_0(y) - 2\mu/y - \tilde{V}_{v_- - 2\mu/r_0, v_+}(y - d_0))\, dy, \quad r \geq r_0.$$

(VI) **Case** $v_- - 2\mu/r_0 > v_+,\ v_- + v_+ = 2\mu/r_0$ **:**
Assume that $v_0 - 2\mu/r - v_+ \in H^2 \cap L^1$ *and* $\int_{r_0}^{\infty}(v_0(r) - 2\mu/r - v_+)\, dr > 0$. *Then there exist positive constants* ϵ_0 *and* β_0 *such that if* $(1/d_0) + \|W_0\|_{H^{3,\beta_0}} < \epsilon_0$, *then the initial-boundary value problem* (1.2) *has a unique time-global solution* v *satisfying*

$$v - v_+ \in C^0([0,\infty); H^2), \qquad v_r \in L^2(0,T; H^2), \quad T > 0,$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{r>r_0} \left| v(t,r) - \left(2\mu/r + \tilde{V}_{-v_+, v_+}(r - d(t))\right) \right| = 0,$$

for a function $d(t) \in C^0([0,\infty))$ *which has the property* $d(t) \sim \log t,\ t \to \infty$.

The proofs for cases (I) and (II) are based on the arguments in Liu-Matsumura-Nishihara [5]. The proof for case (III) is based on the arguments in Nakamura [8], and the proofs for cases (IV) and (V) are based on the arguments in Liu-Nishihara [6]. The proof for case (VI), which is the most subtle case, is based on the arguments in Nishihara [9] and Liu-Yu [7]. Note that the arguments in [7] are made in a classical function space by using the maximum principle.

Because Theorem 2.3 is proved by combining the results of [1] and [4], we only show the rough sketch of the proof of Theorem 2.4.

3 Asymptotic stability of stationary wave in the case $v_+ < 0$

3.1 Reformulation of the problem

Recall the non-monotonic stationary wave which satisafies (2.1) with $v_- \leq v_+ < 0$. Integrating the equation in (2.1) with respect to r once, we have

$$\begin{cases} \mu\phi_r + \dfrac{\mu(n-1)}{r}\phi = \dfrac{1}{2}(\phi^2 - v_+^2), & r > r_0, \\ \phi(r_0) = v_-, \quad \phi(\infty) = v_+. \end{cases} \tag{3.1}$$

Letting us introduce the perturbation $w(t,r)$ from $\phi(r)$ by

$$v(t,r) = \phi(r) + w(t,r),$$

we rewrite our original problem (1.2) in terms of w as

$$\begin{cases} w_t + \frac{1}{2}(w^2 + 2\phi w)_r = \mu\left(w_{rr} + (n-1)\left(\frac{w}{r}\right)_r\right), & r > r_0, \quad t > 0, \\ w(t, r_0) = 0, & t > 0, \\ w(0,r) = w_0(r) := v_0(r) - \phi(r), & r > r_0. \end{cases} \tag{3.2}$$

Now we further define a new unknown function z by

$$z(t,r) = r^{\frac{n-1}{2}} w(t,r).$$

Then the problem (3.2) is again rewritten in terms of z as in the form

$$\begin{cases} z_t + (\phi z)_r + \Big(-\dfrac{n-1}{2r}\phi + \dfrac{\mu(n^2-1)}{4r^2}\Big) z - \mu z_{rr} & \\ \quad = R(z) := \dfrac{n-1}{2r^{\frac{n+1}{2}}} z^2 - \dfrac{1}{2r^{\frac{n-1}{2}}}(z^2)_r, & r > r_0, \quad t > 0, \\ z(t, r_0) = 0, & t > 0, \\ z(0,r) = z_0(r) := r^{\frac{n-1}{2}}(v_0(r) - \phi(r)), & r > r_0. \end{cases} \tag{3.3}$$

The theorem corresponding to Theorem 2.4 for the reformulated problem (3.3) is written as follows.

Theorem 3.1. *Suppose that $n \geq 3$, $v_- \leq v_+ < 0$ and $z_0 \in H^1$. Then there exists a positive constant ϵ_0 such that if $\|z_0\|_{H^1} \leq \epsilon_0$, then the initial-boundary value problem* (3.3) *has a unique time-global solution z satisfying*

$$z \in C^0([0,\infty); H^1), \quad z_r \in L^2(0,T; H^1), \quad T > 0,$$

and the asymptotic behavior

$$\lim_{t\to\infty} \sup_{r > r_0} |z(r,t)| = 0.$$

Proof. We only show a desired *a priori* estimate for the solution. First, put

$$N(T) = \sup_{0 \leq t \leq T} \|v(t)\|_{H^1},$$

and then we suppose $N(T) \leq 1$ in what follows. Multiplying the equation in (3.3) by z and integrating the resultant equality in terms of r over $[r_0, \infty)$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{r_0}^{\infty} z^2\, dr + \frac{1}{2}\int_{r_0}^{\infty} \Big(\phi_r - \frac{n-1}{r}\phi + \frac{\mu(n^2-1)}{2r^2}\Big) z^2\, dr$$

$$+\mu\int_{r_0}^{\infty}|z_r|^2\,dr=\int_{r_0}^{\infty}zR(z)\,dr. \tag{3.4}$$

Hence, due to Proposition 2.2, we have

$$\frac{1}{2}\frac{d}{dt}\int_{r_0}^{\infty}z^2\,dr+\frac{1}{2}\int_{r_0}^{\infty}(\frac{n-1}{r}\nu_0+\frac{\mu(n-1)}{r^2})z^2\,dr$$

$$+\mu\int_{r_0}^{\infty}|z_r|^2\,dr\leq\int_{r_0}^{\infty}zR(z)\,dr,$$

which implies

$$\begin{aligned}&||z(t)||_{L^2}^2+\int_0^t\Big(\int_{r_0}^{\infty}\frac{z^2(\tau,r)}{r}\,dr+||z_r(\tau)||^2\Big)\,d\tau\\&\leq C\Big(||z_0||_{L^2}^2+N(t)\int_0^t\Big(\int_{r_0}^{\infty}\frac{z^2(\tau,r)}{r}\,dr+||z_r(\tau)||^2\Big)d\tau\Big).\end{aligned} \tag{3.5}$$

Next we proceed to the higher order estimate. Multiplying the equation in (3.3) by $-z_{rr}$ and integrating the resultant equality in terms of r and t over $[r_0,\infty)\times[0,t]$, we get

$$\begin{aligned}&||z_r(t)||_{L^2}^2+\int_0^t||z_{rr}(\tau)||^2\,d\tau\\&\leq C\Big(||z_{0,r}||_{L^2}^2+N(t)\int_0^t\Big(\int_{r_0}^{\infty}\frac{z^2(\tau,r)}{r}\,dr+||z_r(\tau)||^2\Big)d\tau\Big),\end{aligned} \tag{3.6}$$

where we used the equation in (3.1) and basic estimate (3.5). Combining (3.5) and (3.6) and taking $N(t)$ suitably small, we have the desired estimate

$$||z(t)||_{H^1}^2+\int_0^t(||\frac{z}{r}(\tau)||_{L^2}^2+||z_r(\tau)||_{H^1}^2)\,d\tau\leq C||z_0||_{H^1}^2. \tag{3.7}$$

Once the *a priori* estimate (3.7) is established, we can show the existence of time-global solution and its asymptotic behavior as in the same way as the previous papers. Thus the proof of the Theorem 3.1 is completed. □

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