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On a conjugation and a linear operator II

by

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Abstract

Last year, we showed the study of some classes of operators concerning with conjugations on a complex Hilbert space with title "On a conjugation and a linear operator". In this time, we show some results after that.

1. ∞-isometric operators

Definition 1.1 T is said to be ∞-isometric if

$$\lim_{m\to\infty}\sup \|\beta_m(T)\|^{\frac{1}{m}} = 0,$$

where

$$\beta_m(T) = \sum_{j=1}^{m}(-1)^j \binom{m}{j} T^{*m-j}T^{m-j}.$$

T is said to be *m-isometric* if and only if $\beta_m(T) = 0$.

It holds: T : m-isometric $\Longrightarrow$ T : ∞-isometric.

Theorem 1.1 *Let T be ∞-isometric. Then*
(1) $\sigma_a(T) \subset \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$,
(2) *For sequences of unit vectors* $\{x_n\}, \{y_n\}$, *if* $(T-a)x_n \to 0$ *and* $(T-b)y_n \to 0$ $(a \neq b)$, *then* $\langle x_n, y_n\rangle \to 0$.

Hence if $Tx = ax, Ty = by$ $(a \neq b)$, then $\langle x, y\rangle = 0$.

Theorem 1.2 *Let T and T_n be ∞-isometric.*
(1) *If Q is quasinilpotent and $TQ = QT$, then $T + Q$ is ∞-isometric.*
(2) *If $T_n \to S$ in operator norm, then S is ∞-isometric.*
(3) *If T_1 and T_2 are doubly commuting, then T_1T_2 is ∞-isometric.*

Hence it holds that T, S are ∞-isometric, then so is $T \otimes S$.

Definition 1.2 For $T \in \mathcal{L}(\mathcal{H})$, put

$$K_m(T) := \bigcap_{k \geq 0} \ker(\beta_m(T)\, T^k),$$

$$K_\infty(T) := \{x : \lim_{m\to\infty} \sup \|\beta_m(T) T^k x\|^{\frac{1}{m}} = 0 \text{ for all } k \geq 0\}.$$

It holds

$$K_m(T) \subset K_\infty(T).$$

Theorem 1.3 *For all T, it holds:*
(1) *K_m is invariant for T and $T_{|K_m}$ is m-isometric.*
(2) *K_∞ is invariant for T and $T_{|K_\infty}$ is ∞-isometric.*

2. Conjugation and examples

Definition 2.1
$C: \mathcal{H} \to \mathcal{H}$ is said to be *conjugation* on $\mathcal{H}$ if the following conditions hold:
(1) C is antilinear; $C(ax + by) = \bar{a}Cx + \bar{b}Cy$ for all $a, b \in \mathbb{C}$ and $x, y \in \mathcal{H}$.
(2) C is isometric; $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$
(3) C is involutive; $C^2 = I$.

Example 2.1 The followings are examples:
(1) $C(x_1, x_2, x_3, \cdots, x_n) := (\overline{x_1}, \overline{x_2}, \overline{x_3}, \cdots, \overline{x_n})$ on $\mathbb{C}^n$.
(2) $C(x_1, x_2, x_3, \cdots, x_n) := (\overline{x_n}, \overline{x_{n-1}}, \overline{x_{n-2}}, \cdots, \overline{x_1})$ on $\mathbb{C}^n$.
(3) $(Cf)(x) := \overline{f(x)}$ on $\mathcal{L}^2(\mathcal{X}, \mu)$.
(4) $(Cf)(x) := \overline{f(1-x)}$ on $L^2([0,1])$.
(5) $(Cf)(x) := \overline{f(-x)}$ on $L^2(\mathbb{R}^n)$.

3. m-complex symmetric operators

Definition 3.1
(1) An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be an *m-complex symmetric operator* if there exists some conjugation C such that

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} C T^{m-j} C = 0$$

for some positive integer m.
(2) If $m = 1$, we say that T is *complex symmetric* with conjugation C (i.e., $T^* = CTC$).

Set $\Delta_m(T) := \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j} T^{*j}CT^{m-j}C$.
Then T is an m-complex symmetric operator with conjugation C if and only if $\Delta_m(T) = 0$. Note that

$$T^*\Delta_m(T) - \Delta_m(T)(CTC) = \Delta_{m+1}(T).$$

If T is m-complex symmetric with conjugation C, then T is n-complex symmetric with conjugation C for all $n \geq m$.

4. $[m, C]$-isometric operators

Definition 4.1 An operator $T \in \mathcal{L}(\mathcal{H})$ is called an $[m, C]$-*isometric operator* with conjugation C if $\lambda_m(T; C) := \sum_{j=0}^{m}(-1)^j \binom{m}{j} CT^{m-j}C \cdot T^{m-j} = 0$.

It holds

$$CTC \cdot \lambda_m(T; C) \cdot T - \lambda_m(T; C) = \lambda_{m+1}(T; C).$$

Theorem 4.1 *Let T be an $[m, C]$-isometric operator. Then the following statements hold:*
(1) T is bounded below.
(2) $0 \notin \sigma_a(T)$.
(3) T is injective and $R(T)$ is closed.

Theorem 4.2 *Let T be an $[m, C]$-isometric operator. If $a \in \sigma_a(T)$, then $\overline{a}^{-1} \in \sigma_a(T)$.*

Hence we have $\|T\| \geq 1$ if T is $[m, C]$-isometric.

Theorem 4.3 *Let T be an $[m, C]$-isometric operator. Then the following statements hold:*
(1) If T is invertible, then T^{-1} is $[m, C]$-isometric.
(2) T^n is $[m, C]$-isometric for all $n \in \mathbb{N}$.

Theorem 4.4 *Let T be an $[m, C]$-isometric operator and N be n-nilpotent. If $TN = NT$, then $T + N$ is $[m + 2n - 2, C]$-isometric.*

Theorem 4.5 *Let T be an $[m, C]$-isometric operator and S be an $[n, C]$-isometric operator. If $TS = ST$ and $S \cdot CTC = CTC \cdot S$, then TS is $[m + n - 1, C]$-isometric.*

- If C and D are conjugations on $\mathcal{H}$, then $C \otimes D$ is a conjugation on $\mathcal{H} \otimes \mathcal{H}$.

Theorem 4.6 *Let T be an $[m, C]$-isometric operator and S be an $[n, D]$-isometric operator. Then $T \otimes S$ is $[m+n-1, C \otimes D]$-isometric on $\mathcal{H} \otimes \mathcal{H}$.*

5. ∞-complex symmetric operators

Definition 5.1 An operator $T \in \mathcal{L}(\mathcal{H})$ is called an ∞-*complex symmetric operator* with conjugation C if $\limsup_{m \to \infty} \|\Delta_m(T)\|^{\frac{1}{m}} = 0$.

$$\begin{aligned}\{1\text{–CSO}\} \subset \{2\text{–CSO}\} &\subset \{3\text{–CSO}\} \subset \cdots \\ &\subset \{m\text{–CSO}\} \subset \cdots \subset \{\infty\text{–CSO}\}.\end{aligned}$$

Example 5.1 Let C be the canonical conjugation on $\mathcal{H}$ given by

$$C(\sum_{n=0}^{\infty} x_n e_n) = \sum_{n=0}^{\infty} \overline{x_n} e_n$$

where $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$. Given any $\epsilon > 0$, choose a $N > 0$ such that $\frac{1}{N} < \epsilon$. Fix any $m > N$. If W is the weighted shift on $\mathcal{H}$ defined by $We_n = \frac{1}{2^{m+n}} e_{n+1}$ $(n = 0, 1, 2, ...)$ for such m, then $T = I + W$ is an ∞-complex symmetric operator.

Example 5.2 Let C_n be the conjugation on $\mathbb{C}^n$ defined by $C_n(z_1, z_2, \cdots, z_n) := (\overline{z_1}, \overline{z_2}, \cdots, \overline{z_n})$ and let $T = \oplus_{n=1}^{\infty} T_n$ where T_n has the following form;

$$T_n = \begin{pmatrix} \alpha_n & \frac{1}{n} & 0 & \cdots & 0 \\ 0 & \alpha_n & \frac{1}{n} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & \frac{1}{n} \\ 0 & 0 & 0 & \cdots & \alpha_n \end{pmatrix}$$

for a bounded set $\{\alpha_1, \alpha_2, \alpha_3, ...\}$. Then T is an ∞-*complex symmetric operator* with conjugation $C = \oplus_{n=1}^{\infty} C_n$.

Two vectors x and y are C-*orthogonal* if $\langle Cx, y \rangle = 0$.

Theorem 5.3 *Let $T \in \mathcal{L}(\mathcal{H})$ be an ∞-complex symmetric operator with conjugation C and let λ and μ be any distinct eigenvalues of T.*
(1) *Eigenvectors of T corresponding to λ and μ are C-orthogonal.*
(2) *If $\{x_n\}$ and $\{y_n\}$ are sequences of unit vectors such that $\lim_{n\to\infty}(T-\lambda)x_n = 0$ and $\lim_{n\to\infty}(T-\mu)y_n = 0$, then $\lim_{k\to\infty}\langle Cx_{n_k}, y_{n_k}\rangle = 0$, where $\langle Cx_{n_k}, y_{n_k}\rangle$ is any convergent subsequence of $\langle Cx_n, y_n\rangle$.*

Theorem 5.4 *Let Q be a quasinilpotent operator. Then $T = aI + Q$ is an ∞-complex symmetric operator for all $a \in \mathbb{C}$.*

Theorem 5.5 Let T be an m-complex symmetric operator with a conjugation C. If λ is an eigenvalue of T, then $\overline{\lambda}$ is an eigenvalue of T^*.

However, if T is an ∞-complex symmetric operator, this does not hold.

Example 5.3 Let C be the conjugation on $\mathcal{H}$ given by

$$C(\sum_{n=0}^{\infty} x_n e_n) = \sum_{n=0}^{\infty}(-1)^{n+1}\overline{x_n} e_n$$

where $\{e_n\}$ is an orthonormal basis of $\mathcal{H}$ and let W be the weighted shift on $\mathcal{H}$ defined by $We_n = \frac{1}{n+1}e_{n+1}$ $(n = 0, 1, 2, ...)$.
If $T = \lambda I + W^*$, then T is an ∞-complex symmetric operator. Moreover, $(T - \lambda I)e_0 = W^*e_0 = 0$, but $(T^* - \overline{\lambda} I)Ce_0 = WCe_0 = We_0 = e_1 \neq 0$.

Theorem 5.6 If $\{T_n\}$ is a sequence of commuting ∞-complex symmetric operators with conjugation C such that $\lim_{n\to\infty}\|T_n - T\| = 0$, then T is also ∞-complex symmetric with conjugation C.

Theorem 5.7 Let C be a conjugation on $\mathcal{H}$. Assume that $T \in \mathcal{L}(\mathcal{H})$ is a complex symmetric operator with conjugation C and $R \in \mathcal{L}(\mathcal{H})$ commutes with T.
(1) RT is an m-complex symmetric operator with conjugation C if and only if R is an m-complex symmetric operator on $\overline{ran(T^m)}$.
(2) If R is an ∞-complex symmetric operator with conjugation C, then RT is an ∞-complex symmetric operator with conjugation C.

Corollary 5.8 *If T is normal or algebraic operator of order 2 and $R = I + Q$ where Q is quasinilpotent with $QT = TQ$, then $QT + T$ is an ∞-complex symmetric operator.*

Theorem 5.9 *Let S and T be in $\mathcal{L}(\mathcal{H})$ and let C be a conjugation on $\mathcal{H}$. Suppose that $TS = ST$ and $S^*(CTC) = (CTC)S^*$ for a conjugation C.*
(1) *If T and S are m-complex symmetric and n-complex symmetric, respectively, then $T + S$ is $(m+n-1)$-complex symmetric.*
(2) *If T is complex symmetric and S is an ∞-complex symmetric operator, then $T + S$ is ∞-complex symmetric operator.*

• $X \in \mathcal{L}(\mathcal{H})$ is called a *quasiaffinity* if it has trivial kernel and dense range.
• $S \in \mathcal{L}(\mathcal{H})$ is said to be a *quasiaffine transform* of an operator $T \in \mathcal{L}(\mathcal{H})$ if there is a quasiaffinity $X \in \mathcal{L}(\mathcal{H})$ such that $XS = TX$.
• Two operators S and T are *quasisimilar* if there are quasiaffinities X and Y such that $XS = TX$ and $SY = YT$.

Corollary 5.10 *Let $T \in \mathcal{L}(\mathcal{H})$ be an ∞-complex symmetric operator and T have the decomposition property (δ).*
(1) *If T has real spectrum on $\mathcal{H}$, then $exp(iT)$ is decomposable.*
(2) *If $\sigma(T)$ is not singleton and $S \in \mathcal{L}(\mathcal{H})$ is quasisimilar to T, then S has a nontrivial hyperinvariant subspace.*

Corollary 5.11
(1) *If $F \subset \mathbb{C}$ is closed, then the operator $S =: T/_{H_T(F)}$, induced by T, on the quotient space $\mathcal{H}/H_T(F)$ satisfies $\sigma(S) \subset \overline{\sigma(T) \setminus F}$.*
(2) *If $\mathcal{M}$ is a spectral maximal space of T, then $\mathcal{M} = H_T(\sigma(T|_{\mathcal{M}}))$.*
(3) *$f(T)$ is decomposable where f is any analytic function on some open neighborhood of $\sigma(T)$.*
(4) $\sigma(T) = \sigma_{ap}(T) = \sigma_{su}(T) = \cup\{\sigma_T(x) : x \in \mathcal{H}\}$.

Theorem 5.12 *Let T and S be m-complex symmetric and n-complex symmetric with conjugation C, respectively. If T commutes with S and $S^*(CTC) = (CTC)S^*$, then TS is $(m+n-1)$-complex symmetric with conjugation C.*

Theorem 5.13 *Let T and S be an m-complex symmetric operator and n-complex symmetric operator with conjugations C and D, respectively. If T commutes with S and $S^*(CTC) = (CTC)S^*$, then $T \otimes S$ is an $(m+n-1)$-complex symmetric operator with conjugation $C \otimes D$.*

• $T \in \mathcal{L}(\mathcal{H})$ is called a 2-*normal* operator if T is unitarily equivalent to an operator matrix of the form $\begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}$, where N_1, N_2, N_3, N_4 are mutually commuting normal operators.

Corollary 5.14 *If T is an m-complex symmetric operator with a conjugation C and S is a 2-normal operator with $TS = ST$, then $T \otimes U^*NU$ is an m-complex symmetric operator, where $S = U^*NU$ with $N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix}$ and a unitary operator U.*

Example 5.4 Let C be a conjugation given by $C(z_1, z_2, z_3) = (\overline{z_1}, \overline{z_2}, \overline{z_3})$ on $\mathbb{C}^3$. If N is normal and $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ on $\mathbb{C}^3$ with $TN = NT$, then T is a 5-complex symmetric operator with conjugation C. Hence $T \otimes N = \begin{pmatrix} 0 & N & 0 \\ 0 & 0 & 2N \\ 0 & 0 & 0 \end{pmatrix}$ is 5-complex symmetric from Corollary.

Theorem 5.15 *Let T and S be ∞-complex symmetric operators with conjugation C. Assume that $TS = ST$ and $S^*(CTC) = (CTC)S^*$. Then TS is an ∞-complex symmetric operator with conjugation C.*

Theorem 5.16 *Let T and S be ∞-complex symmetric operators with conjugations C and D, respectively. Suppose that T commutes with S and $S^*(CTC) = (CTC)S^*$. Then $T \otimes S$ is an ∞-complex symmetric operator with conjugation $C \otimes D$.*

Theorem 5.17 *Let T and S be ∞-complex symmetric operators with conjugations C and D, respectively. If T commutes with S and $S^*(CTC) = (CTC)S^*$, then $(T \otimes S)^*$ has the property (β) if and only if $T \otimes S$ is decomposable.*

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