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# Operator Norm Inequality and Positive Definiteness of Some Functions

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## 1. Notation

The continuous function $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ is said to be positive definite if for any positive integer $n \in \mathbb{N}$ and for any $x_1, x_2, \ldots, x_n \in \mathbb{R}$, the following $n \times n$ matrix

$$\begin{pmatrix} \varphi(x_1 - x_1) & \varphi(x_1 - x_2) & \cdots & \varphi(x_1 - x_n) \\ \varphi(x_2 - x_1) & \varphi(x_2 - x_2) & \cdots & \varphi(x_2 - x_n) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(x_n - x_1) & \varphi(x_n - x_2) & \cdots & \varphi(x_n - x_n) \end{pmatrix} \geq 0,$$

that is, $\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \varphi(x_i - x_j) \geq 0$ for all $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. By definition, $\varphi(0) \geq 0$, $\varphi(-x) = \overline{\varphi(x)}$, and $|\varphi(x)| \leq \varphi(0)$ for any $x \in \mathbb{R}$.

Typical example of positive definite function is $\varphi(x) = e^{\sqrt{-1}ax}$, where $a \in \mathbb{R}$. This can be seen because of the identity

$$(\varphi(x_i - x_j))_{i,j=1}^{n} = \left(e^{\sqrt{-1}a(x_i - x_j)}\right)_{i,j=1}^{n} = \begin{pmatrix} e^{\sqrt{-1}ax_1} \\ e^{\sqrt{-1}ax_2} \\ \vdots \\ e^{\sqrt{-1}ax_n} \end{pmatrix} \begin{pmatrix} e^{\sqrt{-1}ax_1} \\ e^{\sqrt{-1}ax_2} \\ \vdots \\ e^{\sqrt{-1}ax_n} \end{pmatrix}^{*}.$$

It is known as Bochner's theorem that the function $\varphi$ is positive definite if there exists a positive finite measure $\mu$ on $\mathbb{R}$ such that

$$\varphi(x) = \int_{-\infty}^{\infty} e^{ixt} d\mu(t).$$

## 2. Introduction and Main Results

It is known that

$$\left\| H^{\frac{1}{2}} X K^{\frac{1}{2}} \right\| \leq \frac{1}{2} \| HX + XK \|,$$

where $H, K, X$ are operators on Hilbert space and $H, K$ are positive and invertible. To show this operator norm inequality, we consider two functions as follows:

$$M(s,t) = (st)^{\frac{1}{2}} = t\left(\frac{s}{t}\right)^{\frac{1}{2}} = tf\left(\frac{s}{t}\right)$$

and

$$N(s,t) = \frac{s+t}{2} = t\left(\frac{\frac{s}{t}+1}{2}\right) = tg\left(\frac{s}{t}\right),$$

that is $f(t) = t^{\frac{1}{2}}$ and $g(t) = \frac{t+1}{2}$. Then, we have

$$\frac{f(e^{2x})}{g(e^{2x})} = \frac{e^x}{\frac{e^{2x}+1}{2}} = \frac{1}{\frac{e^x+e^{-x}}{2}} = \frac{1}{\cosh x} = \frac{2\sinh x}{\sinh 2x}.$$

It is known that $\frac{2\sinh x}{\sinh 2x}$ is positive definite. By [3], this fact is equivalent to the operator norm inequality

$$\left\| H^{\frac{1}{2}} X K^{\frac{1}{2}} \right\| \leq \frac{1}{2} ||| HX + XK |||,$$

where $||| \bullet |||$ is any unitarily invariant norm and usual operator norm $\| \bullet \|$ is one of example of unitarily invariant norms. Borrowing notation in [3], we can write the above operator norm inequality as follows:

$$||| M(H,K)X ||| \leq \frac{1}{2} ||| N(H,K)X |||.$$

For $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ and $b_1 \geq b_2 \geq \cdots \geq b_n > 0$, we define the function

$$h(x) = \prod_{i=1}^{n} \frac{b_i \sinh a_i x}{a_i \sinh b_i x}.$$

The following statements had proved in [1]:

(1) If $\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i$ for any $k = 1, 2, \cdots, n$, then $h$ is positive definite.

(2) If $a_1 > b_1$, then $h$ is not positive definite.

(3) If $\sum_{i=1}^{n} a_i > \sum_{i=1}^{n} b_i$, then $h$ is not positive definite.

For $a \geq b > 0$ and $c \geq d > 0$, we set

$$M(s,t) = (st)^{\frac{1-a+b}{2}} \frac{b(s^a - t^a)}{a(s^b - t^b)}$$

and

$$N(s,t) = (st)^{\frac{1-c+d}{2}} \frac{d(s^c - t^c)}{c(s^d - t^d)}.$$

By the facts (1), (2) and (3), we have that the following three statements are equivalent.

(a) $c \geq a$ and $b + c \geq a + d$.

(b) $\dfrac{\sinh ax \sinh dx}{\sinh bx \sinh cx}$ is positive definite.

(c) $|||M(H,K)X||| \leq \frac{1}{2} |||N(H,K)X|||$, where $H, K, X$ are operators on Hilbert space and $H, K$ are positive and invertible.

The following two functions does not satisfies the assumption of (1), (2) and (3):

$$h_1(x) = \frac{\sinh 8x \sinh 6x \sinh x}{\sinh 9x \sinh 4x \sinh 4x}$$

and

$$h_2(x) = \frac{\sinh 8x \sinh 6x \sinh 3x}{\sinh 9x \sinh 4x \sinh 4x}.$$

It is proved in [1] that $h_1$ is positive definite whereas $h_2$ is not positive definite. We can get the following statement and the non positive definiteness of $h_2$ can be extended as stated in Corollary 2.

**Theorem 1.** [2] If $\varphi$ (not necessarily continuous) is positive definite on $\mathbb{R}$ and $\lim_{n\to\infty} \varphi(nx) = \varphi(0)$ for all $x \in \mathbb{R}$, then $\varphi$ is constant.

**Corollary 2.** If $h$ is non-constant satisfying

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$$

and

$$a_1 \times a_2 \times \cdots \times a_n = b_1 \times b_2 \times \cdots \times b_n,$$

then $h$ is not positive definite.

## 3. Proof of the Main Results

We define that the function $\varphi$ is positive definite on $\mathbb{Z}$ if for any natural numbers $n$, $\sum_{i,j=1}^{n} \alpha_i \overline{\alpha_j} \varphi(x_i - x_j) \geq 0$, where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}$ and $x_1, x_2, \ldots, x_n \in \mathbb{Z}$. It is known as Herglotz's theorem that $\varphi$ is positive definite function on $\mathbb{Z}$ if there exists positive finite measure $\mu$ on $\mathbb{T}$ (as a dual of $\mathbb{Z}$) such that

$$\varphi(n) = \int_{\mathbb{T}} e^{2\pi\sqrt{-1}nx} d\mu(x) \ \text{ for all } n \in \mathbb{Z}.$$

Here, $[0,1) \cong \mathbb{R}/\mathbb{Z}$ will identify $\mathbb{T}$.

Before we begin our proof, we will show that

$$\mu(\{0\}) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \varphi(n). \qquad (\dagger)$$

To prove this, it suffices to show that if $\mu(\{0\}) = 0$, then

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \varphi(n) = 0$$

by considering $\mu - \mu(\{0\})\delta_0$ instead of $\mu$, where $\delta_0$ is a Dirac measure at 0. So, we are going to assume that $\mu(\{0\}) = 0$.

Since $|\sin x| \leq |x|$ and $\frac{2x}{\pi} \leq \sin x$ $(0 \leq x \leq \frac{\pi}{2})$, then

$$\frac{1}{2N+1} \left| \frac{\sin(2N+1)\pi x}{\sin \pi x} \right| \leq \frac{1}{2N+1} \times \frac{(2N+1)\pi x}{\frac{2\pi x}{\pi}} = \frac{\pi}{2} \text{ if } 0 \leq x \leq \frac{1}{2}.$$

For a sufficiently small $\varepsilon > 0$, we have

$$\begin{aligned}
\left|\frac{1}{2N+1}\sum_{n=-N}^{N}\varphi(n)\right| &= \left|\frac{1}{2N+1}\sum_{n=-N}^{N}\int_0^1 e^{2\pi\sqrt{-1}nx}d\mu(x)\right| \\
&= \left|\int_0^1 \frac{1}{2N+1}\frac{\sin(2N+1)\pi x}{\sin\pi x}d\mu(x)\right| \\
&\le \int_{-\varepsilon}^{\varepsilon}\left|\frac{1}{2N+1}\frac{\sin(2N+1)\pi x}{\sin\pi x}\right|d\mu(x) \\
&\quad + \int_{\varepsilon}^{1-\varepsilon}\left|\frac{1}{2N+1}\frac{\sin(2N+1)\pi x}{\sin\pi x}\right|d\mu(x) \\
&\le \frac{\pi}{2}\mu((-\varepsilon,\varepsilon)) + \frac{1}{(2N+1)\sin\pi\varepsilon}\mu(\mathbb{T}).
\end{aligned}$$

Hence, $\limsup_{N\to\infty}\left|\frac{1}{2N+1}\sum_{n=-N}^{N}\varphi(n)\right| \le \frac{\pi}{2}\mu((-\varepsilon,\varepsilon))$. Since $\varepsilon$ is arbitrary and $\mu(\{0\}) = 0$, then $\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\varphi(n) = 0$.

**Lemma 3.** Let $\varphi : \mathbb{Z} \longrightarrow \mathbb{C}$ be positive definite and $\lim_{n\to\infty}\varphi(n) = \varphi(0)$. Then, $\varphi$ is constant.

*Proof.* Let $\varphi(n) = \int_0^1 e^{2\pi\sqrt{-1}nx}d\mu(x)$ for all $n \in \mathbb{Z}$. Then, $\varphi(0) = \mu(\mathbb{T})$. Furthermore, by (†) and since $\lim_{n\to\infty}\varphi(n) = \varphi(0)$, then $\mu(\{0\}) = \varphi(0)$. This means that $\mu$ is a non-negative scalar multiple of Dirac measure at 0 and so, $\varphi(n) = \varphi(0)$ for all $n \in \mathbb{Z}$. □

**Proof of Theorem 1.** Given $x \in \mathbb{R}$, we can consider the complex-valued function $\varphi_x$ on $\mathbb{Z}$ in which

$$\varphi_x(n) = \varphi(nx).$$

We have that $\varphi_x$ is positive definite and since $\lim_{n\to\infty}\varphi(nx) = \varphi(0)$, then by Lemma 3, $\varphi_x$ is constant and since $x$ is arbitrarily chosen, then $\varphi$ is constant.

**Proof of Corollary 2.** By the assumption we have

$$h(x) = \frac{\sinh a_1 x \sinh a_2 x \cdots \sinh a_n x}{\sinh b_1 x \sinh b_2 x \cdots \sinh b_n x}.$$

Consider that

$$\begin{aligned} h(0) &= \lim_{x\to 0} h(x) \\ &= \lim_{x\to 0} \frac{\sinh a_1 x \sinh a_2 x \cdots \sinh a_n x}{\sinh b_1 x \sinh b_2 x \cdots \sinh b_n x} \\ &= \frac{a_1 \times a_2 \times \cdots \times a_n}{b_1 \times b_2 \times \cdots \times b_n} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to\infty} h(x) &= \lim_{x\to\infty} \frac{(e^{a_1 x} - e^{-a_1 x})(e^{a_2 x} - e^{-a_2 x})\dots(e^{a_n x} - e^{-a_n x})}{(e^{b_1 x} - e^{-b_1 x})(e^{b_2 x} - e^{-b_2 x})\dots(e^{b_n x} - e^{-b_n x})} \\ &= \lim_{x\to\infty} \frac{e^{(a_1+a_2+\cdots+a_n)x}(1 - e^{-2a_1 x})(1 - e^{-2a_2 x})\dots(1 - e^{-2a_n x})}{e^{(b_1+b_2+\cdots+b_n)x}(1 - e^{-2b_1 x})(1 - e^{-2b_2 x})\dots(1 - e^{-2b_n x})} \\ &= 1. \end{aligned}$$

This means

$$\lim_{n\to\infty} h(nx) = 1 \text{ for all } x \in \mathbb{R}.$$

Since $h$ is non-constant, then by Theorem 1, $h$ is not positive definite.

# References

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[4] Hiai. F and Petz. D, *Introduction to Matrix Analysis and Applications*, Springer, 2014.

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